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Existence of periodic solutions for nonautonomous second-order discrete Hamiltonian systems

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Abstract
In this paper, we consider the existence of periodic solutions for a class of nonautonomous second-order discrete Hamiltonian systems in case the sum on the time variable of potential is periodic. The tools used in our paper are the direct variational minimizing method and Rabinowitz's saddle point theorem.

MSC: 34C25; 58E50

Keywords: variational minimizers; saddle point theorem; discrete Hamiltonian systems; periodic

1 Introduction and main results

Consider the following discrete Hamiltonian system:

$$\triangle^2 u(t-1) + \nabla F\bigl(t, u(t)\bigr) = 0, \quad t \in \mathbb{Z}, \tag{1.1}$$

where $\triangle^2 u(t) = \triangle(\triangle u(t))$, and $\nabla F(t,x)$ denotes the gradient of $F(t,x)$ in x. In this paper, we always suppose that the following condition is satisfied:

(A) $F(t,x) \in C^1(\mathbb{R}^N, \mathbb{R})$ for any $t \in \mathbb{Z}$, and $F(t+T,x) = F(t,x)$ for $(t,x) \in \mathbb{Z} \times \mathbb{R}^N$, where $T > 0$ is a integer.

In the last years, a great deal of work has been devoted to the study of the existence and multiplicity of periodic solutions for discrete Hamiltonian system (1.1); see [1–16] and the references therein. In particular, Guo and Yu [7] considered the existence of one periodic solution to system (1.1) in case $\nabla F(t,x)$ is bounded. Xue and Tang [12, 13] generalized these results when the gradient of potential energy does not exceed sublinear growth.

Tang and Zhang [11] completed and extended the results obtained in [12, 13] under a more weaker assumption on $F(t,x)$.

Recently, Yan *et al.* [15] obtained multiple periodic solutions for system (1.1) when the growth of $\nabla F(t,x)$ is sublinear and there exists an integer $r \in [0,N]$ such that:

(i) $F(t,x)$ is T_i-periodic in x_i, $1 \le i \le r$.

(ii)

$$|x|^{-2\alpha} \sum_{t=1}^{T} F(t,x) \to \pm\infty \quad \text{as } |x| \to \infty, x \in \{0\} \times \mathbb{R}^{N-r}.$$

In this paper, motivated by the results mentioned and [17], we further study the existence of periodic solutions to the discrete Hamiltonian system (1.1).

Our main results are the following theorems.

Theorem 1.1 *Suppose that* (A) *holds and*

(H_1) $\sum_{t=1}^{T} F(t, x + T_i e_i) = \sum_{t=1}^{T} F(t,x)$, $1 \leq i \leq N$, *where* $T_i > 0$, *and* $\{e_i | 1 \leq i \leq N\}$ *is an orthogonal basis in* $\mathbb{R}^N$;

(H_2) *there exist* $0 < C_1 < 2\sin^2 \frac{\pi}{T}$ *and* $C_2 > 0$ *such that*

$$\left|F(t,x)\right| \leq C_1|x|^2 + C_2.$$

Then system (1.1) *has at least one* T*-periodic solution.*

Corollary 1.1 *Let* $F(t,x) = -a\cos x - e(t)x$. *If* $e(t)$ *satisfies*

$$e(t+T) = e(t), \qquad \sum_{t=1}^{T} e(t) = 0,$$

then system (1.1) *has at least one* T*-periodic solution.*

Remark 1.1 When $F(t,x) = -a\cos x - e(t)x$ ($a \geq 0$), system (1.1) is a discrete form of forced equations studied by Mawhin and Willem [18–20], in which they require the assumption that the forced potential is periodic on spatial variables. So, our results, Theorem 1.1 and Corollary 1.1, generalize their results in discrete situation.

Theorem 1.2 *Suppose that* (A) *and* (H_1) *hold and*

(H_3) *there exist* $\mu_1 < 2$ *and* $\mu_2 \in \mathbb{R}$ *such that*

$$\left(\nabla F(t,x), x\right) \leq \mu_1 F(t,x) + \mu_2;$$

(H_4) *there exists* $\delta > 0$ *such that, for* $t \in \mathbb{Z}$, *we have*

$$F(t,x) > \delta, \quad |x| \to +\infty;$$

(H_5) *there exists* $0 < b < 2\sin^2 \frac{\pi}{T}$ *such that*

$$F(t,x) \leq b|x|^2.$$

Then system (1.1) *has at least one* T*-periodic solution. Furthermore, system* (1.1) *has at least one nonconstant* T*-periodic solution if* $\sum_{t=1}^{T} F(t,x) \geq 0$ *for all* $x \in \mathbb{R}^N$.

2 Some important lemmas

Let

$$H_T = \left\{u : \mathbb{Z} \to \mathbb{R}^N \mid u(t) = u(t+T) \text{ for all } t \in \mathbb{Z}\right\}$$

with norm

$$\|u\| = \left(\sum_{t=1}^{T}\left|\triangle u(t)\right|^2\right)^{\frac{1}{2}} + \left|\sum_{t=1}^{T} u(t)\right|.$$

Set

$$\Phi(u) = \frac{1}{2}\sum_{t=1}^{T}\left|\triangle u(t)\right|^2 - \sum_{t=1}^{T} F\bigl(t, u(t)\bigr)$$

and

$$\bigl\langle\Phi'(u), v\bigr\rangle = \sum_{t=1}^{T}\bigl(\triangle u(t), \triangle v(t)\bigr) - \sum_{t=1}^{T}\bigl(\nabla F\bigl(t, u(t)\bigr), v(t)\bigr)$$

for $u, v \in H_T$.

According to assumption (A), it is well known that Φ is continuously differentiable and the T-periodic solutions of problem (1.1) correspond to the critical points of the functional Φ.

Definition 2.1 ([21]) Assume that X is a Banach space and $f \in C^1(X, \mathbb{R})$. If $\{u_n\} \subset X$ satisfies

$$f(u_n) \to C, \qquad \bigl(1 + \|u_n\|\bigr)f'(u_n) \to 0,$$

then we say that $\{u_n\}$ is a $(CPS)_C$ sequence of f. For any $(CPS)_C$ sequence $\{u_n\}$, if there exists a subsequence of $\{u_n\}$ convergent in X, then we say that f satisfies $(CPS)_C$ condition.

Lemma 2.1 ([19, 22]) *Assume that X is a Banach space and $f \in C^1(X, \mathbb{R})$. Let $X = X_1 \oplus X_2$ and*

$$\dim X_1 < +\infty, \qquad \sup_{S_R^1} f < \inf_{X_2} f,$$

where $S_R^1 = \{u \in X_1 \mid |u| = R\}$.

Set $B_R^1 = \{u \in X_1, |u| \le R\}$, $M = \{g \in C(B_R^1, X) \mid g(s) = s, s \in S_R^1\}$,

$$C = \inf_{g \in M} \max_{s \in B_R^1} f\bigl(g(s)\bigr).$$

Then $C > \inf_{X_2} f$. Furthermore, if f satisfies $(CPS)_C$ condition, then C is a critical value of f.

Lemma 2.2 ([11]) *If $u \in H_T$ and $\sum_{t=1}^{T} u(t) = 0$, then*

$$\sum_{t=1}^{T}\left|u(t)\right|^2 \le \frac{1}{4\sin^2\frac{\pi}{T}}\sum_{t=1}^{T}\left|\triangle u(t)\right|^2$$

and

$$\|u\|_\infty^2 := \Bigl(\max_{t\in\mathbb{Z}[1,T]}\bigl|u(t)\bigr|\Bigr)^2 \le \frac{T^2-1}{6T}\sum_{t=1}^{T}\bigl|\triangle u(t)\bigr|^2.$$

3 Proof of main results

Proof of Theorem 1.1 Let

$$H_T = \mathbb{R}^N \oplus \widetilde{H}_T,$$

where $\widetilde{H}_T = \{u \in H_T : \overline{u} = \frac{1}{T}\sum_{t=1}^{T} u(t) = 0\}$.

For any $u \in H_T$, there are $\tilde{u} \in \widetilde{H}_T$ and $\overline{u} \in \mathbb{R}^N$ such that $u = \tilde{u} + \overline{u}$.

According to (H_2), we have that

$$\begin{aligned}\Phi(\tilde{u}) &= \frac{1}{2}\sum_{t=1}^{T}\bigl|\triangle\tilde{u}(t)\bigr|^2 - \sum_{t=1}^{T}F\bigl(t,\tilde{u}(t)\bigr)\\ &\ge \frac{1}{2}\sum_{t=1}^{T}\bigl|\triangle\tilde{u}(t)\bigr|^2 - C_1\sum_{t=1}^{T}\bigl|\tilde{u}(t)\bigr|^2 - TC_2\\ &\ge \frac{1}{2}\sum_{t=1}^{T}\bigl|\triangle\tilde{u}(t)\bigr|^2 - \frac{C_1}{4\sin^2\frac{\pi}{T}}\sum_{t=1}^{T}\bigl|\triangle\tilde{u}(t)\bigr|^2 - TC_2\\ &= \Bigl(\frac{1}{2} - \frac{C_1}{4\sin^2\frac{\pi}{T}}\Bigr)\sum_{t=1}^{T}\bigl|\triangle\tilde{u}(t)\bigr|^2 - TC_2.\end{aligned}$$

So,

$$\Phi(\tilde{u}) \to +\infty \quad \text{as } \|\tilde{u}\| \to \infty. \tag{3.1}$$

Suppose that $\{u_k\}$ is a minimizing sequence for Φ, that is,

$$\Phi(u_k) \to \inf\Phi, \quad k \to \infty.$$

Then $u_k = \tilde{u}_k + \overline{u}_k$, where $\tilde{u}_k \in \widetilde{H}_T$, $\overline{u}_k \in \mathbb{R}^N$. By (3.1) there exists $c > 0$ such that

$$\|\tilde{u}_k\| \le c. \tag{3.2}$$

By (H_1) we have that

$$\Phi(u + T_i e_i) = \Phi(u), \quad u \in H_T, 1 \le i \le N.$$

Hence, if $\{u_k\}$ is a minimizing sequence for Φ, then

$$(\tilde{u}_k \cdot e_1 + \overline{u}_k \cdot e_1 + k_1 T_1, \dots, \tilde{u}_k \cdot e_N + \overline{u}_k \cdot e_N + k_N T_N)$$

is also a minimizing sequence of Φ.

Therefore, we can assume that

$$0 \le \overline{u}_k \cdot e_i \le T_i, \quad 1 \le i \le N. \tag{3.3}$$

By (3.2) and (3.3), $\{u_k\}$ is a bounded minimizing sequence of Φ in H_T.

Going to a subsequence if necessary, since H_T is finite dimensional, we can assume that $\{u_k\}$ converges to some $u_0 \in H_T$.

Since Φ is continuously differentiable, we have

$$\Phi(u_0) = \inf \Phi(u), \qquad \Phi'(u_0) = 0.$$

Therefore, the proof is finished. □

Proof of Theorem 1.2 For the proof, we will apply Rabinowitz's saddle point theorem. First, to prove that Φ satisfies the $(CPS)_C$ condition. Suppose that for C, a sequence $\{u_k\} \in H_T$ satisfies

$$\Phi(u_k) \to C, \qquad \bigl(1 + \|u_k\|\bigr)\Phi'(u_k) \to 0.$$

Since $\Phi(u_k) \to C$, we have that

$$\frac{1}{2}\sum_{t=1}^{T}\bigl|\triangle u_k(t)\bigr|^2 - \sum_{t=1}^{T} F\bigl(t, u_k(t)\bigr) \to C. \tag{3.4}$$

From (H_3) we have that

$$\begin{aligned}\bigl\langle \Phi'(u_k), u_k\bigr\rangle &= \sum_{t=1}^{T}\bigl|\triangle u_k(t)\bigr|^2 - \sum_{t=1}^{T}\bigl(\nabla F\bigl(t, u_k(t)\bigr), u_k(t)\bigr) \\ &\ge \sum_{t=1}^{T}\bigl|\triangle u_k(t)\bigr|^2 - \mu_1 \sum_{t=1}^{T} F\bigl(t, u_k(t)\bigr) - \mu_2 T.\end{aligned}$$

By (3.4) we have that

$$-\sum_{t=1}^{T} F\bigl(t, u_k(t)\bigr) = C - \frac{1}{2}\sum_{t=1}^{T}\bigl|\triangle u_k(t)\bigr|^2 + \varepsilon.$$

So, we have

$$\begin{aligned}\bigl\langle \Phi'(u_k), u_k\bigr\rangle &= \sum_{t=1}^{T}\bigl|\triangle u_k(t)\bigr|^2 - \sum_{t=1}^{T}\bigl(\nabla F\bigl(t, u_k(t)\bigr), u_k(t)\bigr) \\ &\ge \left(1 - \frac{\mu_1}{2}\right)\sum_{t=1}^{T}\bigl|\triangle u_k(t)\bigr|^2 + C\mu_1 - \mu_2 T + \varepsilon.\end{aligned}$$

Therefore,

$$\left(1 - \frac{\mu_1}{2}\right)\sum_{t=1}^{T}\bigl|\triangle u_k(t)\bigr|^2 + C\mu_1 - \mu_2 T \le 0.$$

From this inequality we have that $\sum_{t=1}^{T} | \triangle u_k(t)|^2$ is bounded.

By (H_1) we have that

$$\Phi(u + T_i e_i) = \Phi(u), \quad u \in H_T, 1 \leq i \leq N.$$

Therefore, if $\{u_k\}$ is a $(CPS)_C$ sequence of Φ, then

$$(\tilde{u}_k \cdot e_1 + \overline{u}_k \cdot e_1 + k_1 T_1, \ldots, \tilde{u}_k \cdot e_N + \overline{u}_k \cdot e_N + k_N T_N)$$

is also a $(CPS)_C$ sequence of Φ.

So, we can assume that

$$0 \leq \overline{u}_k \cdot e_i \leq T_i, \quad 1 \leq i \leq N,$$

that is, $|\overline{u}_k|$ is bounded.

From these results we have that $\{u_k\}$ is bounded.

Since H_T is a finite-dimensional Banach space, it is easy to see that Φ satisfies the $(CPS)_C$ condition.

We now prove that the conditions of Rabinowitz's saddle point theorem are satisfied.

Let

$$X_1 = \mathbb{R}^N, \qquad X_2 = \left\{ u \in H_T : \sum_{t=1}^{T} u(t) = 0 \right\}.$$

For any $u \in X_2$, by (H_5) and Lemma 2.2 we have that

$$\begin{aligned}
\Phi(u) &= \frac{1}{2} \sum_{t=1}^{T} \left|\triangle u(t)\right|^2 - \sum_{t=1}^{T} F\bigl(t, u(t)\bigr) \\
&\geq \frac{1}{2} \sum_{t=1}^{T} \left|\triangle u(t)\right|^2 - b \sum_{t=1}^{T} \left|u(t)\right|^2 \\
&\geq \frac{1}{2} \sum_{t=1}^{T} \left|\triangle u(t)\right|^2 - \frac{b}{4 \sin^2 \frac{\pi}{T}} \sum_{t=1}^{T} \left|\triangle u(t)\right|^2 \\
&= \left(\frac{1}{2} - \frac{b}{4 \sin^2 \frac{\pi}{T}} \right) \sum_{t=1}^{T} \left|\triangle u(t)\right|^2 \\
&\geq 0.
\end{aligned}$$

On the other hand, for any $u \in X_1$, by (H_4) we have that

$$\Phi(u) = -\sum_{t=1}^{T} F\bigl(t, u(t)\bigr) \leq -\delta, \quad |u| \to +\infty.$$

From this it follows that the conditions of Rabinowitz's saddle point theorem are all satisfied.

So, by Lemma 2.1 there exists a periodic solution of system (1.1). Furthermore, if $\sum_{t=1}^{T} F(t,x) \geq 0$, then there exists a nonconstant periodic solution $\overline{u}$ of system (1.1) such that $\Phi(\overline{u}) = C > \inf_{X_2} \geq 0$ since otherwise we would have a contradiction with the fact that $\Phi(\overline{u}) = -\sum_{t=1}^{T} F(t,\overline{u}(t)) \leq 0$.

Therefore, the proof is finished. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All the authors have the same contribution. All authors read and approved the final manuscript.

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