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A Mellin transform method for solving fuzzy differential equations

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Abstract

In this paper, we introduce the fuzzy Mellin transform and investigate some of its operator properties. We then establish the connections with the two-side fuzzy Laplace transform. By using a fuzzy Mellin transform, we solve some fuzzy differential equations under strongly generalized differentiable conditions. Finally, some simple applications are given.

Keywords: fuzzy Mellin transform; fuzzy Laplace transform; fuzzy differential equation

1 Introduction

The first definition of differentiability for fuzzy-valued functions was proposed by Puri and Ralesu in [1], which is based on the Hukuhara difference of sets. The definition of fuzzy-valuedness as a generalization of the Hurahara derivative for set-valued mappings is a rather restrictive concept of derivative. In order to overcome the limitations, some new notions of difference were given, such as the generalized difference [2] and the generalized Hurahara difference [3].

Bica *et al.* proved the convergence of the method of successive approximations and used to approximate the solution of the nonlinear Hammerstein fuzzy integral equation, and then they proposed the notion of numerical stability of the algorithm with respect to the choice of the first iteration in [4]. Cabral *et al.* considered fuzzy differential equations with parameters and initial conditions interactive in two differential ways: differential inclusion and the extension principle [5]. Under generalized differentiability Mosleh *et al.* presented an approach for approximating the fuzzy linear system of differential equations in [6]. The airfoil and the Chebyshev polynomials methods for solving the fuzzy Fredholm integro-differential equation with Cauchy kernel under generalized H-differentiability was discussed in [7]. Moreover, Chalco-Cano *et al.* discussed the formulation and procedure for solving fuzzy differential equations via a differential inclusion and gave several examples showing the corrected and incorrect procedure for solving differential equations in [8]. Agarwal *et al.* proposed the notion of a differential equation of fractional order with uncertainty and presented the concept of the solution [9]. Malinowski established mathematical foundations of random fuzzy fractional integral equations which involved a fuzzy integral of fractional order [10]. For more study on fuzzy fractional differential equations one may refer to [11–14].

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Recently, Salahshour *et al.* dealt with the solutions of fuzzy Volterra integral equations with separable kernel by using a fuzzy differential transform method in [15]. Moreover, the fuzzy Laplace transform was expressed by Salahshour *et al.* in [16]. And the existence theorem was given for a fuzzy-valued function which possesses the fuzzy Laplace transform. Ahmadi *et al.* investigated the Laplace transform formula on the fuzzy nth-order derivative by using the strongly generalized differentiability concept in [17]. The fuzzy Laplace transforms method for solving fuzzy fractional differential equations was proposed by Salahshour *et al.* in [18].

By the change of variables $x = \exp(-t)$ of the classical Mellin transform, one can obtain its Laplace transform. By using this connection with the two-side fuzzy Laplace transform, we can deduce the operator properties of the fuzzy Mellin transform. Then we may use the Mellin transform technology to solve some kinds of fuzzy differential equations. That is the main result of this paper. Some simple applications are given in the last section.

2 Preliminaries

We denote the sets of all nonempty convex compact subsets of metric space $(\mathcal{X}, d)$ by $\mathcal{K}(\mathcal{X})$. The Hausdorff metric for $X_1, X_2 \in \mathcal{K}(\mathcal{X})$ is defined as

$$D(X_1, X_2) = \inf\bigl\{X_1 \subset N(X_2, \varepsilon) \text{ and } X_2 \subset N(X_1, \varepsilon)\bigr\},$$

where $N(X_1, \varepsilon) = \{x_1 \in \mathcal{X} \mid d(x_1, x_2) < \varepsilon \text{ for some } x_2 \in X_1\}$.

A real-valued mapping $\nu : \mathcal{X} \to [0,1]$ is called a fuzzy set.

Let us denote by $\mathcal{E}$ the set of all fuzzy sets satisfying the following four conditions:

(i) ν is normal, *i.e.*, $\exists x_1 \in \mathcal{X}$ with $\nu(x_1) = 1$;

(ii) ν is convex (*i.e.*, $\nu(\lambda x_1 + (1-\lambda)x_2) \geq \min\{\nu(x_1), \nu(x_2)\}$, $\forall \lambda \in [0,1]$, $x_1, x_2 \in \mathcal{X}$);

(iii) ν is semicontinuous on $\mathcal{X}$;

(iv) $\overline{\{x_1 \in \mathcal{X}; \nu(x_1) > 0\}}$ is compact, where $\bar{B}$ denotes the closure of B.

For $0 < r \leq 1$, the r-level set of ν is denoted by $[\nu]^r = \{x_1 \in \mathcal{X} \mid \nu(x_1) \geq r\}$ and $[\nu]^0 = \overline{\{x_1 \in \mathcal{X} \mid \nu(x_1) > 0\}}$.

Zadeh's extension principle implies

$$(\nu_1 + \nu_2)(x_1) = \sup_{y \in \mathcal{X}} \min\bigl\{\nu_1(y), \nu_2(x_1 - y)\bigr\}, \quad x \in \mathcal{X},$$

and

$$(k\nu)(x_1) = \nu(x_1/k), \quad k > 0, \quad \text{and} \quad (k\nu)(x_1) = \tilde{0} \in \mathcal{E}, \quad k = 0.$$

In particular, for $\nu_1, \nu_2 \in \mathcal{E}$, $\mathcal{X} \subset \mathbb{R}$, and $\lambda \in \mathbb{R}$, we have

$$[\nu_1 \oplus \nu_2]^r = [\nu_1]^r + [\nu_2]^r, \qquad [\lambda \nu_1]^r = \lambda[\nu_1]^r, \quad \forall r \in [0,1],$$

where $[\nu_1]^r + [\nu_2]^r$ means the usual addition of two closed convex intervals of $\mathbb{R}$ and $\lambda[\nu]^r$ means the usual product between a scalar and an intervals of $\mathbb{R}$.

A mapping $f : \mathcal{X} \to \mathcal{E}$, $\mathcal{X} \subset \mathbb{R}$ is called a fuzzy-valued function. Some authors have considered the Laplace transform of fuzzy-valued functions and called it a fuzzy Laplace transform (see [15, 16, 18]). Notice that the kernels functions of Laplace transform and

Mellin transform are both complex-valued functions. In order to compute $k\nu$ ($k \in \mathbb{C}$, $\nu \in \mathcal{E}$), we have to modify the above definitions. The notion of the complex membership function proposed by Tamir *et al.* in [19] seems to be suitable.

The complex membership function is defined as

$$\nu(t) = \nu_1(t) + \mathrm{i}\nu_2(t),$$

where $\nu_1, \nu_2 : \mathbb{R} \to [0,1]$ and $t \in \mathbb{R}$. For convenience, denote (ν_1, ν_2) by ν. Then, for $\nu = (\nu_1, \nu_2)$ the (α_1, α_2)-level set is denoted by

$$[\nu]^{(\alpha_1,\alpha_2)} = [\nu_1]^{\alpha_1} \cap [\nu_2]^{\alpha_2}.$$

It is easy to see that the (α_1, α_2)-level of ν is always compact and convex. In order to ensure that ν is normal, we define the following set:

$$\widehat{\mathcal{E}'} = \bigl\{(\nu_1, \nu_2) \in \mathcal{E} \times \mathcal{E} \mid \exists t_0 \in \mathbb{R} \text{ s.t. } \nu_1(t_0) = \nu_2(t_0) = 1\bigr\}, \tag{2.1}$$

where $\mathcal{E} = \{\nu \mid \nu : \mathbb{R} \to [0,1]\}$.

We define $\hat{0} \in \mathcal{E}$ by $\hat{0}(t) = 1$ when $t = 0$ and $\hat{0}(t) = 0$ otherwise. The zero element on $\widehat{\mathcal{E}'}$ then reads $\hat{0}_2(t) = (\hat{0}(t), \hat{0}(t)) \in \mathcal{E} \times \mathcal{E}$.

For $f = (u_f, \nu_f), g = (u_g, \nu_g) \in \widehat{\mathcal{E}'}$, we give the notions of addition and scalar multiplication as follows:

$$\begin{aligned} &f + g = (u_f + u_g, \nu_f + \nu_g), \\ &c \odot f = (au_f - b\nu_f, a\nu_f + bu_f) \quad (c = a + \mathrm{i}b). \end{aligned} \tag{2.2}$$

The Hausdorff distance $D_1 : \mathcal{E} \times \mathcal{E} \to [0, \infty)$ is defined by

$$D_1(\nu_1, \nu_2) = \sup\bigl\{D\bigl([\nu_1]^r, [\nu_2]^r\bigr) \mid r \in [0,1]\bigr\}. \tag{2.3}$$

For $f = (u_f, \nu_f), g = (u_g, \nu_g) \in \widehat{\mathcal{E}'}$, we then define the Hausdorff distance $D_2 : \widehat{\mathcal{E}'} \to [0, \infty)$ by

$$D_2(f, g) = D_1\bigl((u_f, \nu_f), (u_g, \nu_g)\bigr) = \max\bigl\{D_1(u_f, u_g), D_1(\nu_f, \nu_g)\bigr\}. \tag{2.4}$$

$(\widehat{\mathcal{E}'}, D_2)$ is a complete metric space. Using the method given by Karpenko *et al.* [20], one can show that $(\widehat{\mathcal{E}'}, D_2)$ is isometrically embedded in a Banach space.

A mapping $F : [a, b] \to \widehat{\mathcal{E}'}$ is called complex fuzzy-valued function. In order to investigate differentiability of the complex membership function, we need the notion of the difference of two fuzzy numbers. There are some different definitions such as the Hukuhara difference, the generalized Hukuhara difference, the generalized difference, and so on. But we only consider the following H-difference in this paper.

Let $x, y \in \widehat{\mathcal{E}'}$. If there exists $z \in \widehat{\mathcal{E}'}$ such that $x = y + z$, then z is called the H-difference of x and y and it is denoted by $x - y$.

Let $F(x) = (u_f(x), \nu_f(x))$ and assume that $u_f(x)$ and $\nu_f(x)$ are both fuzzy Riemann-integrable, which was introduced by Stefanini and Bede.

Definition 1 ([21]) Let u be a fuzzy-valued function on $[a,\infty]$, and $u(x;r) = [\underline{u}(x;r), \bar{u}(x;r)]$. Assume that the endpoint functions $\underline{u}(x;r)$ and $\bar{u}(x;r)$ are both Riemann-integrable on $[a,\infty]$ for every $b \geq a$, and assume that there are two positive $\underline{M}(r)$ and $\bar{M}(r)$ such that $\int_a^b |\underline{u}(x;r)|\,\mathrm{d}x \leq \underline{M}(r)$ and $\int_a^b |\bar{u}(x;r)|\,\mathrm{d}x \leq \bar{M}(r)$ for every $b \geq a$. Then $u(x)$ is improper fuzzy Riemann-integrable on $[a,\infty]$, and the improper fuzzy Riemann-integral is denoted by $Iu(x)$.

Furthermore, one can obtain

$$Iu(x;r) = \int_a^\infty u(x;r)\,\mathrm{d}x = \left[\int_a^\infty \underline{u}(x;r)\,\mathrm{d}x, \int_a^\infty \bar{u}(x;r)\,\mathrm{d}x\right]. \tag{2.5}$$

Definition 2 Let $F : [a,b] \to \widehat{\mathcal{E}}'$ be a complex fuzzy-valued function and let $F(x) = (u_f(x), v_f(x))$. Assume that $u_f(x)$ and $v_f(x)$ are both fuzzy Riemann-integrable. Then $F(x)$ is improper fuzzy Riemann-integrable and the improper fuzzy Riemann-integral is denoted by $IF(x) = (Iu_f(x), Iv_f(x))$.

Definition 3 We call a complex fuzzy-valued function $F : (a,b) \to \widehat{\mathcal{E}}'$ strongly generalized differentiable at $x \in (a,b)$ if there exists some $F'(x) \in \widehat{\mathcal{E}}'$ such that

(i) there exist the differences $F(x+x_0) - F(x)$, $F(x) - F(x-x_0)$ and

$$\lim_{x_0\to 0^+} \frac{F(x+x_0) - F(x)}{x_0} = \lim_{x_0\to 0^+} \frac{F(x) - F(x-x_0)}{x_0} = F'(x),$$

or

(ii) there exist the differences $F(x) - F(x+x_0)$, $F(x-x_0) - F(x)$ and

$$\lim_{x_0\to 0^+} \frac{F(x) - F(x+x_0)}{-x_0} = \lim_{x_0\to 0^+} \frac{F(x-x_0) - F(x)}{-x_0} = F'(x),$$

or

(iii) there exist the differences $F(x+x_0) - F(x)$, $F(x-x_0) - F(x)$ and

$$\lim_{x_0\to 0^+} \frac{F(x+x_0) - F(x)}{x_0} = \lim_{x_0\to 0^+} \frac{F(x-x_0) - F(x)}{-x_0} = F'(x),$$

or

(iv) there exist the differences $F(x) - F(x+x_0)$, $F(x) - F(x-x_0)$ and

$$\lim_{x_0\to 0^+} \frac{F(x) - F(x+x_0)}{-x_0} = \lim_{x_0\to 0^+} \frac{F(x) - F(x-x_0)}{x_0} = F'(x).$$

Furthermore, one can obtain

$$DF(x) = F'(x) = \left[Du_f(x), Dv_f(x)\right].$$

3 The fuzzy Mellin transform

Definition 4 The fuzzy Mellin transform of a complex fuzzy-valued function $f(t)$ is defined by

$$\mathcal{M}\left[f(t)\right](s) = \int_0^\infty t^{s-1} \odot f(t)\,\mathrm{d}t, \tag{3.1}$$

where $s \in \mathbb{C}$.

We then have

$$\begin{aligned}\mathcal{M}[f](s) &= \left(\int_0^\infty t^{s-1} \odot u_f(t)\,\mathrm{d}t, \int_0^\infty t^{s-1} \odot v_f(t)\,\mathrm{d}t\right)\\ &= \big(\mathcal{M}[u_f](s), \mathcal{M}[v_f](s)\big).\end{aligned} \tag{3.2}$$

We can establish the connection with the fuzzy Laplace transform defined by the authors in [16, 22, 23]. Let $x = e^{-t}$. We have the Laplace transform

$$\mathcal{L}\big[f(t)\big](s) = \int_{-\infty}^{+\infty} e^{-st} \odot f\big(e^{-t}\big)\,\mathrm{d}t. \tag{3.3}$$

By using this formula, we can obtain the following propositions.

Proposition 1

$$\mathcal{L}\big[f\big(e^{-t}\big)\big](s) = \mathcal{M}\big[f(t)\big](s). \tag{3.4}$$

For the definitions of $|\cdot|$ and converges absolutely we refer the reader to [22]. If the function $F(t) := f(e^{-t})$ satisfies $\delta_- < \delta_+$, then the fuzzy Laplace transform converges absolutely for $\delta_- < \mathrm{Re}(s) < \delta_+$, where

$$\begin{aligned}\delta_- &:= \inf\big\{\delta \mid \big|F(t)\big| = O\big(e^{\delta t}\big), t \to \infty\big\},\\ \delta_+ &:= \sup\big\{\delta \mid \big|F(t)\big| = O\big(e^{\delta t}\big), t \to -\infty\big\}.\end{aligned}$$

By (3.4), if $\delta_- < \delta_+$, then for $\delta_- < \mathrm{Re}(s) < \delta_+$ the fuzzy Mellin transform converges absolutely, where

$$\begin{aligned}\delta_- &:= \inf\big\{\delta \mid \big|f(t)\big| = O\big(t^{-\delta}\big), t \to 0+\big\},\\ \delta_+ &:= \sup\big\{\delta \mid \big|f(t)\big| = O\big(t^{-\delta}\big), t \to \infty\big\}.\end{aligned}$$

If the fuzzy Mellin transform $\mathcal{M}[f(t)](s)$ converges absolutely in the vertical strip $\delta_- < \mathrm{Re}(s) < \delta_+$, then one can obtain

$$\begin{aligned}&\lim_{t\to 0^+} t^s \odot f(t) = \hat{0}, \quad \mathrm{Re}(s) > \delta_-,\\ &\lim_{t\to \infty} t^s \odot f(t) = \hat{0}, \quad \mathrm{Re}(s) < \delta_+.\end{aligned}$$

Furthermore, for $\delta_- < \alpha \le \mathrm{Re}(s) \le \beta < \delta_+$, the fuzzy Mellin transform converges absolutely and uniformly, and

$$\lim_{|\mathrm{Im}(s)|\to\infty} \tilde{f}(s) = \hat{0}$$

for $\alpha < \mathrm{Re}(s) < \beta$. Then we give the inversion formula for the fuzzy Mellin transform

$$f(t) = \mathcal{M}^{-1}\big[\tilde{f}(s)\big](t) = \frac{1}{2\pi i} \odot \int_{c-i\infty}^{c+i\infty} t^{-s} \odot \tilde{f}(s)\,\mathrm{d}s \tag{3.5}$$

with $\delta_- < c < \delta_+$.

Example 1 Let $f(t) = (u_f \odot e^{-t}, v_f \odot e^{-t}) = a \odot e^{-t}$ with $a = (u_f, v_f) \in \widehat{\mathcal{E}}'$, then

$$\begin{aligned}\mathcal{M}\big[a \odot e^{-t}\big](s) &= \left(\int_0^\infty t^{s-1} \odot u_f \odot e^{-t}\,\mathrm{d}t, \int_0^\infty t^{s-1} \odot v_f \odot e^{-t}\,\mathrm{d}t\right)\\ &= \big(u_f \odot \Gamma(s), v_f \odot \Gamma(s)\big) = a \odot \Gamma(s), \quad \operatorname{Re}(s) > 0. \end{aligned} \tag{3.6}$$

Thus, for $t > 0$, $a \in \widehat{\mathcal{E}}'$, $c > 0$,

$$a \odot e^{-t} = \mathcal{M}^{-1}\big[a \odot \Gamma(s)\big](t) = \frac{1}{2\pi i} \odot pv \int_{c-i\infty}^{c+i\infty} t^{-s} \odot a \odot \Gamma(s)\,\mathrm{d}s. \tag{3.7}$$

In fact, the fuzzy Mellin transform is a linear operator in the domain of the strips of convergence. Then we have the following property.

Proposition 2 *If $\mathcal{M}[f_j(t)](s) = \tilde{f}_j(s)$ for $\alpha_j < \operatorname{Re}(s) < \beta_j$, then*

$$\mathcal{M}\left[\sum_{j=1}^{n} c_j \odot f_j(t)\right](s) = \sum_{j=1}^{n} c_j \odot \mathcal{M}\big[f_j(t)\big](s), \tag{3.8}$$

where $\alpha < \Re(s) < \beta$, $\alpha \triangleq \max\{\alpha_1, \ldots, \alpha_n\}$, and $\beta \triangleq \min\{\beta_1, \ldots, \beta_n\}$.

Proposition 3 *Let $\mathcal{M}[f(t)](s) = \tilde{f}(s)$ for $\alpha < \Re(s) < \beta$,*

$$\mathcal{M}\big[f(\lambda t)\big](s) = \lambda^{-s} \odot \mathcal{M}\big[f(t)\big](s).$$

Proof By a fuzzy Laplace transformation we obtain

$$\begin{aligned}\mathcal{M}\big[f(\lambda t)\big](s) &= \mathcal{L}\big[f\big(\lambda e^{-\tau}\big)\big](s) = \mathcal{L}\big[f\big(e^{-(\tau-\log\lambda)}\big)\big](s)\\ &= e^{-s\log\lambda} \odot \mathcal{L}\big[f\big(e^{-\tau}\big)\big](s) = \lambda^{-s} \odot \mathcal{M}\big[f(t)\big](s),\end{aligned} \tag{3.9}$$

where $\lambda > 0$, $t > 0$, $\alpha < \Re(s) < \beta$. □

Example 2

$$\mathcal{M}\big[c \odot e^{-\lambda t}\big](s) = c \odot \lambda^{-s} \odot \Gamma(s), \quad \operatorname{Re}(s) > 0, c \in \widehat{\mathcal{E}}', \lambda > 0.$$

Proposition 4 *For $\lambda \neq 0$ and $t > 0$, we have*

$$\mathcal{M}\big[f\big(t^{\lambda}\big)\big](s) = \frac{1}{|\lambda|} \odot \mathcal{M}\big[f(\tau)\big]\left(\frac{s}{\lambda}\right).$$

Proof For $\lambda \neq 0$ and $t > 0$, by using the fuzzy Laplace transformation, we have

$$\begin{aligned}\mathcal{M}\big[f\big(t^{\lambda}\big)\big](s) &= \mathcal{L}\big[f\big(e^{-\lambda\tau}\big)\big](s) = \frac{1}{|\lambda|} \odot \mathcal{L}\big[f\big(e^{-\tau}\big)\big]\left(\frac{s}{\lambda}\right)\\ &= \frac{1}{|\lambda|} \odot \mathcal{M}\big[f(\tau)\big]\left(\frac{s}{\lambda}\right),\end{aligned} \tag{3.10}$$

where $\lambda\alpha < \operatorname{Re}(s) < \lambda\beta$, and $\lambda\beta < \operatorname{Re}(s) < \lambda\alpha$, according to $\lambda > 0$ or $\lambda < 0$, respectively. □

Example 3 Let $\lambda > 0$, $c \in \widehat{\mathcal{E}}'$, and $\gamma \neq 0$,

$$\mathcal{M}\big[c \odot e^{-\lambda t^{\gamma}}\big](s) = c \odot \frac{\lambda^{-s/\gamma}}{|\gamma|} \odot \Gamma\left(\frac{s}{\gamma}\right). \tag{3.11}$$

Proposition 5

$$\mathcal{M}\big[(\log t)^n \odot f(t)\big](s) = \frac{\mathrm{d}^n}{\mathrm{d}s^n}\mathcal{M}\big[f(t)\big](s).$$

Proof Let $\mathcal{M}[f(t)](s) = \tilde{f}(s)$ for $\alpha < \mathrm{Re}(s) < \beta$. Then, by using a fuzzy Laplace transformation, we have

$$\begin{aligned}\mathcal{M}\big[(\log t)^n \odot f(t)\big](s) &= \mathcal{L}\big[f\big(e^{-\tau}\big) \odot \log\big(e^{-\tau}\big)^n\big](s) = \mathcal{L}\big[(-\tau)^n \odot f\big(e^{-\tau}\big)\big](s) \\ &= \frac{\mathrm{d}^n}{\mathrm{d}s^n}\mathcal{L}\big[f\big(e^{-\tau}\big)\big](s) = \frac{\mathrm{d}^n}{\mathrm{d}s^n}\mathcal{M}\big[f(t)\big](s),\end{aligned} \tag{3.12}$$

where $\mathrm{Re}(s) \in (\alpha, \beta)$ and $n \in \mathbb{N}$. □

Example 4 For $a > 0$, $c \in \widehat{\mathcal{E}}'$

$$\begin{aligned}&\mathcal{M}\big[c \odot (\log t)^n \odot \delta(t-a)\big](s) = c \odot \frac{\mathrm{d}^n}{\mathrm{d}s^n}a^{s-1} = c \odot (\log a)^n \odot a^{s-1}, \\ &\mathcal{M}\big[c \odot (\log t)^n \odot e^{-t}\big](s) = c \odot \frac{\mathrm{d}^n}{\mathrm{d}s^n}\Gamma(s) \quad \big(0 < \mathrm{Re}(s)\big).\end{aligned} \tag{3.13}$$

Proposition 6

$$\mathcal{M}\big[t^{\lambda} \odot f(t)\big](s) = \mathcal{M}\big[f(t)\big](s+\lambda).$$

Proof By using the fuzzy Laplace transformation, one can obtain

$$\begin{aligned}\mathcal{M}\big[t^{\lambda} \odot f(t)\big](s) &= \mathcal{L}\big[e^{-\lambda\tau} \odot f\big(e^{-\tau}\big)\big](s) \\ &= \mathcal{L}\big[f\big(e^{-\tau}\big)\big](s+\lambda) = \mathcal{M}\big[f(t)\big](s+\lambda),\end{aligned} \tag{3.14}$$

where $\mathrm{Re}(s) \in (\alpha - \mathrm{Re}(\lambda), \beta - \mathrm{Re}(\lambda))$. □

Example 5 For $a > 0$, $c \in \widehat{\mathcal{E}}'$ we have

$$\begin{aligned}&\mathcal{M}\big[c \odot t^{\lambda} \odot \delta(t-a)\big](s) = c \odot a^{s+\lambda-1}, \\ &\mathcal{M}\big[c \odot t^{\lambda} \odot e^{-t}\big](s) = c \odot \Gamma(s+\lambda) \quad \big(-\mathrm{Re}(\lambda) < \mathrm{Re}(s)\big).\end{aligned} \tag{3.15}$$

In particular, when $\lambda = 1$, we have

$$\mathcal{M}\big[t \odot f(t)\big](s) = \mathcal{M}\big[f(t)\big](s+1), \quad \mathrm{Re}(s) \in (\alpha - 1, \beta - 1).$$

Moreover, we can obtain the following proposition.

Proposition 7

$$\mathcal{M}\big[t^{-1}\odot f\big(t^{-1}\big)\big](s)=\mathcal{M}\big[f(t)\big](1-s),\quad \Re(s)\in\big(-(\beta-1),-(\alpha-1)\big)$$

and

$$\mathcal{M}\left[f\left(\frac{1}{t}\right)\right](s)=\mathcal{M}\big[f(t)\big](-s),\quad \mathrm{Re}(s)\in(-\beta,-\alpha).$$

Example 6

$$\mathcal{M}\left[c\odot\frac{e^{-a/t}}{t}\right](s)=c\odot a^{s-1}\odot\Gamma(1-s),\quad \mathrm{Re}(s)<1.$$

Proposition 8

$$\mathcal{M}\big[f'(t)\big](s)=-(s-1)\odot\mathcal{M}\big[f(t)\big](s-1). \tag{3.16}$$

Proof Let $\mathcal{M}[f(t)](s)=\tilde{f}(s)$ for $\alpha<\mathrm{Re}(s)<\beta$. By using the fuzzy Laplace transformation,

$$\begin{aligned}\mathcal{M}\big[f'(t)\big](s)&=\mathcal{L}\big[f'\big(e^{-t}\big)\big](s)=\mathcal{L}\big[-e^{t}\odot Df\big(e^{-t}\big)\big](s)\\&=-\mathcal{L}\big[Df\big(e^{-t}\big)\big](s)|_{s\to s-1}\\&=-s\odot\mathcal{L}\big[f\big(e^{-t}\big)\big](s)|_{s\to s-1}\\&=-(s-1)\odot\mathcal{M}\big[f(t)\big](s-1),\end{aligned} \tag{3.17}$$

where $\alpha+1<\mathrm{Re}(s)<\beta+1$. □

Moreover, we have:

(1) $\mathcal{M}[D^n f(t)](s)=(1-s)_n\odot\tilde{f}(s-n)$ where $\mathrm{Re}(s)\in(\alpha+n,\beta+n)$, $(a_n):=a(a+1)\cdots(a+n-1)$.

(2) $\mathcal{M}[t\odot f'(t)](s)=-s\odot\mathcal{M}[f(t)](s)$, and the strip of convergence is unchanged.

(3) $\mathcal{M}[(t\odot Df(t))^n](s)=(-s)^n\odot\mathcal{M}[f(s)](s)$. For $n=2$ it implies

$$\mathcal{M}\big[t^2\odot D^2f(t)+t\odot Df(t)\big](s)=s^2\odot\mathcal{M}\big[f(t)\big](s). \tag{3.18}$$

(4) $\mathcal{M}[D^2f(t)+\frac{1}{t}\odot Df(t)](s)=(s-2)^2\odot\mathcal{M}[f(t)](s-2)$ for $\mathrm{Re}(s)\in(\alpha-2,\beta-2)$.

We consider the following integral operator:

$$I_-f(t):=\int_0^t f(\tau)\,\mathrm{d}\tau,\qquad I_+f(t)=\int_t^\infty f(\tau)\,\mathrm{d}\tau. \tag{3.19}$$

Proposition 9

$$\mathcal{M}\big[I_-f(t)\big](s)=-\frac{1}{s}\mathcal{M}\big[f(t)\big](s+1),\quad \alpha-1<\mathrm{Re}(s)<\min\{\beta-1,-1\}. \tag{3.20}$$

Proof

$$\begin{aligned}\mathcal{M}\big[I_{-}f(t)\big](s) &= \mathcal{L}\big[I_{-}f\big(e^{-\tau}\big)\big](s) = \mathcal{L}\big[I_{+}\big(e^{-\tau}\odot f\big(e^{-\tau}\big)\big)\big](s)\\ &= \frac{1}{s}\odot\mathcal{L}\big[e^{-\tau}\odot f\big(e^{-\tau}\big)\big](s) = -\frac{1}{s}\odot\mathcal{M}\big[f(t)\big](s+1)\end{aligned}\tag{3.21}$$

holds for $\alpha-1<\mathrm{Re}(s)<\min\{\beta-1,-1\}$.

Similarly, we can obtain

$$\mathcal{M}\big[I_{+}f(t)\big](s) = -\frac{1}{s}\odot\mathcal{M}\big[f(t)\big](s+1)\tag{3.22}$$

for $\alpha-1<\mathrm{Re}(s)<\beta-1$. □

Example 7 We have

$$\begin{aligned}\mathcal{M}\big[c\odot e^{-t^2}\big](s) &= \big(\mathcal{M}\big[u_c\odot e^{-t^2}\big](s),\mathcal{M}\big[v_c\odot e^{-t^2}\big](s)\big)\\ &= \left(\frac{1}{2}\Gamma\left(\frac{s}{2}\right)\right)\odot c\end{aligned}$$

for $\alpha=0<\mathrm{Re}(s)<\infty=\beta$. Thus,

$$\mathcal{M}\big[I_{+}\big(c\odot e^{-t^2}\big)\big](s) = -c\odot\frac{1}{2s}\odot\Gamma\left(\frac{s+1}{2}\right).$$

Definition 5 The fuzzy Mellin convolution product, denoted by $f*g$, of a real-valued function $f(t)$ and a fuzzy-valued function $g(t)$, is defined by

$$f(t)*g(t) = \int_0^\infty\frac{1}{\tau}\odot f\left(\frac{t}{\tau}\right)\odot g(\tau)\,\mathrm{d}\tau.\tag{3.23}$$

Suppose that the real-valued function $f(t)$ and the fuzzy-valued function $g(t)$ are both defined on the positive part of the real axis and that both of them have the Mellin transform $\tilde{f}(s)$ and $\tilde{g}(s)$ for $\alpha<\mathrm{Re}(s)<\beta$. Then we can obtain the following convolution rule.

Proposition 10

$$\mathcal{M}\big[f(t)*g(t)\big](s) = \mathcal{M}\big[f(t)\big](s)\odot\mathcal{M}\big[g(t)\big](s)\tag{3.24}$$

for $\alpha<\mathrm{Re}(s)<\beta$.

Proof

$$\begin{aligned}\mathcal{M}\big[f(t)*g(t)\big](s) &= \int_0^\infty t^{s-1}\odot\int_0^\infty f\left(\frac{r}{\tau}\right)\odot g(\tau)\,\mathrm{d}\tau\,\mathrm{d}t\\ &= \int_0^\infty t^{s-1}\odot f\left(\frac{t}{\tau}\right)\mathrm{d}t\int_0^\infty\frac{1}{\tau}\odot g(\tau)\,\mathrm{d}\tau\\ &= \mathcal{M}\big[f(t)\big](s)\odot\mathcal{M}\big[g(t)\big](s).\end{aligned}\tag{3.25}$$

In fact, by using the fuzzy Laplace transformation, we can also obtain

$$\begin{aligned}\mathcal{M}\big[f(t)*g(t)\big](s)&=\mathcal{L}\big[f\big(e^{-t}\big)\star g\big(e^{-t}\big)\big]=\mathcal{L}\big[f\big(e^{-t}\big)\big]\odot\mathcal{L}\big[g\big(e^{-t}\big)\big]\\&=\mathcal{M}\big[f(t)\big](s)\odot\mathcal{M}\big[g(t)\big](s).\end{aligned}\tag{3.26}$$

□

4 Applications

Consider the inhomogeneous fuzzy differential system (Euler's differential equation)

$$\big\{t^2\odot D^2+a_1\odot t\odot D+a_2\big\}U(t)=f(t),\quad t>0.\tag{4.1}$$

By using the Mellin transform, the equation becomes

$$\big(s^2-(a_1-1)s+a_2\big)\odot\widetilde{U}(s)=\tilde{f}(s).\tag{4.2}$$

Example 8 Consider the fuzzy Euler's differential equation

$$\big\{t^2\odot D^2+2\odot t\odot D-2\big\}U(t)=c\odot\big(u(a-t)t^{-3}\big),\quad a>0, c\in\widehat{\mathcal{E}}'.$$

By (4.2), a particular solution of the system is found to be

$$\widetilde{U}(s)=c\odot\frac{a^{s-3}}{(s+1)(s-2)(s-3)}=\left(\frac{1}{4}\frac{a^{s-3}}{s-3}-\frac{1}{3a}\frac{a^{s-2}}{s-2}+\frac{1}{12a^4}\frac{a^{s+1}}{s+1}\right)\odot c\tag{4.3}$$

for $\operatorname{Re}(s)>3$. By the inverse Mellin transform, we get

$$U(t)=\frac{1}{4}\odot u(a-t)\odot\left(t^{-3}-\frac{1}{3a}t^{-2}+\frac{1}{12a^4}t\right)\odot c.$$

Example 9 Consider the fuzzy Euler differential equation

$$\big\{t^2\odot D^2+2\odot t\odot D-2\big\}U(t)=c\odot\big(\delta(x-a)\big),\quad a>0, c\in\widehat{\mathcal{E}}'.$$

Using the same Mellin transform method, we have

$$\widetilde{U}(s)=\frac{a^{s-1}}{(s+1)(s-2)}\odot c=\left(\frac{a}{3}\frac{a^{s-2}}{s-2}-\frac{1}{3a^2}\frac{a^{s+1}}{s+1}\right)\odot c.\tag{4.4}$$

There exist three solutions corresponding to different strips of convergence:

(i) For $\operatorname{Re}(s)<-2$, one can get

$$U(t)=c\odot u(t-a)\odot\left\{\frac{t}{3a^2}-\frac{a}{3t^2}\right\}.$$

(ii) For $-1<\operatorname{Re}(s)<2$, we obtain

$$U(t)=c\odot\left\{-\frac{tu(a-t)}{3a^2}-\frac{au(t-a)}{3t^2}\right\}.$$

(iii) For $\mathrm{Re}(s) > 2$, we have

$$U(t) = c \odot u(a-t) \odot \left\{ \frac{a}{3t^2} - \frac{t}{3a^2} \right\}.$$

Example 10 Consider the following abstract integral equation:

$$\int_0^\infty \kappa(t\tau) \odot U(\tau)\,\mathrm{d}\tau = \psi(t), \quad x > 0$$

with a real-valued function $\kappa(t)$ and a given fuzzy-valued function $\psi(t)$.

By using the fuzzy Mellin transform and the convolution rule, we have

$$U(t) = \mathcal{M}^{-1}\big[\tilde{\varphi}(s) \odot \tilde{\psi}(1-s)\big](t) = \int_0^\infty \varphi(t\tau) \odot f(\tau)\,\mathrm{d}\tau,$$

where $\tilde{\varphi}(s) = 1/\tilde{\kappa}(1-s)$.

Example 11 Consider the following fuzzy partial differential equation:

$$\big(D_{0+}^{\alpha} U(\cdot,y)\big)(x) = -\frac{\mathrm{d}}{\mathrm{d}y} U(x,y) \quad \big(x,y \in \mathbb{R}^+\big) \tag{4.5}$$

with initial valued condition $U(x,0) = f(x) \in \widehat{\mathcal{E}}'$, $x > 0$. The operator D_{0+}^{α} is defined by

$$\big(D_{0+}^{\alpha} U(\cdot,y)\big)(x) = \left(x\frac{\mathrm{d}}{\mathrm{d}x}\right)\left[\frac{1}{\Gamma(1-\alpha)} \odot \int_0^x \left(\log\frac{x}{u}\right)^{-\alpha} \odot U(\cdot,y)\,\mathrm{d}u\right], \quad \alpha \in (0,1). \tag{4.6}$$

Assume that there is a given function $\mathcal{K}(x)$ such that, for every $x, y > 0$,

$$\left|\frac{\partial}{\partial y} U(x,y)\right| \le \mathcal{K}(x),$$

where $\mathcal{K}(x)$ has a Mellin transform.

By the property of the Mellin transform, the equation can be transformed into a first order ordinary fuzzy differential equation

$$(-\nu - \mathrm{i}t)^{\alpha} \odot \mathcal{M}\big[U(\cdot,y)\big](\nu + \mathrm{i}t) = -\frac{\partial}{\partial y}\mathcal{M}\big[U(\cdot,y)\big](\nu + \mathrm{i}t) \quad (s = \nu + \mathrm{i}t), \tag{4.7}$$

which has the solution

$$\mathcal{M}\big[U(\cdot,y)\big](\nu + \mathrm{i}t) = A(\nu + \mathrm{i}t) \odot e^{-(-\nu-\mathrm{i}t)^{\alpha} y}, \tag{4.8}$$

where $A(\nu + \mathrm{i}t) = \mathcal{M}[f](\nu + \mathrm{i}t)$.

By using the convolution formula, the solution of the fuzzy differential equation is given by

$$U(x,y) = \int_0^\infty f(t) \odot G\left(\frac{x}{t}, y\right) \odot \frac{\mathrm{d}t}{t}, \quad x, y > 0, \tag{4.9}$$

where the function

$$G(x,y) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-(-\nu-\mathrm{i}t)^{\alpha}} x^{-\nu-\mathrm{i}t}\,\mathrm{d}t.$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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