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# Stability of delayed impulsive stochastic differential equations driven by a fractional Brown motion with time-varying delay

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## Abstract

We study the stability problem of mild solutions of impulsive stochastic differential equations driven by a fractional Brown motion with finite time-varying delay. The Hurst parameter  $H$  of the fractional Brown motion belongs to  $(\frac{1}{2}, 1)$ . In terms of fractional power of operators and semigroup theory, we obtain sufficient conditions that guarantee the stability of the mild solution of such an equation in two cases: the impulse depends on current states of the system and the impulse depends not only on current states but also on historical states of the system. We give two examples illustrating the theorems.

**Keywords:** fractional Brownian motion; stability; delayed impulse

## 1 Introduction

Fractional Brownian motions (fBMs) play a central role in the modeling and analysis of many complex phenomena in applications where the systems are subject to “rough” external forcing. An fBm with Hurst parameter  $H \in (0, 1)$  is a zero-mean Gaussian process denoted by  $W^H = \{W^H(t), t \geq 0\}$ . If  $H \in (0, \frac{1}{2})$ , then it is regarded as a short-memory process; if  $H = \frac{1}{2}$ , then it reduces to the standard Brownian motion; if  $H \in (\frac{1}{2}, 1)$ , then it is regarded as a long-memory process. It is easy to see that fBm is a generalization of Brownian motion, but it behaves differently significantly from the standard Brownian motion. In particular, it is neither a semimartingale nor a Markov process. It is characterized by stationary increments and memory property, which make an fBm a potential candidate to model noise in biology and finance (see, e.g., [1–5], in geophysics ([6]), in communication networks ([7]), in electricity markets ([8]), and so on.

In recent years, stochastic differential equations driven by fBMs have attracted increasing interest because of their applications in a variety of fields (see [4, 9–19] and references therein). However, most of the existing literature is focused on the existence and uniqueness of mild solutions for stochastic differential equations driven by fBMs (see, e.g., [9–15]), but the existing results on the stability of mild solutions for stochastic differential equations driven by fBMs are relatively few. We only found a few stories in the literature [4, 16–19]. In [4], the authors provided sufficient conditions to guarantee the exponential asymptotic behavior of solutions of general linear stochastic differential equations driven

by fBms with time-varying delays. In [16], the authors provided the conditions for the existence, uniqueness, and exponential asymptotic behavior of mild solutions to stochastic delay evolution equations perturbed by an fBm. In [17], the authors provided conditions to ensure the exponential decay to zero in mean square of solutions of neutral stochastic differential equations driven by fBms in a Hilbert space. However, in [4, 16, 17], the impulses are not considered in the systems. In [18], the author gave asymptotic stability conditions for mild solutions of neutral stochastic differential equations driven by fBms with finite delays and nonlinear impulsive effects. In [19], the authors gave mean-square exponential stability conditions for mild solutions of neutral stochastic differential equations driven by fBms with infinite delays and impulses. However, the impulses in stochastic differential equations in [18, 19] only depend on the current states of the systems, which are supposed to be of the form  $I_k(x(t_k))$  at impulsive moments  $t_k$  ( $k = 1, 2, \dots$ ). Here, the delayed impulses we consider describe the impulsive transients depending not only on their current states but also on historical states of the system. Delayed impulses exist in many practical problems, for example, in communication security systems based on impulsive synchronization. During the information transmission process, the sampling delay created from sampling the impulses at some discrete instances causes the impulsive transients depend on their historical states. There are some results ([20–25]) on delayed impulsive differential equation, where the delays in impulsive perturbations are fixed as constants or vary in a finite interval. However, most of these studies of stability for such delayed impulsive differential equations consider deterministic equations ([20–25]), whereas stochastic differential equations are rarely considered ([26]). In [26], the stochastic term was supposed to be of the form  $g(x_t, t) dw(t)$ , where  $w(t)$  is a standard Brownian motion, not a fractional one.

In view of the above discussion, we investigate the stability of impulsive stochastic delay differential equations driven by fBms with Hurst parameter  $H \in (\frac{1}{2}, 1)$ . Firstly, we assume that the impulse in such an equation depends on the current states and give mean-square exponential stability conditions. Secondly, we assume that the impulse in such an equation depends on the historical states and give mean-square asymptotic stability conditions.

The rest of this paper is organized as follows. In Section 2, we introduce some notation, concepts, and lemmas. In Section 3, we give mean-square exponential stability conditions for stochastic differential equations driven by fBms with impulses and time-varying delays, where the impulses only depend on the current states of the system. In Section 4, we study a delayed impulsive differential equation driven by an fBm with impulses and time-varying delay, where the impulses depend not only on the current states but also on the historical states of the system, and provide mean-square asymptotic stability conditions for such an equation. In Section 5, we give two illustrative examples. Conclusions are given in Section 6.

## 2 Preliminaries

In the present paper, we study stability behavior of mild solutions of an impulsive stochastic delay differential equation driven by an fBm in a Hilbert space of the form

$$\begin{cases} dx(t) = [Ax(t) + F(t, x(t - \sigma(t)))] dt + G(t) dW^H(t), & t \geq t_0, t \neq t_k, \\ \Delta x(t_k) = I_k(x(t_k^-), x(t_k^- - \delta)), & k \in \mathbb{N}, \\ x(t_0 + \theta) = \phi(\theta), & \theta \in [-\alpha, 0], \end{cases} \tag{1}$$

where  $A$  is the infinitesimal generator of an analytic semigroup  $\{S(t)\}_{t \geq t_0}$  of bounded linear operators in a Hilbert space  $X$ ,  $W^H(t)$  is an fBm with Hurst parameter  $H \in (\frac{1}{2}, 1)$  on a real separable Hilbert space  $Y$ , the delay  $\sigma(t) : [t_0, +\infty) \rightarrow [0, \alpha]$  ( $\alpha > 0$ ) is continuous,  $\mathbb{N}$  denotes the set of positive integers, impulsive moments satisfy  $0 \leq t_0 < t_1 < t_2 < \dots < t_k < t_{k+1} < \dots$  and  $\lim_{k \rightarrow \infty} t_k = \infty$ ,  $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$  represents the jump size in the state  $x$  at  $t_k$ ,  $x(t_k^+)$  and  $x(t_k^-)$  are respectively the right and left limits of  $x(t)$  at  $t = t_k$ ,  $\delta$  is a constant delay in impulses,  $I_k : X \times X \rightarrow X$  is continuous, the initial data  $\phi \in C([-\alpha, 0], X)$  (the space of all continuous functions from  $[-\alpha, 0]$  to  $X$ ) has finite second moments, and  $F : [t_0, +\infty) \times X \rightarrow X$  and  $G : [t_0, +\infty) \rightarrow \mathcal{L}_2^0(Y, X)$ , where  $\mathcal{L}_2^0(Y, X)$  is the space of all Q-Hilbert-Schmidt operators  $\psi : Y \rightarrow X$ .

In the rest of this section, we recall the definition of Wiener integrals with respect to fBMs and two useful lemmas.

Let  $(\Omega, \mathfrak{F}, P)$  be a complete probability space. An fBm  $\{W^H(t), t \geq t_0\}$  with Hurst parameter  $H \in (\frac{1}{2}, 1)$  is a continuous centered Gaussian process with covariance function

$$R_H(t, s) = \mathbb{E}[W^H(t)W^H(s)] = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t - s|^{2H})$$

and has the following Wiener integral representation:

$$W^H(t) = \int_{t_0}^t K_H(t, s) dW(s),$$

where  $W(t) = \{W(t) : t \geq t_0\}$  is a standard Wiener process,  $K_H(t, s)$  is the kernel given by

$$K_H(t, s) = c_H s^{\frac{1}{2}-H} \int_s^t (u - s)^{H-\frac{1}{2}} u^{H-\frac{1}{2}} du, \quad t > s,$$

with  $c_H = \sqrt{\frac{H(2H-1)}{B(2-2H, H-\frac{1}{2})}}$ , and  $B$  denotes the beta function. Let  $K_H(t, s) = 0$  for  $t \leq s$ . For the deterministic function  $\varphi \in L^2([t_0, +\infty))$ , it is known from [16] that the fractional Wiener integral with respect to  $W^H(t)$  can be defined by

$$\int_{t_0}^\infty \varphi(t) dW^H(t) = \int_{t_0}^\infty (K_H^* \varphi)(t) dW(t),$$

where  $(K_H^* \varphi)(s) = \int_s^\infty \varphi(t) \frac{\partial K_H(t, s)}{\partial t} dt$ .

Next, we are interested in considering an fBm with values in a Hilbert space and giving the definition of the corresponding stochastic integral.

Let  $X$  and  $Y$  be real separable Hilbert spaces, and let  $\mathcal{L}(Y, X)$  denote the space of all bounded linear operators from  $Y$  to  $X$ . Let  $Q \in \mathcal{L}(Y, X)$  be the operator defined by  $Qe_n = \lambda_n e_n$  with finite trace  $\text{tr} Q = \sum_{n=1}^\infty \lambda_n$ , where  $\lambda_n, n = 1, 2, \dots$ , are nonnegative real numbers, and  $\{e_n, n = 1, 2, \dots\}$  is a complete orthonormal basis in  $Y$ . Define a  $Y$ -valued Gaussian process as

$$W^H(t) = \sum_{n=1}^\infty \sqrt{\lambda_n} e_n W_n^H(t),$$

where  $W_n^H(t)$  ( $n = 1, 2, \dots$ ) are real independent fBms. It has the covariance

$$\mathbb{E}\langle W^H(t, x) | W^H(s, y) \rangle = R(t, s) \langle Q(x), y \rangle$$

for all  $x, y \in Y$  and  $t, s \geq t_0$ . The space  $\mathcal{L}_2^0(Y, X)$  the space of all Q-Hilbert-Schmidt operators  $\psi : Y \rightarrow X$ , equipped with the following norm and inner product

$$\|\psi\|_{\mathcal{L}_2^0(Y, X)} = \sum_{n=1}^{\infty} \|\sqrt{\lambda_n} \psi e_n\|^2$$

and

$$\langle \phi, \psi \rangle_{\mathcal{L}_2^0(Y, X)} = \sum_{n=1}^{\infty} \langle \phi e_n, \psi e_n \rangle.$$

The space  $\mathcal{L}_2^0(Y, X)$  is a separable Hilbert space. Then (see [16]) the fractional Wiener integral of the function  $\psi : [t_0, +\infty) \rightarrow \mathcal{L}_2^0(Y, X)$  with respect to the fBm is defined by

$$\int_{t_0}^t \psi(s) dW^H(s) = \sum_{n=1}^{\infty} \int_{t_0}^t \sqrt{\lambda_n} \psi(s) e_n dW_n^H(s) = \sum_{n=1}^{\infty} \int_{t_0}^t \sqrt{\lambda_n} K_H^*(\psi e_n)(s) dW_n(s).$$

**Lemma 1** ([16]) *For any  $\psi : [t_0, \infty) \rightarrow \mathcal{L}_2^0(Y, X)$  satisfying  $\int_{t_0}^{\infty} \|\psi(s)\|_{\mathcal{L}_2^0(Y, X)}^2 ds < +\infty$ , the integral  $\int_{t_0}^t \psi(s) dW^H(s)$  is well defined as an  $X$ -valued random variable, and*

$$\mathbb{E} \left\| \int_{t_0}^t \psi(s) dW^H(s) \right\|^2 \leq cH(2H - 1)(t - t_0)^{2H-1} \int_{t_0}^t \|\psi(s)\|_{\mathcal{L}_2^0(Y, X)}^2 ds, \quad t > t_0,$$

where  $c = c(H)$  is a constant.

**Lemma 2** ([19]) *For any  $\mu > 0$ , assume that there exist positive constants  $\alpha_1, \alpha_2, \beta_k, k \in \mathbb{N}$ , and a function  $\psi : [-\alpha, \infty) \rightarrow [0, \infty)$  such that*

$$\psi(t) \leq \alpha_1 e^{-\mu t}, \quad t \in [-\alpha, t_0],$$

and, for each  $t \geq t_0$ ,

$$\psi(t) \leq \alpha_1 e^{-\mu t} + \alpha_2 \int_{t_0}^t e^{-\mu(t-s)} \sup_{\theta \in [-\alpha, 0]} \psi(s + \theta) ds + \sum_{t_k < t} \beta_k e^{-\mu(t-t_k)} \psi(t_k^-)$$

if  $\frac{\alpha_2}{\mu} + \sum_{k=1}^{\infty} \beta_k < 1$ . Then we have

$$\psi(t) \leq M_0 e^{-\lambda t}, \quad t \geq -\alpha,$$

where  $\lambda > 0$  is the unique solution of the equation  $\frac{\alpha_2 e^{\lambda \alpha}}{\mu - \lambda} + \sum_{k=1}^{\infty} \beta_k = 1$ , and  $M_0 = \max\{\alpha_1, \frac{\alpha_1(\mu - \lambda)}{\alpha_2 e^{\lambda \alpha}}\}$ .

**Definition 2.1** A mild solution of system (1) is said to be exponentially stable in mean square if there is a pair of positive constants  $\lambda, M_0$  such that

$$\mathbb{E} \|x(t, t_0, \phi)\|^2 \leq M_0 e^{-\lambda(t-t_0)}.$$

### 3 Mean-square exponential stability of the impulsive differential equation driven by an fBm with time-varying delay

In this section, we first assume that the impulses in Eq. (1) only depend on the current states of the system, that is, the impulses can be described as  $I_k(x(t_k^-), x(t_k^- - \delta)) = I_k(x(t_k^-))$ . Then Eq. (1) reduces to the following equation:

$$\begin{cases} dx(t) = [Ax(t) + F(t, x(t - \sigma(t)))] dt + G(t) dW^H(t), & t \geq t_0, t \neq t_k, \\ \Delta x(t_k) = I_k(x(t_k^-)), & k \in \mathbb{N}, \\ x(t_0 + \theta) = \phi(\theta), & \theta \in [-\alpha, 0]. \end{cases} \tag{2}$$

Next, we present the definition of a mild solution of Eq. (2).

**Definition 3.1** An  $X$ -valued stochastic process  $\{x(t), t \in [-\alpha, \infty)\}$  is called a mild solution of Eq. (2) if  $x(t_0 + \theta) = \phi(\theta)$  on the interval  $[-\alpha, 0]$  and the following conditions hold:

- (i)  $x(\cdot)$  is continuous on  $[t_0, t_1]$  and on each interval  $(t_k, t_{k+1}), k \in \mathbb{N}$ ,
- (ii) for each  $k$ , the limits  $x(t_k^+), x(t_k^-)$  exist, and  $x(t_k^-) = x(t_k)$ ,
- (iii) for each  $t \geq t_0$ , we have a.s.

$$x(t) = \begin{cases} \int_{t_0}^t S(t-s)F(s, x(s - \sigma(s))) ds + \int_{t_0}^t S(t-s)G(s) dW^H(s), & t \in [t_0, t_1], \\ S(t-t_k)[x(t_k^-) + I_k(x(t_k^-))] + \int_{t_k}^t S(t-s)F(s, x(s - \sigma(s))) ds \\ + \int_{t_k}^t S(t-s)G(s) dW^H(s), & t \in (t_k, t_{k+1}), k \in \mathbb{N}. \end{cases} \tag{3}$$

Our aim in this section is to obtain mean-square exponential stability conditions for a mild solution for Eq. (2). For this, we need the following assumptions.

- (H<sub>1</sub>) The strongly continuous semigroup  $\{S(t)\}_{t \geq t_0}$  is exponentially stable, that is, there exist a constant  $M > 0$  and a real number  $r > 0$  such that  $\|S(t)\| \leq Me^{-rt}, t \geq t_0$ .
- (H<sub>2</sub>) There exists a nonnegative real number  $R_1 > 0$  such that

$$\|F(t, x)\| \leq R_1 \|x\|, \quad t \geq t_0, x \in X.$$

- (H<sub>3</sub>) There exist nonnegative real numbers  $d_k, k \in \mathbb{N}$ , such that

$$\|I_k(x) - I_k(y)\| \leq d_k \|x - y\|$$

for all  $x, y \in X$  and  $\sum_{k=1}^{\infty} d_k < \infty$

- (H<sub>4</sub>) The function  $G : [t_0, \infty) \rightarrow \mathcal{L}_2^0(Y, X)$  satisfies, for some  $r > 0$ ,

$$\int_{t_0}^{\infty} e^{rs} \|G(s)\|_{\mathcal{L}_2^0(Y, X)}^2 ds < \infty.$$

**Theorem 3.1** *Suppose that  $(H_1)$ - $(H_4)$  hold and*

$$\frac{3M^2R_1^2}{r^2} + 3M^2\left(\sum_{i=1}^{\infty} d_i\right)^2 < 1. \tag{4}$$

*Then, the mild solution of Eq. (2) is mean-square exponentially stable.*

*Proof* For  $t \in [t_0, t_1]$ , we have

$$x(t) = \int_{t_0}^t S(t-s)F(s, x(s-\sigma(s))) \, ds + \int_{t_0}^t S(t-s)G(s) \, dW^H(s). \tag{5}$$

For  $t \in (t_1, t_2]$ , from (3) we get

$$\begin{aligned} x(t) &= S(t-t_1)(x(t_1^-) + I_1(x(t_1^-))) + \int_{t_1}^t S(t-s)G(s) \, dW^H(s) \\ &\quad + \int_{t_1}^t S(t-s)F(s, x(s-\sigma(s))) \, ds. \end{aligned} \tag{6}$$

By (ii) of Definition 3.1,  $x(t_k^-) = x(t_k)$ , and combining (5) with (6), for  $t \in (t_1, t_2]$ , we get

$$\begin{aligned} x(t) &= S(t-t_1)I_1(x(t_1^-)) + S(t-t_1) \int_{t_0}^{t_1} S(t_1-s)F(s, x(s-\sigma(s))) \, ds \\ &\quad + S(t-t_1) \int_{t_0}^{t_1} S(t_1-s)G(s) \, dW^H(s) + \int_{t_1}^t S(t-s)G(s) \, dW^H(s) \\ &\quad + \int_{t_1}^t S(t-s)F(s, x(s-\sigma(s))) \, ds \\ &= \int_{t_0}^t S(t-s)F(s, x(s-\sigma(s))) \, ds + S(t-t_1)I_1(x(t_1^-)) \\ &\quad + \int_{t_0}^t S(t-s)G(s) \, dW^H(s). \end{aligned} \tag{7}$$

In the same manner, for  $t \in (t_2, t_3]$ , we get

$$\begin{aligned} x(t) &= S(t-t_2)(x(t_2^-) + I_2(x(t_2^-))) \\ &\quad + \int_{t_2}^t S(t-s)G(s) \, dW^H(s) \\ &\quad + \int_{t_2}^t S(t-s)F(s, x(s-\sigma(s))) \, ds \\ &= \int_{t_0}^t S(t-s)F(s, x(s-\sigma(s))) \, ds \\ &\quad + \int_{t_0}^t S(t-s)G(s) \, dW^H(s) \\ &\quad + S(t-t_1)I_1(x(t_1^-)) + S(t-t_2)I_2(x(t_2^-)). \end{aligned} \tag{8}$$

Assuming that, for  $t \in (t_m, t_{m+1}]$ , where  $m \in \mathbb{N}$ ,

$$\begin{aligned}
 x(t) &= \int_{t_0}^t S(t-s)F(s, x(s-\sigma(s))) \, ds + \int_{t_0}^t S(t-s)G(s) \, dW^H(s) \\
 &\quad + \sum_{i=1}^m S(t-t_i)I_i(x(t_i^-)),
 \end{aligned} \tag{9}$$

for  $t \in (t_{m+1}, t_{m+2}]$ , by (3) we have

$$\begin{aligned}
 x(t) &= S(t-t_{m+1})[x(t_{m+1}^-) + I_{m+1}(x(t_{m+1}^-))] + \int_{t_{m+1}}^t S(t-s)G(s) \, dW^H(s) \\
 &\quad + \int_{t_{m+1}}^t S(t-s)F(s, x(s-\sigma(s))) \, ds.
 \end{aligned} \tag{10}$$

Combining (9) with (10), for  $t \in (t_{m+1}, t_{m+2}]$ , we have

$$\begin{aligned}
 x(t) &= S(t-t_{m+1}) \left[ \int_{t_0}^{t_{m+1}} S(t_{m+1}-s)F(s, x(s-\sigma(s))) \, ds \right. \\
 &\quad \left. + \int_{t_0}^{t_{m+1}} S(t_{m+1}-s)G(s) \, dW^H(s) + \sum_{i=1}^m S(t_{m+1}-t_i)I_i(x(t_i^-)) \right] \\
 &\quad + S(t-t_{m+1})I_{m+1}(x(t_{m+1}^-)) + \int_{t_{m+1}}^t S(t-s)G(s) \, dW^H(s) \\
 &\quad + \int_{t_{m+1}}^t S(t-s)F(s, x(s-\sigma(s))) \, ds.
 \end{aligned} \tag{11}$$

By the properties of a semigroup and (11), for  $t \in (t_{m+1}, t_{m+2}]$ , we have

$$\begin{aligned}
 x(t) &= \int_{t_0}^{t_{m+1}} S(t-s)F(s, x(s-\sigma(s))) \, ds \\
 &\quad + \int_{t_0}^{t_{m+1}} S(t-s)G(s) \, dW^H(s) \\
 &\quad + \sum_{i=1}^m S(t-t_i)I_i(x(t_i^-)) + S(t-t_{m+1})I_{m+1}(x(t_{m+1}^-)) \\
 &\quad + \int_{t_{m+1}}^t S(t-s)G(s) \, dW^H(s) + \int_{t_{m+1}}^t S(t-s)F(s, x(s-\sigma(s))) \, ds.
 \end{aligned} \tag{12}$$

Calculating (12), then we get that, for  $t \in (t_{m+1}, t_{m+2}]$ ,

$$\begin{aligned}
 x(t) &= \int_{t_0}^t S(t-s)F(s, x(s-\sigma(s))) \, ds + \int_{t_0}^t S(t-s)G(s) \, dW^H(s) \\
 &\quad + \sum_{i=1}^{m+1} S(t-t_i)I_i(x(t_i^-)).
 \end{aligned} \tag{13}$$

From Eqs. (9) and (13), by mathematical induction, for all  $t \in (t_k, t_{k+1}]$ , we have

$$\begin{aligned}
 x(t) &= \int_{t_0}^t S(t-s)F(s, x(s-\sigma(s))) \, ds + \int_{t_0}^t S(t-s)G(s) \, dW^H(s) \\
 &\quad + \sum_{i=1}^k S(t-t_i)I_i(x(t_i^-)).
 \end{aligned}
 \tag{14}$$

Taking the mathematical expectation of (14), by the  $C_p$  inequality, for  $t \in (t_k, t_{k+1}]$ ,  $k \in \mathbb{N}$ , we have

$$\begin{aligned}
 \mathbb{E} \|x(t)\|^2 &\leq 3\mathbb{E} \left\| \int_{t_0}^t S(t-s)F(s, x(s-\sigma(s))) \, ds \right\|^2 \\
 &\quad + 3\mathbb{E} \left\| \int_{t_0}^t S(t-s)G(s) \, dW^H(s) \right\|^2 \\
 &\quad + 3\mathbb{E} \left\| \sum_{i=1}^k S(t-t_i)I_i(x(t_i^-)) \right\|^2 \\
 &= 3 \sum_{j=1}^3 Q_j.
 \end{aligned}
 \tag{15}$$

Next, we will estimate each item separately. Denote  $\mu = r - \varepsilon$ ,  $\varepsilon > 0$ . Using the Hölder inequality and  $(H_1)$ ,  $(H_2)$ , and  $(H_4)$ , we have

$$\begin{aligned}
 Q_1 &= \mathbb{E} \left\| \int_{t_0}^t S(t-s)F(s, x(s-\sigma(s))) \, ds \right\|^2 \\
 &\leq \mathbb{E} \left( \int_{t_0}^t \|S(t-s)\| \|F(s, x(s-\sigma(s)))\| \, ds \right)^2 \\
 &\leq M^2 \mathbb{E} \left( \int_{t_0}^t e^{-\frac{r}{2}(t-s)} e^{-\frac{\varepsilon}{2}(t-s)} \|F(s, x(s-\sigma(s)))\| \, ds \right)^2 \\
 &\leq M^2 \int_{t_0}^t e^{-r(t-s)} \, ds \int_{t_0}^t e^{-r(t-s)} \mathbb{E} \|F(s, x(s-\sigma(s)))\|^2 \, ds \\
 &\leq M^2 \times \frac{1}{r} (1 - e^{-r(t-t_0)}) \int_{t_0}^t e^{-r(t-s)} R_1^2 \mathbb{E} \|x(s-\sigma(s))\|^2 \, ds \\
 &\leq \frac{M^2 R_1^2}{r} \int_{t_0}^t e^{-\mu(t-s)} \sup_{\theta \in [-\alpha, 0]} \mathbb{E} \|x(s+\theta)\|^2 \, ds.
 \end{aligned}
 \tag{16}$$

Using Lemma 1 and  $(H_4)$ , we have

$$\begin{aligned}
 Q_2 &= \mathbb{E} \left\| \int_{t_0}^t S(t-s)G(s) \, dW^H(s) \right\|^2 \\
 &\leq cH(2H-1)(t-t_0)^{2H-1} \int_{t_0}^t \|S(t-s)G(s)\|_{\mathbb{S}_2^0(Y, X)}^2 \, ds \\
 &\leq M^2 cH(2H-1)(t-t_0)^{2H-1} e^{-rt} \int_{t_0}^t e^{rs} e^{-r(t-s)} \|G(s)\|_{\mathbb{S}_2^0(Y, X)}^2 \, ds
 \end{aligned}$$

$$\begin{aligned} &\leq M^2 cH(2H - 1)(t - t_0)^{2H-1} e^{-(\mu+\varepsilon)t} \int_{t_0}^t e^{rs} \|G(s)\|_{\mathcal{L}_0^2(Y, X)}^2 ds \\ &\leq e^{-\mu t} \left( M^2 cH(2H - 1)(t - t_0)^{2H-1} e^{-\varepsilon t} \int_{t_0}^t e^{rs} \|G(s)\|_{\mathcal{L}_0^2(Y, X)}^2 ds \right). \end{aligned} \tag{17}$$

In view of  $(H_4)$ , there exists a constant  $R_2 > 0$  such that

$$M^2 cH(2H - 1)(t - t_0)^{2H-1} e^{-\varepsilon t} \int_{t_0}^t e^{rs} \|G(s)\|_{\mathcal{L}_0^2(Y, X)}^2 ds \leq R_2. \tag{18}$$

Then, combining (17) and (18), we have

$$Q_2 \leq R_2 e^{-\mu t}. \tag{19}$$

By the Hölder inequality,  $(H_1)$ , and  $(H_4)$ , we have

$$\begin{aligned} Q_3 &= \mathbb{E} \left\| \sum_{i=1}^k S(t - t_i) I_i(x(t_i^-)) \right\|^2 \\ &\leq \mathbb{E} \left( \sum_{i=1}^k \|S(t - t_i) I_i(x(t_i^-))\| \right)^2 \\ &\leq M^2 \mathbb{E} \left( \sum_{i=1}^k (d_i)^{\frac{1}{2}} (d_i)^{\frac{1}{2}} e^{-r(t-t_i)} x(t_i^-) \right)^2 \\ &\leq M^2 \left( \sum_{i=1}^k d_i \right) \left( \sum_{i=1}^k d_i e^{-2r(t-t_i)} \mathbb{E} \|x(t_i^-)\|^2 \right) \\ &\leq M^2 \left( \sum_{i=1}^k d_i \right) \sum_{i=1}^k d_i e^{-\mu(t-t_i)} \mathbb{E} \|x(t_i^-)\|^2. \end{aligned} \tag{20}$$

By (15), (16), (19), and (20), for all  $t \geq t_0$ , we have

$$\begin{aligned} \mathbb{E} \|x(t)\|^2 &\leq 3 \frac{M^2 R_1^2}{r} \int_{t_0}^t e^{-\mu(t-s)} \sup_{\theta \in [-\alpha, 0]} \mathbb{E} \|x(s + \theta)\|^2 ds + 3R_2 e^{-\mu t} \\ &\quad + 3M^2 \left( \sum_{i=1}^{\infty} d_i \right) \sum_{t_i < t} d_i e^{-\mu(t-t_i)} \mathbb{E} \|x(t_i^-)\|^2, \end{aligned} \tag{21}$$

and it is easy to see that, for  $t \in [-\alpha, t_0]$ ,

$$\mathbb{E} \|x(t)\|^2 \leq \alpha_1 e^{-\mu t}, \tag{22}$$

where  $\alpha_1 = \max(3R_2, \sup_{\theta \in [-\alpha, 0]} \mathbb{E} \|\phi(\theta)\|^2)$ . By (21), (22), (4), and Lemma 2, for all  $t \geq -\alpha$ , we have

$$\mathbb{E} \|x(t)\|^2 \leq M_0 e^{-\lambda t}, \tag{23}$$

where  $\lambda$  is the unique solution to the equation  $\frac{\alpha_2 e^{\lambda\alpha}}{\mu-\lambda} + \sum_{k=1}^{\infty} \beta_k = 1$ , with  $\alpha_2 = \frac{3M^2 R_1^2}{r}$ ,  $\beta_k = 3M^2 d_k (\sum_{k=1}^{\infty} d_k)$ , and  $M_0 = \max\{\alpha_1, \frac{\alpha_1(\mu-\lambda)}{\alpha_2 e^{\lambda\alpha}}\}$ . This means that the mild solution of Eq. (2) is exponentially stable in mean square, and the proof is completed.  $\square$

**Remark 3.1** We would like to mention that our method used in Theorem 3.1 can be extended to the neutral case.

**Remark 3.2** If  $(H_2)$  is replaced by  $(\overline{H}_2)$ , then Theorem 3.1 remains true.

$(\overline{H}_2)$  There exist two nonnegative real numbers  $R_1 > 0$  and  $l > 0$  such that

$$\|F(t, x)\|^2 \leq R_1 \|x\|^2 + l e^{-rt}, \quad \forall t \geq t_0, x \in X.$$

Note that  $\alpha_1$  in (22) should be  $\alpha_1 = \max(3R_2 + 3\frac{M^2 l}{r(r-\mu)}, \sup_{\theta \in [-\alpha, 0]} \mathbb{E}\|\phi(\theta)\|^2)$ .

**Remark 3.3** If the impulses in (2) are removed, that is,  $\Delta x(t_k) = I_k(\cdot) = 0, k \in \mathbb{N}$ , then Eq. (2) reduces to the following equation:

$$\begin{cases} dx(t) = [Ax(t) + F(t, x(t - \sigma(t)))] dt + G(t) dW^H(t), & t \geq t_0, \\ x(t_0 + \theta) = \phi(\theta), & \theta \in [-\alpha, 0]. \end{cases} \tag{24}$$

By using the same technique as in Theorem 3.1, we can easily deduce the following corollary.

**Corollary 3.1** Suppose that  $(H_1), (H_2), (H_4)$  or  $(H_1), (\overline{H}_2), (H_4)$  and  $2M^2 R_1 < r^2$  hold. Then the mild solution of Eq. (24) is exponentially stable in mean square.

**Remark 3.4** In [16], the authors studied Eq. (24) and proved the mean-square exponential stability of the mild solution for (24), assuming that the delay function  $\sigma(t) : [0, +\infty) \rightarrow [0, r]$  is differentiable and  $|1 - \sigma'(t)|^{-1} < \rho^*$ , where  $\rho^*$  is a constant. It is obvious that Corollary 3.1 is less conservative than Theorem 4 in [16].

#### 4 Mean-square exponential stability of delayed impulsive differential equation driven by an fBm with time-varying delay

In this section, we assume that the impulses in Eq. (1) depend on the historical states of the system, that is, the impulses can be described as  $I_k(x(t_k^-), x(t_k^- - \delta)) = I_k(x(t_k^- - \delta))$ . In addition, we assume that the impulses are linear. Then Eq. (1) reduces to the following equation:

$$\begin{cases} dx(t) = [Ax(t) + F(t, x(t - \sigma(t)))] dt + G(t) dW^H(t), & t \geq t_0, t \neq t_k, \\ \Delta x(t_k) = \rho_k x(t_k^- - \delta), & k \in \mathbb{N}, \\ x(t_0 + \theta) = \phi(\theta), & \theta \in [-\alpha, 0], \end{cases} \tag{25}$$

where the delay  $\delta$  in impulses is a constant, and  $0 \leq \delta < \underline{\tau} = \inf_{k \in \mathbb{N}} \{t_k - t_{k-1}\}, \rho_k > 0, k \in \mathbb{N}$ , are constants.

Next, we give the definition of a mild solution of Eq. (25).

**Definition 4.1** An  $X$ -valued stochastic process  $\{x(t), t \in [-\alpha, \infty)\}$  is called a mild solution of Eq. (25) if  $x(t_0 + \theta) = \phi(\theta)$  on the interval  $[-\alpha, 0]$  and the following conditions hold:

- (i)  $x(\cdot)$  is continuous on  $[t_0, t_1]$  and on each interval  $(t_k, t_{k+1}), k \in \mathbb{N}$ ,
- (ii) for each  $k$ , the limits  $x(t_k^+), x(t_k^-)$  exist, and  $x(t_k^-) = x(t_k)$ ,
- (iii) for each  $t \geq t_0$ , we have a.s.

$$x(t) = \begin{cases} \int_{t_0}^t S(t-s)F(s, x(s-\sigma(s))) ds + \int_{t_0}^t S(t-s)G(s) dW^H(s), & t \in [t_0, t_1], \\ S(t-t_k)[x(t_k^-) + \rho_k x(t_k^- - \delta)] + \int_{t_k}^t S(t-s)F(s, x(s-\sigma(s))) ds \\ + \int_{t_k}^t S(t-s)G(s) dW^H(s), & t \in (t_k, t_{k+1}], k \in \mathbb{N}. \end{cases} \tag{26}$$

In order to obtain the mean-square asymptotic stability conditions of for the mild solution for Eq. (25), we need the following assumption.

( $H_5$ ) The impulses satisfy

$$\lim_{k \rightarrow \infty} \Sigma < \infty,$$

where

$$\begin{aligned} \Sigma = k & \left( C_k^1 \sum_{i_1=1}^k \rho_{i_1}^2 e^{-r(k-i_1)\tau} + C_k^2 \sum_{i_1=1}^{k-1} \sum_{i_2 > i_1}^k \rho_{i_1}^2 \rho_{i_2}^2 e^{-r((k-i_1)\tau - \delta)} + \dots \right. \\ & \left. + C_k^k \sum_{i_1=1}^1 \sum_{i_2 > i_1}^2 \dots \sum_{i_k > i_{k-1}}^k \rho_{i_1}^2 \rho_{i_2}^2 \dots \rho_{i_k}^2 e^{-r((k-i_1)\tau - (k-1)\delta)} \right) \end{aligned}$$

with binomial coefficients  $C_k^j = \binom{k}{j}, j = 1, 2, \dots, k$ .

**Remark 4.1** Assumption ( $H_5$ ) looks complicated because of the addition of delay in impulses; however, it is not difficult to realize. We give an example here. If  $d_k = 0, k \in \mathbb{N}$ , then  $\Sigma = 0$ . It is easy to see that each term  $e^{-r(k-i_1)\tau} \geq 0, i_1 = 1, 2, \dots, k; e^{-r((k-i_1)\tau - \delta)} \geq 0, i_1 = 1, 2, \dots, k-1; \dots e^{-r((k-i_1)\tau - (k-1)\delta)} \geq 0, i_1 = 1$ . If we choose  $\rho_k = \frac{1}{k^2}$ , then

$$\Sigma \leq k \sum_{j=1}^k \left( C_k^j \frac{1}{k^{2j}} \right)^2 \leq k \left( \sum_{j=1}^k C_k^j \frac{1}{k^{2j}} \right)^2 = k \left[ \left( 1 + \frac{1}{k^2} \right)^k - 1 \right]^2.$$

Taking the limit on both sides of this inequality, we get

$$\lim_{k \rightarrow \infty} \Sigma \leq \lim_{k \rightarrow \infty} \frac{\left\{ \left( 1 + \frac{1}{k^2} \right)^{k^2} - 1 \right\}^2}{\frac{1}{k}} = 0,$$

so that  $\lim_{k \rightarrow \infty} \Sigma = 0$ .

**Theorem 4.1** Suppose that ( $H_1$ ), ( $H_2$ ), ( $H_4$ ), and ( $H_5$ ) hold. Then, the mild solution of Eq. (25) is asymptotically stable in mean square.

*Proof* Denote by  $\mathfrak{F}$  the space of all stochastic processes  $x(t, \omega) : [-\alpha, +\infty) \times \Omega \rightarrow X$  satisfying  $x(t_0 + \theta) = \phi(\theta)$ ,  $\theta \in [-\alpha, 0]$  and conditions (i)-(ii) in Definition 4.1 and

$$\lim_{t \rightarrow \infty} \mathbb{E} \|x(t)\|^2 = 0. \tag{27}$$

Our goal is to estimate the limit of  $\mathbb{E} \|x(t)\|^2$  as  $t \rightarrow \infty$ . For  $t \in [t_0, t_1]$ , we have

$$x(t) = \int_{t_0}^t S(t-s)F(s, x(s-\sigma(s))) \, ds + \int_{t_0}^t S(t-s)G(s) \, dW^H(s). \tag{28}$$

For  $t \in (t_1, t_2]$ , by (26), (28), and Definition 4.1 we have

$$\begin{aligned} x(t) &= S(t-t_1)(x(t_1^-) + \rho_1 x(t_1 - \delta)) + \int_{t_1}^t S(t-s)G(s) \, dW^H(s) \\ &\quad + \int_{t_1}^t S(t-s)F(s, x(s-\sigma(s))) \, ds \\ &= \int_{t_0}^t S(t-s)F(s, x(s-\sigma(s))) \, ds + \int_{t_0}^t S(t-s)G(s) \, dW^H(s) \\ &\quad + \rho_1 \int_{t_0}^{t_1-\delta} S(t-\delta-s)F(s, x(s-\sigma(s))) \, ds \\ &\quad + \rho_1 \int_{t_0}^{t_1-\delta} S(t-\delta-s)G(s) \, dW^H(s). \end{aligned} \tag{29}$$

For  $t \in (t_2, t_3]$ , from (26) we have

$$\begin{aligned} x(t) &= S(t-t_2)(x(t_2^-) + \rho_2 x(t_2 - \delta)) + \int_{t_2}^t S(t-s)F(s, x(s-\sigma(s))) \, ds \\ &\quad + \int_{t_2}^t S(t-s)G(s) \, dW^H(s). \end{aligned} \tag{30}$$

By (29) and (30) we get, for  $t \in (t_2, t_3]$ ,

$$\begin{aligned} x(t) &= \int_{t_0}^t S(t-s)F(s, x(s-\sigma(s))) \, ds + \int_{t_0}^t S(t-s)G(s) \, dW^H(s) \\ &\quad + \rho_1 \int_{t_0}^{t_1-\delta} S(t-\delta-s)F(s, x(s-\sigma(s))) \, ds \\ &\quad + \rho_1 \int_{t_0}^{t_1-\delta} S(t-\delta-s)G(s) \, dW^H(s) \\ &\quad + \rho_2 \int_{t_0}^{t_2-\delta} S(t-\delta-s)F(s, x(s-\sigma(s))) \, ds \\ &\quad + \rho_2 \int_{t_0}^{t_2-\delta} S(t-\delta-s)G(s) \, dW^H(s) \\ &\quad + \rho_1 \rho_2 \int_{t_0}^{t_1-\delta} S(t-2\delta-s)F(s, x(s-\sigma(s))) \, ds \\ &\quad + \rho_1 \rho_2 \int_{t_0}^{t_1-\delta} S(t-2\delta-s)G(s) \, dW^H(s). \end{aligned} \tag{31}$$

Assume that the following holds for  $t \in (t_m, t_{m+1}]$  (for any  $m \in \mathbb{N}$ ):

$$\begin{aligned}
 x(t) &= \int_{t_0}^t S(t-s)F(s, x(s-\sigma(s))) \, ds + \int_{t_0}^t S(t-s)G(s) \, dW^H(s) \\
 &\quad + \sum_{i_1=1}^m \rho_{i_1} \int_{t_0}^{t_{i_1}-\delta} S(t-\delta-s)F(s, x(s-\sigma(s))) \, ds \\
 &\quad + \sum_{i_1=1}^m \rho_{i_1} \int_{t_0}^{t_{i_1}-\delta} S(t-\delta-s)G(s) \, dW^H(s) \\
 &\quad + \sum_{i_1=1}^{m-1} \sum_{i_2>i_1}^m \rho_{i_1} \rho_{i_2} \int_{t_0}^{t_{i_1}-\delta} S(t-2\delta-s)F(s, x(s-\sigma(s))) \, ds \\
 &\quad + \sum_{i_1=1}^{m-1} \sum_{i_2>i_1}^m \rho_{i_1} \rho_{i_2} \int_{t_0}^{t_{i_1}-\delta} S(t-2\delta-s)G(s) \, dW^H(s) + \dots \\
 &\quad + \sum_{i_1=1}^1 \sum_{i_2>i_1}^2 \dots \sum_{i_m>i_{m-1}}^m \rho_{i_1} \rho_{i_2} \dots \rho_{i_m} \int_{t_0}^{t_{i_1}-\delta} S(t-m\delta-s)F(s, x(s-\sigma(s))) \, ds \\
 &\quad + \sum_{i_1=1}^1 \sum_{i_2>i_1}^2 \dots \sum_{i_m>i_{m-1}}^m \rho_{i_1} \rho_{i_2} \dots \rho_{i_m} \int_{t_0}^{t_{i_1}-\delta} S(t-m\delta-s)G(s) \, dW^H(s). \tag{32}
 \end{aligned}$$

Then, for  $t \in (t_{m+1}, t_{m+2}]$ , by (26) we have

$$\begin{aligned}
 x(t) &= S(t-t_{m+1})(x(t_{m+1}^-) + \rho_{m+1}x(t_{m+1}^- - \delta)) + \int_{t_{m+1}}^t S(t-s)G(s) \, dW^H(s) \\
 &\quad + \int_{t_{m+1}}^t S(t-s)F(s, x(s-\sigma(s))) \, ds. \tag{33}
 \end{aligned}$$

Substituting  $x(t_{m+1}^-)$  and  $x(t_{m+1}^- - \delta)$ , which can be obtained from (32), into (33), by properties of semigroups and integrals, for  $t \in (t_{m+1}, t_{m+2}]$ , we have

$$\begin{aligned}
 x(t) &= \int_{t_0}^t S(t-s)F(s, x(s-\sigma(s))) \, ds \\
 &\quad + \int_{t_0}^t S(t-s)G(s) \, dW^H(s) \\
 &\quad + \sum_{i_1=1}^{m+1} \rho_{i_1} \int_{t_0}^{t_{i_1}-\delta} S(t-\delta-s)F(s, x(s-\sigma(s))) \, ds \\
 &\quad + \sum_{i_1=1}^{m+1} \rho_{i_1} \int_{t_0}^{t_{i_1}-\delta} S(t-\delta-s)G(s) \, dW^H(s) \\
 &\quad + \sum_{i_1=1}^m \sum_{i_2>i_1}^{m+1} \rho_{i_1} \rho_{i_2} \int_{t_0}^{t_{i_1}-\delta} S(t-2\delta-s)F(s, x(s-\sigma(s))) \, ds \\
 &\quad + \sum_{i_1=1}^m \sum_{i_2>i_1}^{m+1} \rho_{i_1} \rho_{i_2} \int_{t_0}^{t_{i_1}-\delta} S(t-2\delta-s)G(s) \, dW^H(s) + \dots
 \end{aligned}$$

$$\begin{aligned}
 & + \sum_{i_1=1}^1 \sum_{i_2>i_1}^2 \cdots \sum_{i_{m+1}>i_m}^{m+1} \rho_{i_1} \rho_{i_2} \cdots \rho_{i_m} \int_{t_0}^{t_{i_1}-\delta} S(t-(m+1)\delta-s) F(s, x(s-\sigma(s))) \, ds \\
 & + \sum_{i_1=1}^1 \sum_{i_2>i_1}^2 \cdots \sum_{i_{m+1}>i_m}^{m+1} \rho_{i_1} \rho_{i_2} \cdots \rho_{i_m} \int_{t_0}^{t_{i_1}-\delta} S(t-(m+1)\delta-s) G(s) \, dW^H(s). \tag{34}
 \end{aligned}$$

By mathematical induction and (32)-(34), for all  $k \in \mathbb{N}$ ,  $t \in (t_k, t_{k+1}]$ , we have

$$\begin{aligned}
 x(t) & = \int_{t_0}^t S(t-s) F(s, x(s-\sigma(s))) \, ds + \int_{t_0}^t S(t-s) G(s) \, dW^H(s) \\
 & + \left( \sum_{i_1=1}^k \rho_{i_1} \int_{t_0}^{t_{i_1}-\delta} S(t-\delta-s) F(s, x(s-\sigma(s))) \, ds \right. \\
 & + \sum_{i_1=1}^k \rho_{i_1} \int_{t_0}^{t_{i_1}-\delta} S(t-\delta-s) G(s) \, dW^H(s) \\
 & + \sum_{i_1=1}^{k-1} \sum_{i_2>i_1}^k \rho_{i_1} \rho_{i_2} \int_{t_0}^{t_{i_1}-\delta} S(t-2\delta-s) F(s, x(s-\sigma(s))) \, ds \\
 & + \sum_{i_1=1}^{k-1} \sum_{i_2>i_1}^k \rho_{i_1} \rho_{i_2} \int_{t_0}^{t_{i_1}-\delta} S(t-2\delta-s) G(s) \, dW^H(s) + \cdots \\
 & + \sum_{i_1=1}^1 \sum_{i_2>i_1}^2 \cdots \sum_{i_k>i_{k-1}}^k \rho_{i_1} \rho_{i_2} \cdots \rho_{i_k} \int_{t_0}^{t_{i_1}-\delta} S(t-k\delta-s) F(s, x(s-\sigma(s))) \, ds \\
 & \left. + \sum_{i_1=1}^1 \sum_{i_2>i_1}^2 \cdots \sum_{i_k>i_{k-1}}^k \rho_{i_1} \rho_{i_2} \cdots \rho_{i_k} \int_{t_0}^{t_{i_1}-\delta} S(t-k\delta-s) G(s) \, dW^H(s) \right) \\
 & = \int_{t_0}^t S(t-s) F(s, x(s-\sigma(s))) \, ds + \int_{t_0}^t S(t-s) G(s) \, dW^H(s) + \mathcal{E}. \tag{35}
 \end{aligned}$$

Now we estimate (35). For all  $k \in \mathbb{N}$ ,  $t \in (t_k, t_{k+1}]$ , we have

$$\begin{aligned}
 \mathbb{E} \|x(t)\|^2 & \leq 3 \mathbb{E} \left\| \int_{t_0}^t S(t-s) F(s, x(s-\sigma(s))) \, ds \right\|^2 \\
 & + 3 \mathbb{E} \left\| \int_{t_0}^t S(t-s) G(s) \, dW^H(s) \right\|^2 + 3 \mathbb{E} \|\mathcal{E}\|^2. \tag{36}
 \end{aligned}$$

By  $(H_1)$ ,  $(H_2)$ , and the Hölder inequality we have

$$\begin{aligned}
 T_1 & = 3 \mathbb{E} \left\| \int_{t_0}^t S(t-s) F(s, x(s-\sigma(s))) \, ds \right\|^2 \\
 & \leq 3 \mathbb{E} \left( \int_{t_0}^t M e^{-r(t-s)} \|F(s, x(s-\sigma(s)))\| \, ds \right)^2 \\
 & \leq 3M^2 \int_{t_0}^t e^{-r(t-s)} \, ds \int_{t_0}^t e^{-r(t-s)} \|F(s, x(s-\sigma(s)))\|^2 \, ds \\
 & \leq \frac{3M^2 R_1^2}{r} (1 - e^{-r(t-t_0)}) \int_{t_0}^t e^{-r(t-s)} \mathbb{E} \|x(s-\sigma(s))\|^2 \, ds. \tag{37}
 \end{aligned}$$

For any  $x \in \mathfrak{F}$ , by the definition of  $x \in \mathfrak{F}$ , for arbitrary  $\varepsilon > 0$ , there exists  $s^* > t_0$  such that  $\mathbb{E}\|x(s - \sigma(s))\|^2 < \varepsilon$  for all  $s > s^*$ . So we have

$$\begin{aligned} & \int_{t_0}^t e^{-r(t-s)} \mathbb{E}\|x(s - \sigma(s))\|^2 ds \\ &= \int_{t_0}^{s^*} e^{-r(t-s)} \mathbb{E}\|x(s - \sigma(s))\|^2 ds + \frac{1}{r} \varepsilon (1 - e^{-r(t-s^*)}). \end{aligned} \tag{38}$$

Combining (37) with (38), we have

$$\lim_{t \rightarrow \infty} T_1 = \lim_{t \rightarrow \infty} 3\mathbb{E} \left\| \int_{t_0}^t S(t-s)F(s, x(s - \sigma(s))) ds \right\|^2 = 0. \tag{39}$$

By Lemma 1 we get that

$$\begin{aligned} T_2 &= 3\mathbb{E} \left\| \int_{t_0}^t S(t-s)G(s) dW^H(s) \right\|^2 \\ &\leq 3cH(2H-1)(t-t_0)^{2H-1} \int_{t_0}^t \|S(t-s)G(s)\|_{\mathcal{L}_2^0(Y,X)}^2 ds \\ &\leq 3cH(2H-1)(t-t_0)^{2H-1} \int_{t_0}^t e^{-2r(t-s)} \|G(s)\|_{\mathcal{L}_2^0(Y,X)}^2 ds \\ &= 3cH(2H-1)(t-t_0)^{2H-1} e^{-rt} \int_{t_0}^t e^{rs} e^{-r(t-s)} \|G(s)\|_{\mathcal{L}_2^0(Y,X)}^2 ds \\ &\leq 3cH(2H-1)(t-t_0)^{2H-1} e^{-rt} \int_{t_0}^t e^{rs} \|G(s)\|_{\mathcal{L}_2^0(Y,X)}^2 ds. \end{aligned} \tag{40}$$

It is easy to see that  $\lim_{t \rightarrow \infty} (t-t_0)^{2H-1} e^{-rt} = 0$ , and by condition (H4) we get

$$\lim_{t \rightarrow \infty} T_2 = 3\mathbb{E} \left\| \int_{t_0}^t S(t-s)G(s) dW^H(s) \right\|^2 = 0. \tag{41}$$

Next, we estimate  $\mathbb{E}\|\mathcal{E}\|^2$ :

$$\begin{aligned} T_3 &= 3\mathbb{E} \left\| \sum_{i_1=1}^k \rho_{i_1} \int_{t_0}^{t_{i_1}-\delta} S(t-\delta-s)F(s, x(s - \sigma(s))) ds \right. \\ &\quad + \sum_{i_1=1}^k \rho_{i_1} \int_{t_0}^{t_{i_1}-\delta} S(t-\delta-s)G(s) dW^H(s) \\ &\quad + \sum_{i_1=1}^{k-1} \sum_{i_2 > i_1}^k \rho_{i_1} \rho_{i_2} \int_{t_0}^{t_{i_1}-\delta} S(t-2\delta-s)F(s, x(s - \sigma(s))) ds \\ &\quad + \sum_{i_1=1}^{k-1} \sum_{i_2 > i_1}^k \rho_{i_1} \rho_{i_2} \int_{t_0}^{t_{i_1}-\delta} S(t-2\delta-s)G(s) dW^H(s) + \dots \\ &\quad \left. + \sum_{i_1=1}^1 \sum_{i_2 > i_1}^2 \dots \sum_{i_k > i_{k-1}}^k \rho_{i_1} \rho_{i_2} \dots \rho_{i_k} \int_{t_0}^{t_{i_1}-\delta} S(t-k\delta-s)F(s, x(s - \sigma(s))) ds \right\|^2 \end{aligned}$$

$$\begin{aligned}
 & + \left\| \sum_{i_1=1}^1 \sum_{i_2>i_1}^2 \cdots \sum_{i_k>i_{k-1}}^k \rho_{i_1} \rho_{i_2} \cdots \rho_{i_k} \int_{t_0}^{t_{i_1}-\delta} S(t-k\delta-s)G(s) dW^H(s) \right\|^2 \\
 & = 3\mathbb{E}\|\mathcal{E}\|^2.
 \end{aligned} \tag{42}$$

Using the  $C_p$  inequality, we get

$$\begin{aligned}
 T_3 & = 3\mathbb{E}\|\mathcal{E}\|^2 \\
 & \leq 3 \times 2k \left\{ \mathbb{E} \left\| \sum_{i_1=1}^k \rho_{i_1} \int_{t_0}^{t_{i_1}-\delta} S(t-\delta-s)F(s, x(s-\sigma(s))) ds \right\|^2 \right. \\
 & \quad + \mathbb{E} \left\| \sum_{i_1=1}^{k-1} \sum_{i_2>i_1}^k \rho_{i_1} \rho_{i_2} \times \int_{t_0}^{t_{i_1}-\delta} S(t-2\delta-s)F(s, x(s-\sigma(s))) ds \right\|^2 + \cdots \\
 & \quad + \mathbb{E} \left\| \sum_{i_1=1}^1 \sum_{i_2>i_1}^2 \cdots \sum_{i_k>i_{k-1}}^k \rho_{i_1} \rho_{i_2} \cdots \rho_{i_k} \int_{t_0}^{t_{i_1}-\delta} S(t-k\delta-s)F(s, x(s-\sigma(s))) ds \right\|^2 \\
 & \quad + \mathbb{E} \left\| \sum_{i_1=1}^k \rho_{i_1} \int_{t_0}^{t_{i_1}-\delta} S(t-\delta-s)G(s) dW^H(s) \right\|^2 \\
 & \quad + \mathbb{E} \left\| \sum_{i_1=1}^{k-1} \sum_{i_2>i_1}^k \rho_{i_1} \rho_{i_2} \int_{t_0}^{t_{i_1}-\delta} S(t-2\delta-s)G(s) dW^H(s) \right\|^2 + \cdots \\
 & \quad \left. + \mathbb{E} \left\| \sum_{i_1=1}^1 \sum_{i_2>i_1}^2 \cdots \sum_{i_k>i_{k-1}}^k \rho_{i_1} \rho_{i_2} \cdots \rho_{i_k} \int_{t_0}^{t_{i_1}-\delta} S(t-k\delta-s)G(s) dW^H(s) \right\|^2 \right\} \\
 & = 6k \left( \sum_{j=1}^k U_j + \sum_{j=1}^k V_j \right).
 \end{aligned} \tag{43}$$

Firstly, we estimate the terms  $U_j$  ( $j = 1, 2, \dots, k$ ). By  $(H_1)$ ,  $(H_2)$ , the  $C_p$  inequality, and the Hölder inequality we have

$$\begin{aligned}
 U_1 & \leq C_k^1 \sum_{i_1=1}^k \rho_{i_1}^2 \mathbb{E} \left\| \int_{t_0}^{t_{i_1}-\delta} S(t-\delta-s)F(s, x(s-\sigma(s))) ds \right\|^2 \\
 & \leq M^2 R_1^2 C_k^1 \sum_{i_1=1}^k \rho_{i_1}^2 \int_{t_0}^{t_{i_1}-\delta} e^{-r(t-\delta-s)} ds \int_{t_0}^{t_{i_1}-\delta} e^{-r(t-\delta-s)} \mathbb{E} \|x(s-\sigma(s))\|^2 ds \\
 & \leq \frac{M^2 R_1^2}{r} C_k^1 \sum_{i_1=1}^k \rho_{i_1}^2 e^{-2r(t-t_{i_1})} (1 - e^{-r(t_{i_1}-\delta-t_0)}) \\
 & \quad \times \int_{t_0}^{t_{i_1}-\delta} e^{-r(t-\delta-s)} \mathbb{E} \|x(s-\sigma(s))\|^2 ds \\
 & \leq \frac{M^2 R_1^2}{r} C_k^1 \sum_{i_1=1}^k \rho_{i_1}^2 e^{-2r(t-t_{i_1})} \int_{t_0}^{t_{i_1}-\delta} e^{-r(t_{i_1}-\delta-s)} \mathbb{E} \|x(s-\sigma(s))\|^2 ds.
 \end{aligned} \tag{44}$$

In a similar way, we get

$$\begin{aligned}
 U_2 &\leq C_k^2 \sum_{i_1=1}^{k-1} \sum_{i_2>i_1}^k \rho_{i_1}^2 \rho_{i_2}^2 \mathbb{E} \left\| \int_{t_0}^{t_{i_1}-\delta} S(t-2\delta-s) F(s, x(s-\sigma(s))) \, ds \right\|^2 \\
 &\leq M^2 R_1^2 C_k^2 \sum_{i_1=1}^{k-1} \sum_{i_2>i_1}^k \rho_{i_1}^2 \rho_{i_2}^2 \int_{t_0}^{t_{i_1}-\delta} e^{-r(t-2\delta-s)} \, ds \\
 &\quad \times \int_{t_0}^{t_{i_1}-\delta} e^{-r(t-2\delta-s)} \mathbb{E} \|x(s-\sigma(s))\|^2 \, ds \\
 &\leq \frac{M^2 R_1^2}{r} C_k^2 \sum_{i_1=1}^{k-1} \sum_{i_2>i_1}^k \rho_{i_1}^2 \rho_{i_2}^2 e^{-2r(t-t_{i_1}-\delta)} (1 - e^{-r(t_{i_1}-\delta-t_0)}) \\
 &\quad \times \int_{t_0}^{t_{i_1}-\delta} e^{-r(t_{i_1}-\delta-s)} \mathbb{E} \|x(s-\sigma(s))\|^2 \, ds \\
 &\leq \frac{M^2 R_1^2}{r} C_k^2 \sum_{i_1=1}^{k-1} \sum_{i_2>i_1}^k \rho_{i_1}^2 \rho_{i_2}^2 e^{-2r(t-t_{i_1}-\delta)} \int_{t_0}^{t_{i_1}-\delta} e^{-r(t_{i_1}-\delta-s)} \mathbb{E} \|x(s-\sigma(s))\|^2 \, ds. \tag{45}
 \end{aligned}$$

In the same way, one get

$$\begin{aligned}
 U_k &\leq C_k^k \sum_{i_1=1}^1 \sum_{i_2>i_1}^2 \cdots \sum_{i_k>i_{k-1}}^k \rho_{i_1}^2 \rho_{i_2}^2 \cdots \rho_{i_k}^2 \mathbb{E} \left\| \int_{t_0}^{t_{i_1}-\delta} S(t-k\delta-s) F(s, x(s-\sigma(s))) \, ds \right\|^2 \\
 &\leq M^2 R_1^2 C_k^k \sum_{i_1=1}^1 \sum_{i_2>i_1}^2 \cdots \sum_{i_k>i_{k-1}}^k \rho_{i_1}^2 \rho_{i_2}^2 \cdots \rho_{i_k}^2 \int_{t_0}^{t_{i_1}-\delta} e^{-r(t-k\delta-s)} \, ds \\
 &\quad \times \int_{t_0}^{t_{i_1}-\delta} e^{-r(t-k\delta-s)} \mathbb{E} \|x(s-\sigma(s))\|^2 \, ds \\
 &\leq \frac{M^2 R_1^2}{r} C_k^k \sum_{i_1=1}^1 \sum_{i_2>i_1}^2 \cdots \sum_{i_k>i_{k-1}}^k \rho_{i_1}^2 \rho_{i_2}^2 \cdots \rho_{i_k}^2 e^{-2r(t-t_{i_1}-(k-1)\delta)} \\
 &\quad \times (1 - e^{-r(t_{i_1}-\delta-t_0)}) \int_{t_0}^{t_{i_1}-\delta} e^{-r(t_{i_1}-\delta-s)} \mathbb{E} \|x(s-\sigma(s))\|^2 \, ds \\
 &\leq \frac{M^2 R_1^2}{r} C_k^k \sum_{i_1=1}^1 \sum_{i_2>i_1}^2 \cdots \sum_{i_k>i_{k-1}}^k \rho_{i_1}^2 \rho_{i_2}^2 \cdots \rho_{i_k}^2 e^{-2r(t-t_{i_1}-(k-1)\delta)} \\
 &\quad \times \int_{t_0}^{t_{i_1}-\delta} e^{-r(t_{i_1}-\delta-s)} \mathbb{E} \|x(s-\sigma(s))\|^2 \, ds. \tag{46}
 \end{aligned}$$

By  $C_p$ , Lemma 1, and  $(H_1)$  we have

$$\begin{aligned}
 V_1 &\leq C_k^1 \sum_{i_1=1}^k d_{i_1}^2 \mathbb{E} \left\| \int_{t_0}^{t_{i_1}-\delta} S(t-\delta-s) G(s) \, dW^H(s) \right\|^2 \\
 &\leq C_k^1 \sum_{i_1=1}^k \rho_{i_1}^2 cH(2H-1)(t_{i_1}-\delta-t_0)^{2H-1} \int_{t_0}^{t_{i_1}-\delta} \|S(t-\delta-s)G(s)\|^2 \, ds
 \end{aligned}$$

$$\begin{aligned}
 &\leq M^2 cH(2H-1) C_k^1 \sum_{i_1=1}^k \rho_{i_1}^2 (t_{i_1} - \delta - t_0)^{2H-1} \int_{t_0}^{t_{i_1}-\delta} e^{-2r(t-\delta-s)} \|G(s)\|_{\mathcal{L}_2^0(Y,X)}^2 ds \\
 &\leq M^2 cH(2H-1) C_k^1 \sum_{i_1=1}^k \rho_{i_1}^2 (t_{i_1} - \delta - t_0)^{2H-1} e^{-r(t-\delta)} \int_{t_0}^{t_{i_1}-\delta} e^{rs} \|G(s)\|_{\mathcal{L}_2^0(Y,X)}^2 ds \\
 &\leq M^2 cH(2H-1) C_k^1 \sum_{i_1=1}^k \rho_{i_1}^2 (t_{i_1} - \delta - t_0)^{2H-1} e^{-r(t-\delta)} \int_{t_0}^t e^{rs} \|G(s)\|_{\mathcal{L}_2^0(Y,X)}^2 ds \\
 &\leq M^2 cH(2H-1) C_k^1 \sum_{i_1=1}^k \rho_{i_1}^2 (t_{i_1} - \delta - t_0)^{2H-1} e^{-r(t_{i_1}-\delta-t_0)} e^{-r(t-t_{i_1})} \\
 &\quad \times \int_{t_0}^t e^{rs} \|G(s)\|_{\mathcal{L}_2^0(Y,X)}^2 ds. \tag{47}
 \end{aligned}$$

We can do the same calculation for  $V_2$ :

$$\begin{aligned}
 V_2 &\leq C_k^2 \sum_{i_1=1}^{k-1} \sum_{i_2>i_1}^k \rho_{i_1}^2 \rho_{i_2}^2 \mathbb{E} \left\| \int_{t_0}^{t_{i_1}-\delta} S(t-2\delta-s)G(s) dW^H(s) \right\|^2 \\
 &\leq C_k^2 \sum_{i_1=1}^{k-1} \sum_{i_2>i_1}^k \rho_{i_1}^2 \rho_{i_2}^2 cH(2H-1)(t_{i_1} - \delta - t_0)^{2H-1} \int_{t_0}^{t_{i_1}-\delta} \|S(t-2\delta-s)G(s)\|^2 ds \\
 &\leq cH(2H-1)M^2 C_k^2 \sum_{i_1=1}^{k-1} \sum_{i_2>i_1}^k \rho_{i_1}^2 \rho_{i_2}^2 (t_{i_1} - \delta - t_0)^{2H-1} \int_{t_0}^{t_{i_1}-\delta} e^{-2r(t-2\delta-s)} \|G(s)\|^2 ds \\
 &\leq cH(2H-1)M^2 C_k^2 \sum_{i_1=1}^{k-1} \sum_{i_2>i_1}^k \rho_{i_1}^2 \rho_{i_2}^2 (t_{i_1} - \delta - t_0)^{2H-1} e^{-r(t-2\delta)} \int_{t_0}^{t_{i_1}-\delta} e^{rs} \|G(s)\|^2 ds \\
 &\leq cH(2H-1)M^2 C_k^2 \sum_{i_1=1}^{k-1} \sum_{i_2>i_1}^k \rho_{i_1}^2 \rho_{i_2}^2 (t_{i_1} - \delta - t_0)^{2H-1} e^{-r(t-2\delta)} \int_{t_0}^t e^{rs} \|G(s)\|^2 ds \\
 &\leq cH(2H-1)M^2 C_k^2 \sum_{i_1=1}^{k-1} \sum_{i_2>i_1}^k \rho_{i_1}^2 \rho_{i_2}^2 (t_{i_1} - \delta - t_0)^{2H-1} e^{-r(t_{i_1}-\delta-t_0)} e^{-r(t-t_{i_1}-\delta)} \\
 &\quad \times \int_{t_0}^t e^{rs} \|G(s)\|^2 ds. \tag{48}
 \end{aligned}$$

The rest can be done in the same manner as before:

$$\begin{aligned}
 V_k &\leq C_k^k \sum_{i_1=1}^1 \sum_{i_2>i_1}^2 \cdots \sum_{i_k>i_{k-1}}^k \rho_{i_1}^2 \rho_{i_2}^2 \cdots \rho_{i_k}^2 \mathbb{E} \left\| \int_{t_0}^{t_{i_1}-\delta} S(t-k\delta-s)G(s) dW^H(s) \right\|^2 \\
 &\leq C_k^k \sum_{i_1=1}^1 \sum_{i_2>i_1}^2 \cdots \sum_{i_k>i_{k-1}}^k \rho_{i_1}^2 \rho_{i_2}^2 \cdots \rho_{i_k}^2 cH(2H-1)(t_{i_1} - \delta - t_0)^{2H-1} \\
 &\quad \times \int_{t_0}^{t_{i_1}-\delta} \|S(t-k\delta-s)G(s)\|^2 ds \\
 &\leq cH(2H-1)M^2 C_k^k \sum_{i_1=1}^1 \sum_{i_2>i_1}^2 \cdots \sum_{i_k>i_{k-1}}^k \rho_{i_1}^2 \rho_{i_2}^2 \cdots \rho_{i_k}^2 (t_{i_1} - \delta - t_0)^{2H-1}
 \end{aligned}$$

$$\begin{aligned}
 & \times e^{-r(t-k\delta)} \int_{t_0}^{t_{i_1}-\delta} e^{rs} \|G(s)\|^2 ds \\
 & \leq cH(2H-1)M^2 C_k^k \sum_{i_1=1}^1 \sum_{i_2>i_1}^2 \cdots \sum_{i_k>i_{k-1}}^k \rho_{i_1}^2 \rho_{i_2}^2 \cdots \rho_{i_k}^2 (t_{i_1} - \delta - t_0)^{2H-1} \\
 & \quad \times e^{-r(t-k\delta)} \int_{t_0}^t e^{rs} \|G(s)\|^2 ds \\
 & \leq cH(2H-1)M^2 C_k^k \sum_{i_1=1}^1 \sum_{i_2>i_1}^2 \cdots \sum_{i_k>i_{k-1}}^k \rho_{i_1}^2 \rho_{i_2}^2 \cdots \rho_{i_k}^2 (t_{i_1} - \delta - t_0)^{2H-1} \\
 & \quad \times e^{-r(t_{i_1}-\delta-t_0)} e^{-r(t-t_{i_1}-(k-1)\delta)} \int_{t_0}^t e^{rs} \|G(s)\|^2 ds. \tag{49}
 \end{aligned}$$

By (43) to (49) we get

$$\begin{aligned}
 \sum_{j=1}^k U_j & \leq \frac{M^2 R_1^2}{r} \left( C_k^1 \sum_{i_1=1}^k \rho_{i_1}^2 e^{-2r(t-t_{i_1})} \int_{t_0}^{t_{i_1}-\delta} e^{-r(t_{i_1}-\delta-s)} \mathbb{E} \|x(s-\sigma(s))\|^2 ds \right. \\
 & \quad + C_k^2 \sum_{i_1=1}^{k-1} \sum_{i_2>i_1}^k \rho_{i_1}^2 \rho_{i_2}^2 e^{-2r(t-t_{i_1}-\delta)} \int_{t_0}^{t_{i_1}-\delta} e^{-r(t_{i_1}-\delta-s)} \mathbb{E} \|x(s-\sigma(s))\|^2 ds \\
 & \quad + \cdots + C_k^k \sum_{i_1=1}^1 \sum_{i_2>i_1}^2 \cdots \sum_{i_k>i_{k-1}}^k \rho_{i_1}^2 \rho_{i_2}^2 \cdots \rho_{i_k}^2 e^{-2r(t-t_{i_1}-(k-1)\delta)} \\
 & \quad \left. \times \int_{t_0}^{t_{i_1}-\delta} e^{-r(t_{i_1}-\delta-s)} \mathbb{E} \|x(s-\sigma(s))\|^2 ds \right). \tag{50}
 \end{aligned}$$

For any  $x(t) \in \mathfrak{F}$ , by the definition of  $\mathfrak{F}$  and by  $(H_5)$  we get

$$\lim_{k \rightarrow \infty} k \sum_{j=1}^k U_j = 0 \tag{51}$$

and

$$\begin{aligned}
 \sum_{j=1}^k V_j & \leq M^2 cH(2H-1) \int_{t_0}^t e^{rs} \|G(s)\|_{\mathcal{L}_2^0(Y,X)}^2 ds \\
 & \quad \times \left\{ C_k^1 \sum_{i_1=1}^k \rho_{i_1}^2 e^{-r(t-t_{i_1})(t_{i_1}-\delta-t_0)^{2H-1}} e^{-r(t_{i_1}-\delta-t_0)} \right. \\
 & \quad + C_k^2 \sum_{i_1=1}^{k-1} \sum_{i_2>i_1}^k \rho_{i_1}^2 \rho_{i_2}^2 (t_{i_1} - \delta - t_0)^{2H-1} e^{-r(t_{i_1}-\delta-t_0)} e^{-r(t-t_{i_1}-\delta)} \\
 & \quad + C_k^k \sum_{i_1=1}^1 \sum_{i_2>i_1}^2 \cdots \sum_{i_k>i_{k-1}}^k \rho_{i_1}^2 \rho_{i_2}^2 \cdots \rho_{i_k}^2 (t_{i_1} - \delta - t_0)^{2H-1} \\
 & \quad \left. \times e^{-r(t_{i_1}-\delta-t_0)} e^{-r(t-t_{i_1}-(k-1)\delta)} \right\}. \tag{52}
 \end{aligned}$$

By  $(H_4)$  and  $(H_5)$  we get

$$\lim_{k \rightarrow \infty} k \sum_{j=1}^k V_j = 0. \tag{53}$$

By (43), (51), and (53) we get

$$\lim_{t \rightarrow \infty} T_3 = 0. \tag{54}$$

By (36), (37), (40), and (54) we finally get  $\lim_{t \rightarrow \infty} \mathbb{E}\|x(t)\|^2 = 0$  and complete the proof of the theorem.  $\square$

**Remark 4.2** The delay in impulses of Eq. (25) are assume to satisfy  $0 \leq \delta < \underline{\tau} = \inf_{k \in \mathbb{N}}\{t_k - t_{k-1}\}$ , which means that impulsive instances depend on information of the prior interval in the history. In other words, the information of the history of the prior interval  $(t_{k-1}, t_k]$  is needed to decide the impulsive instances at  $t_k$ . In fact, we can assume that the impulsive instance states are determined by the information of the prior several intervals in the history, and the method of Theorem 4.1 is also applicable, although the expression of the states of the system is very complex.

**Remark 4.3**  $\Delta x(t_k) = \rho_k x(t_k^- - \delta)$  in Eq. (25) describe that impulsive moments are linear in structure and dependent on delayed state variables. By the method of Theorem 3.1 and Theorem 4.1 we can get corresponding stability conditions when the impulsive moments are described as  $\Delta x(t_k) = I_k(x(t_k^-)) + \rho_k x(t_k^- - \delta)$ . Here we omit the details.

**Remark 4.4** In [16] and [19], the authors investigated the exponential stability of impulsive stochastic differential equations driven by an fBm. In [16], the impulses were not considered in the systems. In [19], the impulses were considered in the systems, but they only depended on the current states of the systems. However, in our paper, the impulses considered in the systems depend not only on the current states but also on the historical states of the systems.

### 5 Examples

In this section, we provide two examples to illustrate the obtained results.

**Example 1** Let us consider the following impulsive stochastic differential equation driven by an fBm:

$$\begin{cases} dZ(t, x) = \left[ \frac{\partial^2 Z(t, x)}{\partial x^2} + a(t) \frac{Z(t - \frac{\alpha}{2}(1 + \sin t))}{1 + [Z(t - \frac{\alpha}{2}(1 + \sin t))]^2} \right] dt + e^{-\pi^2 t} dW^H(t), \\ t \geq t_0, t \neq t_k, 0 \leq x \leq \pi, \\ \Delta Z(t_k, x) = \frac{b}{k^2} Z(t_k^-), \quad k \in \mathbb{N}, \\ Z(t, 0) = Z(t, \pi) = 0, \\ Z(t_0 + \theta, x) = \phi(\theta, x), \quad \theta \in [-\alpha, 0], \end{cases} \tag{55}$$

where  $a(t) : [t_0, \infty) \rightarrow \mathbb{R}^+$  is a continuous bounded function for  $t \geq t_0$ . Denote by  $a_0$  the smallest upper bound of the function  $a(t)$  and let  $b > 0$  be a constant. Let  $X =$

$L^2[0, \pi]$  and  $Y = L^2[0, \pi]$ , and let the operator  $A : X \rightarrow X$  be defined by  $Ax = x''$  with domain  $\mathfrak{D}(A) = \{x \in X : x, x' \text{ are absolutely continuous, } x'' \in X, x(0) = x(\pi) = 0\}$ . Then,  $Ax = \sum_{n=1}^{\infty} n^2 \langle x, x_n \rangle x_n$ ,  $x \in \mathfrak{D}(A)$ , where  $x_n(t) = \sqrt{\frac{2}{\pi}} \sin(nt)$ ,  $n = 1, 2, \dots$ , is the orthogonal set of eigenvectors in  $A$ . It is easy to see that  $A$  is the infinitesimal generator of an analytic semigroup  $\{S(t)\}_{t \geq 0}$  in  $X$ . Furthermore, we have  $S(t)x = \sum_{n=1}^{\infty} e^{-n^2 t} \langle x, x_n \rangle x_n$  for all  $x \in X, t > 0$ . We have seen that  $\|S(t)\| \leq e^{-\pi^2 t}$ . To define the operator  $Q : Y \rightarrow X$ , we take a sequence  $\{\lambda_n\}_{n \geq 1} \subset \mathbb{R}^+$  and set  $Q\omega_n = \lambda_n \omega_n$ , where  $\omega_n$  is a complete orthonormal basis in  $Y$ . Also, assume that  $\text{tr}(Q) = \sum_{n=1}^{\infty} \lambda_n^{1/2} < \infty$ . Now we define the process  $W^H(t)$  by  $W^H(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} \omega_n W_n^H(t)$ , where  $H \in (\frac{1}{2}, 1)$ , and  $\{W_n^H(t)\}$  is a sequence of independent two-sided one-dimensional fBms.

Now, let us verify the conditions of Theorem 3.1. Since  $\|S(t)\| \leq e^{-\pi^2 t}$ , we can choose  $M = 1$  and  $r = \pi^2$  in  $(H_1)$  and  $\sigma(t) = \frac{\alpha}{2}(1 + \sin t) \in [0, \alpha]$ . The function  $F$  has the form

$$F(t, Z) = a(t) \frac{Z}{1 + Z^2}.$$

So, we can choose  $R_1 = a_0$  in  $(H_2)$  and  $d_k = b \frac{1}{k^2}$  in  $(H_3)$ . Then  $\sum_{k=1}^{\infty} d_k = b \frac{\pi^2}{6} < \infty$ . So, conditions  $(H_2)$  and  $(H_3)$  hold. In Eq. (55), the function  $G(t) = e^{-\pi^2 t}$ , and  $(H_4)$  holds. If  $a_0$  and  $b$  satisfy the inequality

$$\frac{3M^2 R_1^2}{r^2} + 3M^2 \left( \sum_{i=1}^{\infty} d_i \right)^2 = \frac{3a_0^2}{\pi^4} + \frac{3b\pi^2}{6} < 1,$$

then all the assumptions in Theorem 3.1 are satisfied. By Theorem 3.1 the mild solution of Eq. (55) is exponentially stable in mean square.

**Example 2** Let us consider the following stochastic differential equation driven by an fBm with delayed impulses:

$$\begin{cases} dZ(t, x) = \left[ \frac{\partial^2 Z(t, x)}{\partial x^2} + a(t) \frac{Z(t - \frac{\alpha}{2}(1 + \sin t))}{1 + [Z(t - \frac{\alpha}{2}(1 + \sin t))]^2} \right] dt + e^{-\pi^2 t} dW^H(t), \\ t \geq t_0, t \neq t_k, 0 \leq x \leq \pi, \\ \Delta Z(t_k, x) = \frac{b}{k^2} Z(t_k^- - \delta), \quad k \in \mathbb{N}, \\ Z(t, 0) = Z(t, \pi) = 0, \\ Z(t_0 + \theta, x) = \phi(\theta, x), \quad \theta \in [-\alpha, 0], \end{cases} \tag{56}$$

where  $a(t), b, A, F(t, y)$  are the same as in Eq. (55). The conditions  $(H_1), (H_2)$ , and  $(H_4)$  hold, so we only need to verify condition  $(H_5)$ . We assume that  $t_k - t_{k+1} = 1, k \in \mathbb{N}$ , and  $\delta = 0.9$ . Then

$$\Sigma \leq k \left( C_k^1 \sum_{i_1=1}^k \frac{b^2}{k^4} + C_k^2 \sum_{i_1=1}^{k-1} \sum_{i_2 > i_1}^k \frac{b^2}{k^4} + \dots + C_k^k \sum_{i_1=1}^1 \sum_{i_2 > i_1}^2 \dots \sum_{i_k > i_{k-1}}^k \frac{b^2}{k^4} \right), \tag{57}$$

and by binomial expansion we get

$$\Sigma \leq k \sum_{j=1}^k \left( C_k^j \frac{1}{k^{2j}} \right)^2 \leq k \left( \sum_{j=1}^k C_k^j \frac{1}{k^{2j}} \right)^2 = k \left( \left( 1 + \frac{1}{k^2} \right)^k - 1 \right)^2.$$

From an important limit we get that  $\lim_{k \rightarrow \infty} \Sigma = 0 < \infty$ . So all the conditions of Theorem 4.1 are satisfied, and the mild solution of Eq. (25) is asymptotically stable in mean square.

## 6 Conclusion

In this article, we obtained two stability results for stochastic differential equations driven by an fBm (Hurst parameter  $H \in (\frac{1}{2}, 1)$ ) with varying delays and impulses. The mean-square exponential stability conditions for mild solutions under the condition that the impulses only depend on the current states of the systems, and the mean-square asymptotic stability conditions for mild solutions under the condition that the impulses depend not only on the current states but also on the history states of the systems. Two examples were given to illustrate the theorems.

### Competing interests

The authors declare that they have no competing interests.

### Authors' contributions

For the first time, we attempted to investigate the stability of stochastic differential equations driven by an fBm with varying delays and impulses, where the stochastic disturbance is an fBm rather than a Brownian motion, and the impulsive transients depend not only on the current states but also on the historical states of the systems.

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