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Application of measures of noncompactness to the infinite system of second-order differential equations in ℓ_p spaces

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Abstract
In this article, we use the technique based upon measures of noncompactness in conjunction with a Darbo-type fixed point theorem with a view to studying the existence of solutions of infinite systems of second-order differential equations in the Banach sequence space ℓ_p. An illustrative example is also given in support of our existence result.

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1 Introduction

Measures of noncompactness endow helpful information, which is extensively used in the theory of integral and integro-differential equations. Besides, it is very helpful in the study of optimization, differential equations, functional equations, fixed point theory, *etc.* Some of the well-known measures of noncompactness are the Kuratowski measure (α), the Hausdorff measure (χ), and the Istrăţescu measure (β), which were introduced by Kuratowski [1], Goldenštein *et al.* [2] (also studied by Goldenštein and Markus [3]), and Istrăţescu [4], respectively. Darbo [5] was the first who presented a fixed point theorem by using the idea of Kuratowski measures of noncompactness, the function α, which is popularly called the Darbo fixed point theorem. This fixed point theorem generalized two very important and famous fixed point theorems, namely, (i) the classical Schauder fixed point theorem and (ii) special variant of the Banach fixed point theorem. The Darbo fixed point theorem has been generalized in many different directions. In fact, there is a vast amount of literature dealing with extensions and/or generalizations of this remarkable theorem. Recently, Aghajani *et al.* [6] presented a generalization of the Darbo fixed point theorem and used it to investigate the existence result concerning a general system of nonlinear integral equations. For some other recent works related to these concepts, we refer the interested reader to (for example) [7–12], and [13]. We also refer to the recent work by

Srivastava *et al.* [14] for some applications of fixed point theorems to *fractional* differential equations (for details, see [15]).

Mursaleen and Mohiuddine [16] earlier reported the existence theorems in the classical sequence space ℓ_p for an infinite system of differential equations. On the other hand, existence theorems for infinite systems of linear equations in ℓ_1 and ℓ_p were given by Alotaibi *et al.* [17]. Our main object in this sequel is to determine sufficient conditions for the solvability of an infinite system of second-order differential equations. We use the Dardo-type fixed point theorem given by Aghajani and Pourhadi [18] for a new type of condensing operator and the method based upon the measures of noncompactness to establish the existence theorem for the above-mentioned infinite systems in the Banach sequence space ℓ_p with $1 \leqq p < \infty$. Our existence theorem is an extension of those obtained by Aghajani and Pourhadi [18] in the sequence space ℓ_1.

2 Preliminaries and notation

Let ω denote the space of all complex sequences $x = (x_j)_{j=0}^{\infty}$ or, simply, $x = (x_j)$. Any vector subspace of ω is called a sequence space. We use the standard notation ℓ_∞, c, and c_0 to denote the set of all bounded, convergent, and null sequences of real numbers, respectively. By $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{C}$ we denote the sets of natural, real, and complex numbers, respectively. We recall that the notion of little o is used for comparison of growth of two arbitrary sequences $x_j$and y_j and is defined by

$$x_j = o(y_j) \quad \Longleftrightarrow \quad \lim_{j\to\infty} \frac{x_j}{y_j} = 0 \quad (y_j \neq 0).$$

We introduce the space ℓ_p of all absolutely p-summable series as follows:

$$\ell_p = \left\{ x \in \omega : \sum_{j=0}^{\infty} |x_j|^p < \infty \right\} \quad (1 \leqq p < \infty).$$

Clearly, ℓ_p is a Banach space with norm

$$\|x\|_p = \left(\sum_{j=0}^{\infty} |x_j|^p \right)^{1/p} \quad (1 \leqq p < \infty).$$

By $e^{(j)}$ we denote the sequence with jth term 1 and all other terms zero ($j \in \mathbb{N}$); we also denote $e = (1, 1, 1, \ldots)$. For any sequence $x = (x_j)$, let its n-section be given by

$$x^{[n]} = \sum_{j=0}^{n} x_j e^{(j)}.$$

A sequence space X is called a *BK space* if it is a Banach space with continuous coordinates $p_k : X \to \mathbb{C}$ and $p_k(x) = x_k$ for all $x = (x_j) \in X$ and $k \in \mathbb{N}$. A *BK* space $X \supset \psi$ (that is, the set of all finite sequences that terminate in zeros) is said to have *AK* if every sequence $x = (x_j) \in X$ has a unique representation

$$x = \sum_{j=0}^{\infty} x_j e^{(j)}.$$

We denote by $\mathcal{M}_X$, (X,d), and $B(x,r)$, respectively, the class of all bounded subsets of X, the metric space, and the open ball with center at x and radius r, that is,

$$B(x,r) = \{y \in X : d(x,y) < r\}.$$

Let $F \in \mathcal{M}_X$. Then the *Hausdorff measure of noncompactness* of F is defined by

$$\chi(F) = \inf\left\{\epsilon > 0 : F \subset \bigcup_{j=1}^{n} B(x_j, r_j), x_j \in X, r_j < \epsilon \ (1 \leqq j \leqq n; n \in \mathbb{N})\right\}.$$

The function $\chi : \mathcal{M}_X \to [0,\infty)$ is called the *Hausdorff measure of noncompactness*.

We now recall some basic properties of the Hausdorff measure of noncompactness. Let F, F_1, and F_2 be bounded subsets of the metric space (X,d). Then

(i) $\chi(F) = 0$ if and only if F is totally bounded;
(ii) $\chi(F) = \chi(\bar{F})$, where $\bar{F}$ denotes the closure of F;
(iii) $F_1 \subset F_2$ implies that $\chi(F_1) \leqq \chi(F_2)$;
(iv) $\chi(F_1 \cup F_2) = \max\{\chi(F_1), \chi(F_2)\}$;
(v) $\chi(F_1 \cap F_2) = \min\{\chi(F_1), \chi(F_2)\}$.

In the case of a normed space $(X, \|\cdot\|)$, the function χ has some additional properties connected with the linear structure. For example, we have

$$\chi(F_1 + F_2) \leqq \chi(F_1) + \chi(F_2),$$
$$\chi(F + x) = \chi(F) \quad \text{for all } x \in X,$$
$$\chi(\alpha F) = |\alpha|\chi(F) \quad \text{for all } \alpha \in \mathbb{C}.$$

Theorem 1 (see [19]) *Let X be a BK space with a Schauder basis $(b_j)_{j=0}^{\infty}$ and $F \in \mathcal{M}_X$. Also, let $P_j : X \to X$ $(j \in \mathbb{N})$ be the projector onto the linear span of $\{e^{(1)}, e^{(2)}, \ldots, e^{(j)}\}$. Then*

$$\frac{1}{a}\limsup_{j\to\infty}\left\{\sup_{x\in F}\big\|(I-P_j)(x)\big\|\right\} \leqq \chi(F) \leqq \limsup_{j\to\infty}\left\{\sup_{x\in F}\big\|(I-P_j)(x)\big\|\right\}, \tag{1}$$

where I is the identity operator on X, and

$$a = \limsup_{j\to\infty} \|I - P_j\|.$$

It is known that ℓ_p $(1 \leqq p < \infty)$ is a *BK* space with *AK* with respect to its usual norm $\|\cdot\|_p$. Additionally, $\{e^{(1)}, e^{(2)}, \ldots\}$ as depicted from a Schauder basis for ℓ_p, in view of (1), the following result is derivable by using Theorem 1 (see [19] and [20]).

Theorem 2 *Let F be a bounded subset of $X = \ell_p$. Then*

$$\chi(F) = \lim_{k\to\infty} \sup_{x\in F}\left\{\left(\sum_{j\geqq k} |x_j|^p\right)^{1/p}\right\}. \tag{2}$$

The following generalization of the Darbo fixed point theorem was established by Aghajani *et al.* [21] by using a control function.

Theorem 3 *Let $\mathcal{C}$ be a nonempty, bounded, closed, and convex subset of a Banach space X, and let $T:\mathcal{C}\to\mathcal{C}$ be a continuous function satisfying the inequality*

$$\mu\bigl(T(F)\bigr) \leqq \varphi\bigl(\mu(F)\bigr) \tag{3}$$

for each $F\subset\mathcal{C}$, where μ is an arbitrary measure of noncompactness, and $\varphi:[0,\infty)\to[0,\infty)$ is an increasing (not necessarily continuous) function with

$$\lim_{n\to\infty}\varphi^n(t)=0.$$

Then T has at least one fixed point in the set $\mathcal{C}$.

The notion of (α,ϕ,φ)-μ-condensing operators and α-admissible operators were recently demonstrated by Aghajani and Pourhadi [18] by considering φ and ϕ as follows. We use the notation Ψ to denote the functions $\varphi:[0,+\infty)\to[0,\infty)$ like

$$\liminf_{n\to\infty}\varphi(a_n)=0,$$

conferred that

$$\lim_{n\to\infty}a_n=0,$$

where $(a_n)_{n\in\mathbb{N}}$ is a nonnegative sequence. For $\varphi\in\Psi$, let us consider a function $\phi:[0,+\infty)\to[0,+\infty)$ that satisfies the following conditions:

(i) ϕ is a lower semi-continuous function with $\phi(t)=0$ if and only if $t=0$;
(ii) $\liminf_{n\to\infty}\varphi(a_n)<\phi(a)$, provided that $\lim_{n\to\infty}\{a_n\}=a$.

We use the notation Φ_φ to denote the class of all such functions. Throughout this paper, by $\operatorname{Conv}F$ we denote the *convex hull* of $F\subset X$.

Let $T:W\subseteq X\to X$ is an arbitrary mapping. Further, we state that T is (α,φ,ϕ)-μ-condensing if the functions $\alpha:\mathcal{M}_X\to[0,+\infty)$, $\varphi\in\Psi$, and $\phi=\Phi_\varphi$ are such that

$$\alpha(F)\phi\bigl(\mu(TF)\bigr)=\varphi\bigl(\mu(F)\bigr)\quad(F\in W),$$

where both F and its image TF belong to $\mathcal{M}_X$.

Let T and α be given mappings as before. Then T is *α-admissible* if

$$\alpha(F)\geqq 1\quad\Longrightarrow\quad\alpha(\operatorname{Conv}TF)\geqq 1\quad(F\in W;F,TF\in\mathcal{M}_X).$$

Remark 1 If T follows the Darbo condition with regard to a measure μ and a constant $k\in[0,1)$, that is, if

$$\mu(TF)=k\mu(F)\quad(F\in W;F,TF\in\mathcal{M}_X),$$

then T is an (α,φ,ϕ)-μ-condensing operator, where $\alpha(F)=1$ for any set $F\in W$ such that $F\in\mathcal{M}_X$, ϕ is the identity mapping, and the function $\varphi(t)=kt$, $t\geqq 0$. In this regard, T is a μ-contraction.

Aghajani and Pourhadi [18] also established the following fixed point theorem by using α-admissible and (α,ϕ,φ)-μ-condensing operators.

Theorem 4 *Let $\mathcal{C} \in \mathcal{M}_X$ be a closed convex subset of a Banach space X, and let $T : \mathcal{C} \to \mathcal{C}$ be a continuous (α,φ,ϕ)-μ-condensing operator, where μ is an arbitrary measure of noncompactness. Moreover, T is α-admissible, and $\alpha(\mathcal{C}) \geqq 1$. Then T has at least one fixed point that pertains to* $\ker\mu$.

3 Infinite systems of second-order differential equations

Let us consider the following infinite system of second-order differential equations:

$$-\frac{d^2x_i}{dt^2} = f_i(t,x_0,x_1,x_2,\dots) \tag{4}$$

with the initial conditions given by

$$x_i(0) = x_i(T) = 0 \quad \bigl(i \in \mathbb{N}_0 := \mathbb{N} \cup \{0\} = \{0,1,2,\dots\}; t \in I = [0,T]\bigr).$$

The space of all continuous real functions on I with values in $\mathbb{R}$ and the space of all functions with two continuous derivatives on the interval I are shown by the standard notations $C(I,\mathbb{R})$ and $C^2(I,\mathbb{R})$, respectively. It is evident that $x \in C^2(I,\mathbb{R})$ is a solution of (4) if and only if $x \in C(I,\mathbb{R})$ is a solution of the following system of integral equations:

$$x_i(t) = \int_0^T G(t,s) f_i\bigl(s,x_0(s),x_1(s),x_2(s),\dots\bigr)\,ds \quad (t \in I), \tag{5}$$

where

$$f_i(t,x_0,x_1,x_2,\dots) \in C(I,\mathbb{R}) \quad (i \in \mathbb{N}_0),$$

and the Green function $G(t,s)$ associated with (4) is given by

$$G(t,s) = \begin{cases} \frac{t}{T}(T-s) & (0 \leqq t \leqq s \leqq T), \\ \frac{s}{T}(T-t) & (0 \leqq s \leqq t \leqq T). \end{cases} \tag{6}$$

For more details of green functions, we refer to [22]. We can rewrite (5) with the help of (6) as follows:

$$\begin{aligned} x_i(t) &= \int_0^t \frac{s}{T}(T-t) f_i\bigl(s,x_0(s),x_1(s),x_2(s),\dots\bigr)\,ds \\ &\quad + \int_t^T \frac{t}{T}(T-s) f_i\bigl(s,x_0(s),x_1(s),x_2(s),\dots\bigr)\,ds. \end{aligned} \tag{7}$$

Upon differentiating both sides of (7) with respect to t, we get

$$\begin{aligned} \frac{d}{dt}\bigl\{x_i(t)\bigr\} &= -\frac{1}{T}\int_0^t s f_i\bigl(s,x_0(s),x_1(s),x_2(s),\dots\bigr)\,ds \\ &\quad + \frac{1}{T}\int_t^T (T-s) f_i\bigl(s,x_0(s),x_1(s),x_2(s),\dots\bigr)\,ds. \end{aligned} \tag{8}$$

Again, by differentiating both sides of (8) with respect to t we obtain

$$\begin{aligned}\frac{d^2}{dt^2}\{x_i(t)\} &= -\frac{1}{T}tf_i\bigl(t, x_0(t), x_1(t), x_2(t), \dots\bigr) \\ &\quad + \frac{1}{T}(t-T)f_i\bigl(t, x_0(t), x_1(t), x_2(t), \dots\bigr) \\ &= -f_i\bigl(s, x_0(s), x_1(s), x_2(s), \dots\bigr).\end{aligned}$$

We now investigate the existence result concerning the second-order differential equations for the infinite system given by (4) in the Banach sequence space ℓ_p ($1 \leqq p < \infty$) with the help of measures of noncompactness. For this investigation, we consider the following hypotheses:

(i) The functions f_i ($i \in \mathbb{N}_0$) are defined on $I \times \mathbb{R}^\infty$ and take real values. Furthermore, the operator f is shown on the space $I \times \ell_p$ as

$$(t, x) \mapsto (fx)(t) = \bigl(f_1(t, x), f_2(t, x), f_3(t, x), \dots\bigr),$$

which represent the space of maps from $I \times \ell_p$ into ℓ_p; it is found that the class of all functions $\{(fx)(t)\}_{t \in I}$ is equicontinuous at every point of ℓ_p.

(ii) There are a nonnegative mapping $g : I \to \mathbb{R}_+$, a function $h : I \times \ell_p \to \mathbb{R}$, and a super-additive mapping $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, that is,

$$\varphi(s + t) \geqq \varphi(s) + \varphi(t)$$

for all $s, t \in \mathbb{R}_+$, such that

$$h(t, x) \geqq 0 \quad \Longrightarrow \quad \bigl|f_i(t, x_0, x_1, x_2, \dots)\bigr|^p \leqq g_i(t)\varphi\bigl(|x_i|^p\bigr), \tag{9}$$

where $x = (x_i) \in \ell_p$, $t \in I$, and $i \geqq k$ for some $k \in \mathbb{N}_0$.

(iii) The function $G(t, s)g(s)$ is integrable on I and such that

$$g(s) = \limsup_{i \to \infty}\bigl\{g_i(s)\bigr\} \quad (i \in \mathbb{N}_0)$$

for any fixed element $t \in I$. Additionally, if a nonnegative sequence $(y_n)_{n \in \mathbb{N}}$ converges to some number ℓ, then

$$\liminf_{n \to \infty} \varphi(y_n) < \frac{\ell}{C} \tag{10}$$

such that

$$\sup_{t \in I}\left\{\int_0^T \bigl|G(t, s)\bigr|^p g(s)\, ds\right\} \leqq C$$

for some positive constant C.

(iv) There is a function x such that

$$h\bigl(t, x(t)\bigr) \geqq 0 \quad (\forall t \in I). \tag{11}$$

In addition, for $t \in I$, we have

$$h\big(t, y(t)\big) \geqq 0 \quad \Longrightarrow \quad h\biggl(t, \biggl(\int_0^T G(t,s) f_i\big(s, y_0(s), y_1(s), y_2(s), \ldots\big)\, ds\biggr)\biggr) \geqq 0 \quad (12)$$

for all $y(t) \in \ell_p$.

We are now prepared to formulate our main result.

Theorem 5 *Under assumptions* (i) *to* (iv), *the infinite system of second-order differential equations* (4) *has at least one solution* $x(t) = (x_i(t))$ *such that* $x(t) \in \ell_p$ *for each* $t \in I$.

Proof Let us consider the operator $\mathcal{F} = (\mathcal{F}_i)$ defined on $C(I, \ell_p)$ by

$$(\mathcal{F}x)(t) = \big((\mathcal{F}_i x)(t)\big) = \biggl(\int_0^T G(t,s) f_i\big(s, x_0(t), x_1(t), x_2(t), \ldots\big)\biggr),$$

where $x(t) = (x_i(t)) \in \ell_p$, $x_i \in C(I, \mathbb{R})$, and $t \in I$. Taking into account assumption (i), it is clearly seen that $\mathcal{F}$ is continuous on $C(I, \ell_p)$. Obviously, the function $\mathcal{F}x$ is also continuous, and $(\mathcal{F}x)(t) \in \ell_p$ if $x(t) = (x_i(t)) \in \ell_p$. In view of the fact that φ is superadditive, together with Eq. (9) and hypothesis (iii), it follows that

$$\begin{aligned}
\big\|(\mathcal{F}x)(t)\big\|_p^p &= \sum_{i=0}^{\infty} \biggl| \int_0^T G(t,s) f_i\big(s, x_0(s), x_1(s), x_2(s), \ldots\big)\, ds \biggr|^p \\
&\leqq \sum_{i=0}^{\infty} \biggl| \biggl(\int_0^T \big(G(t,s) f_i\big(s, x_0(s), x_1(s), x_2(s), \ldots\big)\big)^p\, ds\biggr)^{1/p} \biggl(\int_0^T ds\biggr)^{1/p'} \biggr|^p \\
&\leqq T^{p/p'} \sum_{i=0}^{\infty} \int_0^T \big|G(t,s)\big|^p \big|f_i\big(s, x_0(s), x_1(s), x_2(s), \ldots\big)\big|^p\, ds \\
&\leqq \frac{T^{p/p'} T^p}{4^p} \int_0^T \big\|(fx)(s)\big\|_p^p\, ds < \infty,
\end{aligned}$$

where $p > 1$ and $1/p + 1/p' = 1$. We now consider the operator $\mathcal{F} = (\mathcal{F}_i)$ defined on a nonempty bounded set $Q \in \mathcal{M}_{\ell_p}$ (where $\mathcal{M}_{\ell_p}$ denotes the family of all nonempty bounded subsets of ℓ_p) including the functions $x(t) = (x_i(t)) \in \ell_p$ with

$$h\big(t, x(t)\big) \geqq 0$$

for any fixed $t \in I$. Then, clearly, Eq. (2) yields

$$\begin{aligned}
\chi(\mathcal{F}Q) &= \lim_{n\to\infty} \sup_{x(t)\in Q} \biggl\{ \biggl(\sum_{j\geqq n} \biggl| \int_0^T G(t,s) f_j\big(s, x_0(s), x_1(s), x_2(s), \ldots\big)\, ds \biggr|^p \biggr)^{1/p} \biggr\} \\
&\leqq \lim_{n\to\infty} \sup_{x(t)\in Q} \biggl\{ \biggl(T^{p/p'} \sum_{j\geqq n} \int_0^T \big|G(t,s)\big|^p \big|f_j\big(s, x_0(s), x_1(s), x_2(s), \ldots\big)\big|^p\, ds \biggr)^{1/p} \biggr\} \\
&\leqq \lim_{k\to\infty} \sup_{x(t)\in Q} \biggl\{ \biggl(T^{p/p'} \sum_{j\geqq k} \int_0^T \big|G(t,s)\big|^p g_j(s) \varphi\big(\big|x_j(s)\big|^p\big)\, ds \biggr)^{1/p} \biggr\}
\end{aligned}$$

$$\begin{aligned}
&\leqq \lim_{k\to\infty} \sup_{x\in Q}\left\{\left(T^{p/p'}\sum_{j\geqq k} C\varphi\left(|x_j|^p\right)\right)^{1/p}\right\} \\
&\leqq C' \lim_{k\to\infty} \sup_{x\in Q}\left\{\varphi\left(\sum_{j\geqq k}|x_j|^p\right)^{1/p}\right\} \\
&= C' \lim_{k\to\infty}\left\{\varphi\left(\sup_{x\in Q}\left\{\left(\sum_{j\geqq k}|x_j|^p\right)^{1/p}\right\}\right)\right\} \\
&\leqq C'\varphi\bigl(\chi(Q)\bigr).
\end{aligned}$$

This shows that

$$\alpha(Q)\phi\bigl(\chi(\mathcal{F}Q)\bigr) \leqq \varphi\bigl(\chi(Q)\bigr),$$

where $\alpha : \mathcal{M}_{\ell_p} \to [0,\infty)$ is the mapping defined by

$$\alpha(Q) = \begin{cases} 1 & (h(t,x(t)) \geqq 0; x \in Q; t \in I), \\ 0 & (\text{otherwise}), \end{cases}$$

and

$$\phi(b) = \frac{b}{C'} \quad (b \in \mathbb{R}_+).$$

Obviously, $\phi \in \Phi_\varphi$, and it satisfies (10). Interestingly, by hypothesis (iv) we conclude that the operator $\mathcal{F}$ is α-admissible and satisfies all of the conditions of Theorem 5. Therefore, $\mathcal{F}$ has at least one fixed point $x = x(t)$ such that $x(t) \in \ell_p$ for all $t \in I$. Hereof, the function $x = x(t)$ is a solution of the infinite system (4). □

Remark 2 Our existence theorem (Theorem 5) is more general than that proved earlier by Aghajani and Pourhadi [18]. Indeed, if we set $p = 1$ in the sequence space ℓ_p, then it reduces to the sequence space ℓ_1, and so Theorem 4.1 of Aghajani and Pourhadi [18] is a particular case of our Theorem 5.

We now present an interesting illustrative example in support of our result.

Example Consider the following second-order differential equations:

$$-\frac{d^2}{dt^2}\{x_q\} = \frac{t(T-t)e^{-qt}}{(q+1)^4} + \sum_{r=q}^{\infty} \frac{x_r(t)\sqrt{t}}{(1+q^2)(r+1)^2}, \tag{13}$$

where $q \in \mathbb{N}_0$ and $t \in I = [0,T]$ $(0 < T < 2\sqrt{2})$. Obviously, the functions $a_{qr}(t)$ given by

$$a_{qr}(t) = \frac{\sqrt{t}}{(1+q^2)(r+1)^2}$$

are continuous, and the series

$$\sum_{r=q}^{\infty}\bigl|a_{qr}(t)\bigr|^p$$

is absolutely uniformly continuous on I. Since

$$a_q(t) := \sum_{r=q}^{\infty} \left|a_{qr}(t)\right|^p$$

is uniformly bounded on I, for any $t \in I$ and $q \in \mathbb{N}_0$, we consider

$$B = \sup\left\{a_q(t)\right\} < \infty. \tag{14}$$

We note that, if $x(t) = (x_q(t)) \in \ell_p$, then

$$\begin{aligned}(fx)(t) &= \left(f_q(t, x_0, x_1, x_2, \ldots)\right) \\ &= \left(\frac{t(T-t)e^{-qt}}{(q+1)^4} + \sum_{r=q}^{\infty} \frac{x_r(t)\sqrt{t}}{(1+q^2)(r+1)^2}\right) \in \ell_p \end{aligned} \tag{15}$$

because the norm

$$\begin{aligned}\left\|(fx)(t)\right\|_p^p \leqq{}& 2^p \sum_{q=0}^{\infty} \left|\frac{t(T-t)e^{-qt}}{(q+1)^4}\right|^p \\ &+ 2^p \sum_{q=0}^{\infty} \left|\sum_{r=q}^{\infty} \frac{x_r(t)\sqrt{t}}{(1+q^2)(r+1)^2}\right|^p \end{aligned}$$

is finite. We have to demonstrate that the operator $(fx)(t) = ((f_q x)(t))$ is uniformly continuous on ℓ_p. For this, we suppose to prove that the sequence $(f_q(x))$ is equicontinuous. Let $\epsilon > 0$ be given, and $x(t) = (x_q(t)) \in \ell_p$. By considering

$$x'(t) = \left(x'_q(t)\right) \in \ell_p$$

with

$$\left\|x(t) - x'(t)\right\|_p^p \leqq \delta(\epsilon) = \epsilon B^{-1}$$

it follows from (14) that, for any fixed q,

$$\begin{aligned}\left|(f_q x)(t) - \left(f_q x'\right)(t)\right|^p &= \left|\sum_{r=q}^{\infty} \frac{\left(x_r(t) - x'_r(t)\right)\sqrt{t}}{(1+q^2)(r+1)^2}\right|^p \\ &\leqq \left|\left(\sum_{r=q}^{\infty}\left(x_r(t) - x'_r(t)\right)^p\right)^{1/p} \left(\sum_{r=q}^{\infty}\left(\frac{\sqrt{t}}{(1+q^2)(r+1)^2}\right)^{p'}\right)^{1/p'}\right|^p \\ &\leqq \sum_{r=q}^{\infty}\left|x_r(t) - x'_r(t)\right|^p \sum_{r=q}^{\infty}\left|\frac{\sqrt{t}}{(1+q^2)(r+1)^2}\right|^p \\ &\leqq B\left\|x(t) - x'(t)\right\|_p^p \leqq B\epsilon B^{-1} = \epsilon,\end{aligned}$$

where $p > 1$ and $1/p + 1/p' = 1$, which yields the continuity, as desired. Hence hypothesis (i) is satisfied. In order to verify hypotheses from (ii) to (iv), we reckon a function $h : I \times \ell_p \to \mathbb{R}$ that occurs on nonnegative values if and only if

$$x(t) = \bigl(x_q(t)\bigr) \in \ell_p,$$

where $(x_q(t))$ is a nonincreasing sequence in $\mathbb{R}_+$ with

$$x(0) = 0 = x(T).$$

We thus find that

$$\frac{t(T-t)e^{-qt}}{(q+1)^4} = o\bigl(x_q(t)\bigr) \tag{16}$$

uniformly with regard to $t \in (0, T)$. It is convenient to observe that

$$\bigl\{x \in \ell_p : h\bigl(t, x(t)\bigr) \geqq 0 \ (t \in I)\bigr\} \neq \emptyset.$$

Let

$$h\bigl(t, x(t)\bigr) \geqq 0.$$

Now, from the data

$$x(0) = x(T) = 0$$

and (16) it follows that

$$\frac{t(T-t)e^{-qt}}{(q+1)^4} \leqq x_q(t) \tag{17}$$

for all $q > r$, $r \in \mathbb{N}_0$, and $t \in I$. Thus, taking into account (15) and (17), we find that, for all $q > r$ and $t \in I$,

$$\begin{aligned}
\bigl|(f_q x)(t)\bigr|^p &\leqq \left| x_q(t) + \frac{\sqrt{t}}{1+q^2} \sum_{k=q}^{\infty} \frac{x_k(t)}{(k+1)^2} \right|^p \\
&\leqq 2^p \left\{ \bigl|x_q(t)\bigr|^p + \frac{t^{p/2}}{(1+q^2)^p} \left| \sum_{k=q}^{\infty} \frac{x_k(t)}{(k+1)^2} \right|^p \right\} \\
&\leqq 2^p \left\{ \bigl|x_q(t)\bigr|^p + \frac{t^{p/2}}{(1+q^2)^p} \left| \left(\sum_{k=q}^{\infty} \bigl(x_k(t)\bigr)^p \right)^{1/p} \left(\sum_{k=q}^{\infty} \left(\frac{1}{(k+1)^2} \right)^{p'} \right)^{1/p'} \right|^p \right\} \\
&\leqq 2^p \left\{ \bigl|x_q(t)\bigr|^p + \frac{\pi^{2p} t^{p/2}}{6^p (1+q^2)^p} \sum_{k=q}^{\infty} \bigl|x_k(t)\bigr|^p \right\} \\
&\leqq 2^p \left(1 + \frac{\pi^{2p} t^{p/2}}{6^p (1+q^2)^p} \right) \bigl|x_q(t)\bigr|^p,
\end{aligned}$$

which yields

$$\left|(f_q x)(t)\right|^p \leqq g_q(t)\left|x_q(t)\right|^p,$$

where

$$g_q(t) = 2^p\left(1 + \frac{\pi^{2p} t^{p/2}}{6^p(1+q^2)^p}\right).$$

Since

$$g(t) = \limsup_{q\to\infty}\left\{g_q(t)\right\} = 2^p,$$

we obtain

$$\sup_{t\in I}\left\{\int_0^T \left|G(t,s)\right|^p g(s)\,ds\right\} \leqq \frac{T^{p+1}}{2^p} = C.$$

By considering $\varphi(t)$ as a kind of identity mapping, we conclude that conditions (ii), (iii), and (11) are satisfied. It is now left to show that (12) holds. Indeed, if we assume that

$$h\big(t,x(t)\big) \geqq 0 \quad (t\in I)$$

and

$$x(t) = \big(x_q(t)\big) \in \ell_p,$$

then it follows from the term of h that $(x_q(t))$ is a nonincreasing sequence in $\mathbb{R}_+$. Therefore,

$$\begin{aligned} f_{q+1}\big(t,x_0(t),x_1(t),x_2(t),\ldots\big) &= \frac{t(T-t)e^{-(q+1)t}}{(q+2)^4} + \sum_{r=q+1}^{\infty}\frac{x_r(t)\sqrt{t}}{(1+(q+1)^2)(r+1)^2} \\ &\leqq \frac{t(T-t)e^{-qt}}{(q+1)^4} + \sum_{r=q}^{\infty}\frac{x_r(t)\sqrt{t}}{(1+q^2)(r+1)^2}, \end{aligned}$$

which shows that

$$0 \leqq f_{q+1}\big(t,x_0(t),x_1(t),x_2(t),\ldots\big) \leqq f_q\big(t,x_0(t),x_1(t),x_2(t),\ldots\big)$$

for all $t\in I$ and $q\in\mathbb{N}_0$. Accordingly, we have

$$\begin{aligned} 0 &\leqq \int_0^T G(t,s)f_{q+1}\big(s,x_0(s),x_1(s),x_2(s),\ldots\big)\,ds \\ &\leqq \int_0^T G(t,s)f_q\big(s,x_0(s),x_1(s),x_2(s),\ldots\big)\,ds \end{aligned}$$

for all $t\in I$ and $q\in\mathbb{N}_0$. It only remains to demonstrate that

$$\frac{t(T-t)e^{-qt}}{(q+1)^4} = o\left(\int_0^T G(t,s)f_q\big(s,x_0(s),x_1(s),x_2(s),\ldots\big)\,ds\right) \tag{18}$$

uniformly with respect to $t \in (0, T)$. In order to verify (18), we have to show that

$$\frac{(q+1)^4}{t(T-t)} \int_0^T G(t,s) e^{qt} f_q\big(s, x_0(s), x_1(s), x_2(s), \ldots\big)\, ds \to \infty \quad (q \to \infty) \tag{19}$$

uniformly in $(0, T)$. By straightforward calculation we obtain

$$\begin{aligned}
&\frac{(q+1)^4}{t(T-t)} \int_0^T G(t,s) e^{qt} f_q\big(s, x_0(s), x_1(s), x_2(s), \ldots\big)\, ds \\
&\quad \geqq \frac{(q+1)^4}{tT} \int_0^{t/2} s^2 (T-s) e^{(q(t-s))}\, ds \\
&\quad \geqq \frac{(q+1)^4 e^{qt/2}}{T^2} \left(\frac{2(-\frac{3}{q}+T)e^{qt/2}}{q^3} - \frac{t^2(-\frac{3}{q}+T)}{4q} - \frac{t(-\frac{3}{q}+T)}{q^2} - \frac{2(-\frac{3}{q}+T)}{q^3} \right) \\
&\quad \geqq \frac{(q+1)^4}{T^2} \left(\frac{2(-\frac{3}{q}+T)}{q^3} - \frac{T^2(-\frac{3}{q}+T)}{4q} - \frac{T(-\frac{3}{q}+T)}{q^2} - \frac{2(-\frac{3}{q}+T)}{q^3} \right) \quad \left(q > \frac{3}{T} \right),
\end{aligned}$$

which converges uniformly to zero as $q \to \infty$. This evidently proves (19), so that assumption (iv) is satisfied. Hence, in light of Theorem 5, Eq. (13) has a solution in the space ℓ_p.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

The authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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