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Ren-He's method for solving dropping shock response of nonlinear packaging system

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Abstract

In this paper, Ren-He's method for nonlinear oscillators is adopted to give approximate solutions for the dropping shock response of cubic and cubic-quintic nonlinear equations arising from packaging system. In order to improve the accuracy of the solutions, a novel technique combining Ren-He's method with the energy method is proposed, the maximum values of the displacement response and acceleration response of the system are obtained by the energy method, and the approximate solution is corrected. An analytical expression of the important parameters including the maximum displacement, the maximum acceleration of dropping shock response and the dropping shock duration is obtained. The illustrative examples show that the dropping shock response obtained by this method is very similar to the one by the fourth order Runge-Kutta method. The result provides a new simple and effective method for the dropping shock response of a nonlinear packaging system.

Keywords: Ren-He's method; energy method; displacement maximum value; acceleration maximum value; dropping shock duration

1 Introduction

It is a well known fact that the dynamic model of some engineering problems can be described by nonlinear differential equations. In the field of packaging engineering, due to the nonlinear characteristics of cushion packaging materials, the dynamics model of a packaging system usually have strong nonlinear characteristics for the theoretical analysis process of packaged product damage evaluation. Therefore, it is general difficult to obtain the theoretical analytical solution for this kind of problems. The numerical analysis method such as Runge-Kutta method is often adopted to analyze the dynamics performance and dropping shock damage evaluation of the packaging systems [1–7]. Despite the effectiveness of numerical method, the explicit expression of approximate solutions is still expected to be obtained to conveniently discuss the influence of initial conditions and parameters on the solution.

Finding approximate, and if possible in closed-form, solutions of nonlinear differential equations is the subject of many researchers. Recently, various analytical approaches for solving nonlinear differential equations have been widely applied to analyze engineering problems, such as the variational iteration method (VIM) [8–13], the energy balance method (EBM) [14–16], the homotopy perturbation method (HPM) [17, 18], He's frequency-amplitude formulation (FAF) [16, 19–22], the Hamiltonian approach [23], and many others. Although these methods have been applied to obtain approximate solutions of nonlinear equation with large amplitude of oscillations, the solving process of these methods involve sophisticated derivations and computations, and they are difficult to implement. A simple approximate method to nonlinear oscillators is proposed by Ren and He [24] (Ren-He's method) and the analysis process of this method is very simple, anyone with basic knowledge of advanced calculus can apply the method to finding the approximate solution of a nonlinear oscillator.

In this paper, for getting analytical results of the cubic and cubic-quintic nonlinear equation arising from packaging system, Ren-He's method is used to solve the dropping shock dynamic equations of the system, and the approximate solutions including the displacement response, acceleration response and the dropping shock duration are obtained. In order to improve the accuracy of the solution, the correction method combining the Ren-He's method with the energy method (EM) is developed. The research provides an effective method for solving dropping shock problems of the nonlinear packaging systems.

2 Basic idea of approximate analysis

2.1 Analysis process of Ren-He's method

For a generalized un-damped conservative nonlinear oscillator, the dynamic equation of system can be expressed as

$$\ddot{x} + v(x) = 0 \tag{1}$$

with initial conditions

$$x(0) = A, \qquad \dot{x}(0) = 0, \tag{2}$$

where a dot denotes differentiation with respect to t, $v(x)$ is a nonlinear function of x. In order to satisfy the initial conditions, the trial function can be expressed as

$$x = A\cos(\omega t), \tag{3}$$

where ω is the angular frequency of the nonlinear oscillator. Substituting equation (3) into equation (1) results in

$$\ddot{x} + v\bigl(A\cos(\omega t)\bigr) = 0. \tag{4}$$

Integrating equation (4) twice continuously with respect to t, we have the following two equations:

$$\dot{x} = -\int_0^t v(x)\,\mathrm{d}t = -\int_0^t v\bigl(A\cos(\omega t)\bigr)\,\mathrm{d}t, \tag{5}$$

$$x = \int_0^t \dot{x}\,\mathrm{d}t = \int_0^t \left(-\int_0^t v\bigl(A\cos(\omega t)\bigr)\,\mathrm{d}t\right)\mathrm{d}t. \tag{6}$$

For the trial function, the following expression is obtained by equation (6):

$$x(t)|_{\omega t=\pi/2} = 0. \tag{7}$$

By equation (7) or through other special points, we can obtain amplitude-frequency relationship, and can also get the approximate analytic solution of the equation (1). The analysis process of this method is very simple, and for a more detailed analysis of the process one may refer to [24].

2.2 Basic idea of EM

For the dropping impact problem of a nonlinear packaging system, the energy method is usually used to solve the maximum displacement and maximum acceleration of the system, but it is not convenient to obtain the dropping shock duration and the time course of the system response. In order to illustrate the basic idea of the energy method, system damping is not considered in the process of dropping impact, the gravitational potential energy of the system translates into the elastic potential energy of system completely, while the deformation of the cushion material reaches the maximum. Assume the dropping height of the packaging system is h, the weight of the product is W. When the system is in the position of height h, the gravitational potential energy of the system can be expressed as

$$U = Wh. \tag{8}$$

For the nonlinear cushion packaging system, $f(x)$ denotes the corresponding restoring force of the cushion materials. According to the ideas of the EM, we have

$$Wh = \int_0^{x_m} f(x)\,\mathrm{d}x, \tag{9}$$

where $f(x)$ is a nonlinear function of x, depends upon only the function of x. The maximum deformation x_m can be obtained by solving equation (9).

Due to the dynamics, the equation of the system can be expressed in the form

$$m\ddot{x} + f(x) = 0. \tag{10}$$

The maximum acceleration $\ddot{x}_m$ can be obtained by solving the dynamics equation.

In view of the un-damped conservative systems, when the maximum deformation of the buffer occurs, the restoring force of system will be the maximum, the velocity of system is equal to zero and the acceleration response of the system obtains the maximum value, this is the real physical process. In the following discussion, combining Ren-He's method with EM according to the physical process, a correction method of the approximate analytic solution of Ren-He's method is put forward for the drop impact problems in packaging engineering.

3 Approximate solution of the dropping shock response

In order to illustrate the advantages and the accuracy of Ren-He's method, we take the cubic and cubic-quintic nonlinear equations arising from packaging system as examples.

The approximate solution of the dropping shock response is obtained by using Ren-He's method, and the correction method of approximate solutions is proposed.

3.1 Cubic nonlinear system

First of all, we consider the cubic nonlinear cushion packaging system [2, 25, 26]; the dropping shock dynamics equation can be expressed as

$$m\ddot{x} + k_0 x + r x^3 = 0; \tag{11}$$

equation (11) can also be expressed as

$$\ddot{x} + \omega_0^2 x + k x^3 = 0 \tag{12}$$

with initial conditions

$$x(0) = 0, \qquad \dot{x}(0) = \sqrt{2gh}, \tag{13}$$

where $\omega_0 = \sqrt{k_0/m}$ is the frequency parameter, m is the mass of the packaged product, h is the dropping height of the packaging system, r is the nonlinear constants of cushion packaging material, k_0 is the initial elastic constants of the buffer material, g is the acceleration of gravity, $k = \varepsilon\omega_0^2$, and $\varepsilon = r/k_0$, respectively.

Different from equation (2), the initial condition equation (13) describes the dropping shock. In order to satisfy the initial conditions, the following trial function is chosen:

$$x = A\sin(\omega t), \tag{14}$$

where A and ω denote the amplitude and angular frequency of the nonlinear oscillator to be further determined. Notice the following expression:

$$\sin^3(\omega t) = \frac{3}{4}\sin(\omega t) - \frac{1}{4}\sin(3\omega t). \tag{15}$$

By the equation (5), we have

$$\dot{x} = -\int_0^t v(x)\,\mathrm{d}t = -\int_0^t v\big(A\sin(\omega t)\big)\,\mathrm{d}t,$$

where $v(x) = \omega_0^2 x + kx^3$, the above equation can be expressed as

$$\dot{x} = \frac{1}{\omega}\left(A\omega_0^2 + \frac{3}{4}kA^3\right)\cos(\omega t) - \frac{kA^3}{12\omega}\cos(3\omega t) + C_1. \tag{16}$$

To satisfy the initial condition $\dot{x}(0) = \sqrt{2gh}$, from the equation (16), we have

$$C_1 = \sqrt{2gh} + \frac{kA^3}{12\omega} - \frac{1}{\omega}\left(A\omega_0^2 + \frac{3}{4}kA^3\right). \tag{17}$$

Integrating equation (16) with respect to t, we have the following equation:

$$x = \frac{1}{\omega^2}\left(A\omega_0^2 + \frac{3}{4}kA^3\right)\sin(\omega t) - \frac{kA^3}{36\omega^2}\sin(3\omega t) + C_1 t + C_2. \tag{18}$$

To satisfy the initial condition $x(0) = 0$, we have $C_2 = 0$. In order to obtain the amplitude-frequency relation, setting $x(t)|_{\omega t=\pi} = 0$ in equation (18) yields

$$C_1 = \sqrt{2gh} + \frac{kA^3}{12\omega} - \frac{1}{\omega}\left(A\omega_0^2 + \frac{3}{4}kA^3\right) = 0. \tag{19}$$

Solving the above equation leads to

$$\omega = \omega_0\sqrt{1 + 2\varepsilon A^2/3}. \tag{20}$$

The displacement, velocity and acceleration approximate solution of dropping shock dynamics equation (12) can be obtained as follows, respectively:

$$x = \frac{1}{\omega^2}\left(A\omega_0^2 + \frac{3}{4}kA^3\right)\sin(\omega t) - \frac{kA^3}{36\omega^2}\sin(3\omega t), \tag{21}$$

$$\dot{x} = \frac{1}{\omega}\left(A\omega_0^2 + \frac{3}{4}kA^3\right)\cos(\omega t) - \frac{kA^3}{12\omega}\cos(3\omega t), \tag{22}$$

$$\ddot{x} = -\left(A\omega_0^2 + \frac{3}{4}kA^3\right)\sin(\omega t) + \frac{kA^3}{4}\sin(3\omega t). \tag{23}$$

The dropping shock duration can be expressed as

$$\tau = \pi/\omega. \tag{24}$$

Let $\omega t = \pi/2$, from equation (21) and equation (23), the maximum values of displacement and acceleration of the dropping shock response can be written as follows, respectively:

$$x_m = \frac{1}{\omega^2}\left(A\omega_0^2 + \frac{7}{9}kA^3\right), \tag{25}$$

$$\ddot{x}_m = \left(A\omega_0^2 + kA^3\right). \tag{26}$$

In order to evaluate the accuracy of the approximate solution, the numerical example in [12] is chosen. The parameter values of the example are $m = 10$ kg, $h = 0.6$ m, $k_0 = 600$ Ncm^{-1}, $r = 72$ Ncm^{-3}, $\varepsilon = 0.12$ cm^{-2}, and $\omega_0 = 77.46$ s^{-1}. Using the dropping shock initial conditions $\dot{x}(0) = \sqrt{2gh}$, the two key parameters A and ω can be obtained by solving equation (20) and equation (22) simultaneously. The displacement and acceleration response curves of dropping shock process by Ren-He's method (denoted RHM) are compared with those by the Runge-Kutta method (denoted R-K) as shown in Figures 1 and 2, respectively. Comparison of the important parameters including the maximum displacement, the maximum acceleration of the system response and the dropping shock duration by RHM with those by the R-K method are shown in Table 1, the relative errors of the maximum values of displacement and acceleration and the dropping shock duration are 3.2%, 8.8% and 4.6%, respectively. These results show that the approximate solution needs further discussion as regards the demands of engineering.

For the cubic nonlinear cushion packaging system, the corresponding restoring force of the cushion materials can be expressed as

$$f(x) = k_0 x + r x^3. \tag{27}$$

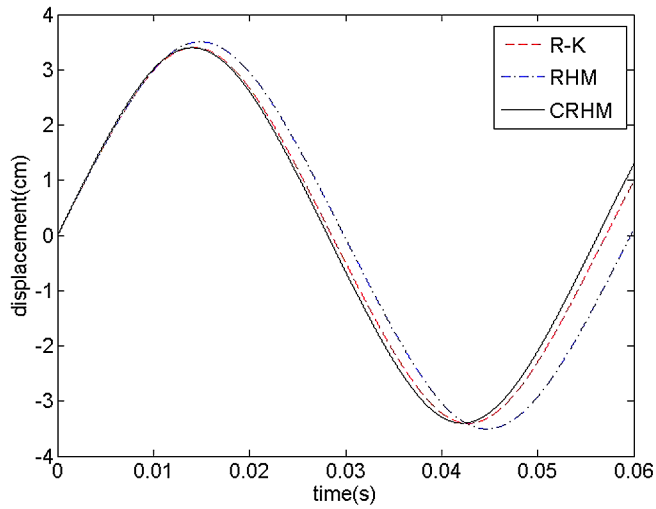

Figure 1 Comparison of the displacement response by RHM and CRHM with R-K method for the cubic nonlinear packaging system.

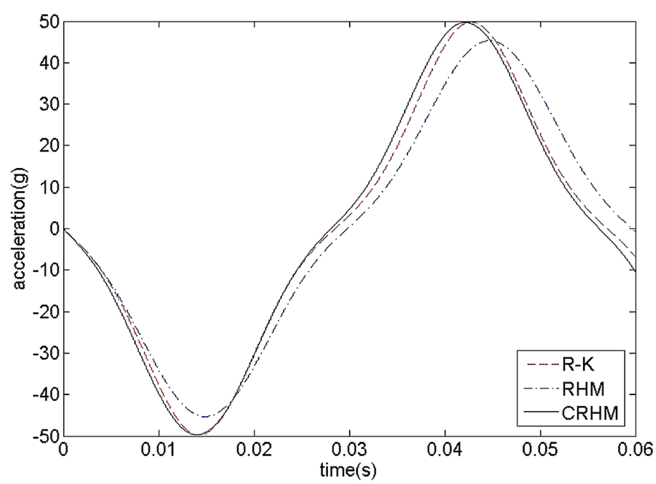

Figure 2 Comparison of the acceleration response by RHM and CRHM with R-K method for the cubic nonlinear packaging system.

Table 1 Comparison of the important parameters by RHM with R-K method for the cubic nonlinear packaging system

	x_m (cm)	$\ddot{x}_m$ (g)	τ (s)
R-K	3.4	49.74	0.02854
RHM	3.51	45.35	0.02984
Relative error (%)	3.2	8.8	4.6

According to the ideas of the EM, we have the following expression:

$$Wh = \int_0^{x_m} \left(k_0 x + rx^3\right) \mathrm{d}x. \tag{28}$$

The maximum value of displacement can be obtained by above equation and can be expressed as

$$x_{\mathrm{m}} = \sqrt{\left(\sqrt{k_0^2 + 4Wrh} - k_0\right)/r}. \tag{29}$$

Under the condition of no system damping, the displacement and the acceleration achieve the maximum value at the same time in the dropping shock process. Through the equation (12), the maximum value of acceleration can be obtained:

$$\ddot{x}_{\mathrm{m}} = \omega_0^2\left(x_{\mathrm{m}} + \varepsilon x_{\mathrm{m}}^3\right). \tag{30}$$

Substituting equation (25) and equation (26) into equation (29) and equation (30), respectively, we get the following two equations:

$$x_m = \frac{1}{\omega^2}\left(A\omega_0^2 + \frac{7}{9}kA^3\right) = \sqrt{\left(\sqrt{k_0^2 + 4Wrh} - k_0\right)/r}, \tag{31}$$

$$\ddot{x}_m = A\omega_0^2 + kA^3 = \omega_0^2\left(x_m + \varepsilon x_m^3\right). \tag{32}$$

Combining equation (31) with equation (32), the parameters A and ω are obtained as follows, respectively:

$$A = x_m = \sqrt{\left(\sqrt{k_0^2 + 4Wrh} - k_0\right)/r}, \tag{33}$$

$$\omega = \omega_0\sqrt{1 + \left(7\varepsilon A^2\right)/9}. \tag{34}$$

The parameter A and ω denote A_r and ω_r, respectively. Substituting A_r and ω_r into the equation (21), equation (22), and equation (23), then the correction solutions of the approximate solutions by RHM (denoted CRHM) are obtained:

$$x = \frac{1}{\omega_r^2}\left(A_r\omega_0^2 + \frac{3}{4}kA_r^3\right)\sin(\omega_r t) - \frac{kA_r^3}{36\omega_r^2}\sin(3\omega_r t), \tag{35}$$

$$\dot{x} = \frac{1}{\omega_r}\left(A_r\omega_0^2 + \frac{3}{4}kA_r^3\right)\cos(\omega_r t) - \frac{kA_r^3}{12\omega_r}\cos(3\omega_r t), \tag{36}$$

$$\ddot{x} = -\left(A_r\omega_0^2 + \frac{3}{4}kA_r^3\right)\sin(\omega_r t) + \frac{kA_r^3}{4}\sin(3\omega_r t). \tag{37}$$

Substituting ω_r into the equation (24), then the correction solutions of the dropping shock duration can be expressed as

$$\tau = \pi/\omega_r. \tag{38}$$

Table 2 Comparison of the important parameters by CRHM with R-K method for the cubic nonlinear packaging system

	x_m (cm)	$\ddot{x}_m$ (g)	τ (s)
R-K	3.40	49.74	0.02854
CRHM	3.40	49.79	0.02811
Relative error (%)	0.0	0.1	1.5

Substituting A_r and ω_r into the equation (25) and equation (26), the correction solutions of the maximum displacement and the maximum acceleration of the system can be obtained, respectively.

The dropping shock displacement response and acceleration response of the CRHM are compared with those by the R-K method as shown in Figures 1 and 2, respectively, the response curve is very close to ones by the R-K method. Comparisons of the important parameters by CRHM with those by the R-K method are shown in Table 2, the relative errors of the maximum displacement, the maximum acceleration, and the dropping shock duration are 0.0%, 0.1% and 1.5%, respectively. These results show the good accuracy.

3.2 Cubic-quintic nonlinear system

We consider the cubic-quintic nonlinear cushion packaging system [11, 27, 28] as the second example, the dropping shock dynamics equation and initial conditions can be written as

$$m\ddot{x} + k_0 x + rx^3 + qx^5 = 0, \tag{39}$$

$$x(0) = 0, \qquad \dot{x}(0) = \sqrt{2gh}, \tag{40}$$

where q is the coefficient of nonlinear term, the physical meaning of other parameters is the same as in the previous example. The restoring force of the system can be represented as

$$f(x) = k_0 x + rx^3 + qx^5. \tag{41}$$

Introducing the new parameters $a = \sigma\omega_0^2$ and $\sigma = q/k_0$, the dropping shock dynamics equation and the initial conditions can also be rewritten as

$$\ddot{x} + \omega_0^2 x + kx^3 + ax^5 = 0, \tag{42}$$

$$x(0) = 0, \qquad \dot{x}(0) = \sqrt{2gh}. \tag{43}$$

In order to satisfy the initial conditions, we chose the trial function

$$x = A\sin(\omega t). \tag{44}$$

Notice the following expressions:

$$\sin^3(\omega t) = \frac{3}{4}\sin(\omega t) - \frac{1}{4}\sin(3\omega t), \tag{45}$$

$$\sin^5(\omega t) = \frac{5}{8}\sin(\omega t) - \frac{5}{16}\sin(3\omega t) + \frac{1}{16}\sin(5\omega t). \tag{46}$$

By equation (5), we have

$$\dot{x} = \frac{1}{\omega}\left(A\omega_0^2 + \frac{3}{4}kA^3 + \frac{5}{8}aA^5\right)\cos(\omega t) - \frac{1}{\omega}\left(\frac{kA^3}{12} + \frac{5aA^5}{48}\right)\cos(3\omega t) + \frac{1}{\omega}\frac{aA^5}{80}\cos(5\omega t) + C_3. \tag{47}$$

To satisfy the initial condition $\dot{x}(0) = \sqrt{2gh}$, by equation (47), we have

$$C_3 = \sqrt{2gh} + \frac{1}{\omega}\left(\frac{kA^3}{12} + \frac{5aA^5}{48}\right) - \frac{1}{\omega}\left(A\omega_0^2 + \frac{3}{4}kA^3 + \frac{5}{8}aA^5\right) - \frac{aA^5}{80\omega}. \tag{48}$$

Integrating equation (47) with respect to t, we have the following equation:

$$x = \frac{1}{\omega^2}\left(A\omega_0^2 + \frac{3}{4}kA^3 + \frac{5}{8}aA^5\right)\sin(\omega t) - \frac{1}{\omega^2}\left(\frac{kA^3}{36} + \frac{5aA^5}{144}\right)\sin(3\omega t) + \frac{1}{\omega^2}\frac{aA^5}{400}\sin(5\omega t) + C_3 t + C_4. \tag{49}$$

To satisfy the initial condition $x(0) = 0$, we have $C_4 = 0$. In order to obtain the amplitude-frequency relations, setting $x(t)|_{\omega t=\pi} = 0$ in equation (49) yields

$$C_3 = \sqrt{2gh} + \frac{1}{\omega}\left(\frac{kA^3}{12} + \frac{5aA^5}{48}\right) - \frac{1}{\omega}\left(A\omega_0^2 + \frac{3}{4}kA^3 + \frac{5}{8}aA^5\right) - \frac{aA^5}{80\omega} = 0. \tag{50}$$

Notice parameters $a = \sigma\omega_0^2$ and $k = \varepsilon\omega_0^2$, we have

$$\omega = \omega_0\sqrt{1 + 2\varepsilon A^2/3 + 8\sigma A^4/15}. \tag{51}$$

The displacement, velocity, and acceleration approximate solutions of dropping shock dynamics equation (42) can be obtained as follows, respectively:

$$x = \frac{1}{\omega^2}\left(A\omega_0^2 + \frac{3}{4}kA^3 + \frac{5}{8}aA^5\right)\sin(\omega t) - \frac{1}{\omega^2}\left(\frac{kA^3}{36} + \frac{5aA^5}{144}\right)\sin(3\omega t) + \frac{1}{\omega^2}\frac{aA^5}{400}\sin(5\omega t), \tag{52}$$

$$\dot{x} = \frac{1}{\omega}\left(A\omega_0^2 + \frac{3}{4}kA^3 + \frac{5}{8}aA^5\right)\cos(\omega t) - \frac{1}{\omega}\left(\frac{kA^3}{12} + \frac{5aA^5}{48}\right)\cos(3\omega t) + \frac{1}{\omega}\frac{aA^5}{80}\cos(5\omega t), \tag{53}$$

$$\ddot{x} = -\left(A\omega_0^2 + \frac{3}{4}kA^3 + \frac{5}{8}aA^5\right)\sin(\omega t) + \left(\frac{kA^3}{4} + \frac{5aA^5}{16}\right)\sin(3\omega t) - \frac{1}{16}aA^5\sin(5\omega t). \tag{54}$$

Let $\omega t = \pi/2$, from equation (52) and equation (54), the maximum displacement and the maximum acceleration of the system can be written as, respectively,

$$x_m = \frac{1}{\omega^2}\left(A\omega_0^2 + \frac{7}{9}kA^3 + \frac{149}{225}aA^5\right), \tag{55}$$

$$\ddot{x}_m = \left(A\omega_0^2 + kA^3 + aA^5\right). \tag{56}$$

In order to facilitate comparison, the related parameter values of the example are set equal to m = 10 kg, h = 0.6 m, k_0 = 500 Ncm^{-1}, r = 72 Ncm^{-3}, σ = 0.0001 cm^{-4}, ε = 0.144 cm^{-2}, and ω_0 = 70.71 s^{-1}, respectively. Using the dropping shock initial condition $\dot{x}(0) = \sqrt{2gh}$, the two key parameters A and ω can be obtained by solving equation (51) and equation (53) simultaneously. The displacement and acceleration response of dropping shock process by RHM are compared with those by the R-K method as shown in Figures 3 and 4, respectively. The comparison of the important parameters including the maximum displacement, the maximum acceleration of the system response and the dropping shock duration by RHM with those by the R-K method are shown in Table 3, the relative errors of maximum displacement, maximum acceleration and the dropping shock duration are 3.7%, 10.4% and 5.2%, respectively. The results shows that the errors still cannot meet the requirement of engineering application, the results need to be further discussed.

For the dropping shock problem of the un-damped cubic-quintic nonlinear packaging system, the correction method of analytical solutions is the same as the above example. According to the ideas of the EM, we have the following expression:

$$Wh = \int_0^{x_m} \left(k_0 x + rx^3 + qx^5\right) \mathrm{d}x. \tag{57}$$

Substituting the related parameter values of example into equation (57), the maximum

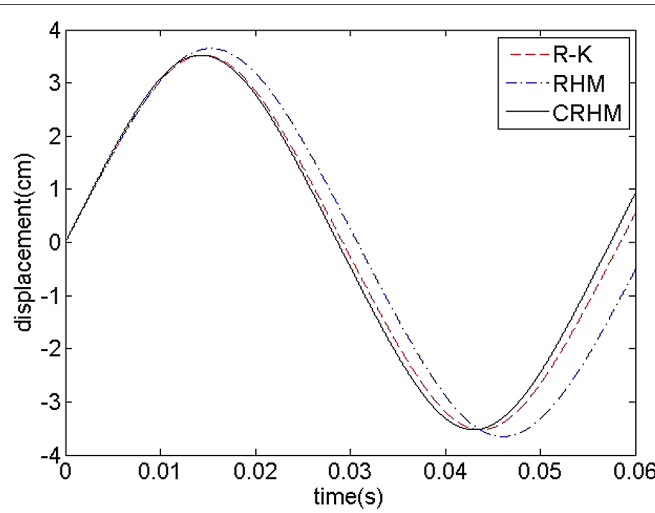

Figure 3 Comparison of the displacement response by RHM and CRHM with R-K method for the cubic-quintic nonlinear packaging system.

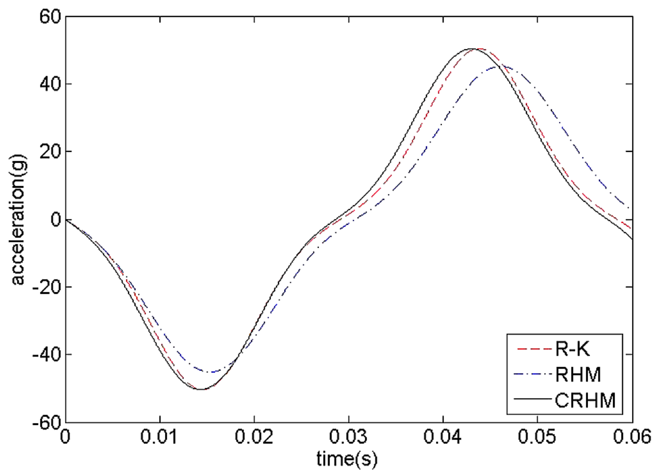

Figure 4 Comparison of the acceleration response by RHM and CRHM with R-K method for the cubic-quintic nonlinear packaging system.

Table 3 Comparison of the important parameters by RHM with R-K method for the cubic-quintic nonlinear packaging system

	x_m (cm)	$\ddot{x}_m$ (g)	τ (s)
R-K	3.52	50.35	0.02920
RHM	3.65	45.13	0.03074
Relative error (%)	3.7	10.4	5.2

displacement can be solved by the above equation,

$$x_{\mathrm{m}} = 3.52\ \mathrm{cm}. \tag{58}$$

By equation (42), the maximum acceleration can be obtained:

$$\ddot{x}_{\mathrm{m}} = \omega_0^2\left(x_{\mathrm{m}} + \varepsilon x_{\mathrm{m}}^3 + \sigma x_{\mathrm{m}}^5\right). \tag{59}$$

Substituting equation (58) and equation (59) into equation (55) and equation (56), respectively, the following two relations are obtained:

$$x_m = \frac{1}{\omega^2}\left(A\omega_0^2 + \frac{7}{9}kA^3 + \frac{149}{225}aA^5\right), \tag{60}$$

$$\ddot{x}_m = A\omega_0^2 + kA^3 + aA^5 = \omega_0^2\left(x_m + \varepsilon x_m^3 + \sigma x_m^5\right). \tag{61}$$

Combining equation (60) with equation (61), the parameter A and the parameter ω are obtained as follows, respectively:

$$A = x_m = 3.52\ \mathrm{cm}, \tag{62}$$

$$\omega = \omega_0\sqrt{1 + \left(7\varepsilon A^2\right)/9 + 149\sigma A^4/225} = 109.49\ \mathrm{s}^{-1}. \tag{63}$$

The parameters A and ω denote A_r and ω_r, respectively. Substituting A_r and ω_r into equation (52), equation (53), and equation (54), then the correction solutions of the approximate solutions by using RHM (denoted CRHM) are obtained:

$$x = \frac{1}{\omega_r^2}\left(A_r\omega_0^2 + \frac{3}{4}kA_r^3 + \frac{5}{8}aA_r^5\right)\sin(\omega_r t) - \frac{1}{\omega_r^2}\left(\frac{kA_r^3}{36} + \frac{5aA_r^5}{144}\right)\sin(3\omega_r t) + \frac{1}{\omega_r^2}\frac{aA_r^5}{400}\sin(5\omega_r t), \tag{64}$$

$$\dot{x} = \frac{1}{\omega_r}\left(A_r\omega_0^2 + \frac{3}{4}kA_r^3 + \frac{5}{8}aA_r^5\right)\cos(\omega_r t) - \frac{1}{\omega_r}\left(\frac{kA_r^3}{12} + \frac{5aA_r^5}{48}\right)\cos(3\omega_r t) + \frac{1}{\omega_r}\frac{aA_r^5}{80}\cos(5\omega_r t), \tag{65}$$

$$\ddot{x} = -\left(A_r\omega_0^2 + \frac{3}{4}kA_r^3 + \frac{5}{8}aA_r^5\right)\sin(\omega_r t) + \left(\frac{kA_r^3}{4} + \frac{5aA_r^5}{16}\right)\sin(3\omega_r t) - \frac{1}{16}aA_r^5\sin(5\omega_r t). \tag{66}$$

The correction solutions of the dropping shock duration can be expressed as

$$\tau = \pi/\omega_r. \tag{67}$$

Substituting A_r and ω_r into the equation (55) and equation (56), then the correction solutions of the maximum displacement and the maximum acceleration of system can be written as, respectively,

$$x_m = \frac{1}{\omega_r^2}\left(A_r\omega_0^2 + \frac{7}{9}kA_r^3 + \frac{149}{225}aA_r^5\right), \tag{68}$$

$$\ddot{x}_m = \left(A_r\omega_0^2 + kA_r^3 + aA_r^5\right). \tag{69}$$

The dropping shock displacement response and acceleration response of the correction solutions by CRHM compared with those by R-K method are given in Figures 3 and 4, respectively, the response curve is very close to the ones by R-K method. Comparison of the important parameters by CRHM with those by R-K method are shown in Table 4, the relative errors of the maximum displacement, the maximum acceleration, and the dropping shock duration are 0.0%, 0.06% and 1.7%, respectively. These results show that the approximate solution by CRHM has accuracy enough for engineering application.

4 Conclusions

The evaluation of the dropping impact dynamics, the maximum displacement, the maximum acceleration of the system response, and the dropping shock duration is in focus as

Table 4 Comparison of the important parameters by the CRHM with the R-K method for the cubic-quintic nonlinear packaging system

	x_m (cm)	$\ddot{x}_m$ (g)	τ (s)
R-K	3.52	50.35	0.02920
CRHM	3.52	50.32	0.02869
Relative error (%)	0.00	0.06	1.7

regards the important parameters. Ren-He's method has been successfully used for the cubic and cubic-quintic nonlinear equations arising from a packaging system; the analytical solutions of the system are obtained, the analytic expression of the above three important parameters is obtained at the same time. In order to further improve the accuracy of the solution, a new method combining the RHM with the energy method, which is defined as a correction method of the RHM (denoted CRHM), is proposed and successfully used to study the cubic and the cubic-quintic nonlinear packaging system. The numerical examples indicate that the lowest order analytical solution obtained by the CRHM has higher accuracy, which outperforms a similar analytical method. The CRHM is obviously advantageous in that it is simple and it can avoid solving the complicated nonlinear algebraic equation. The results show that the CRHM is effective and easy for solving the dropping shock problem of a nonlinear packaging system.

Competing interests
The author declares that they have no competing interests.

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