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Multiple solutions of discrete Schrödinger equations with growing potentials

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Abstract

Under some weaker conditions than elsewhere, we obtain infinitely many homoclinic solutions for a class of discrete Schrödinger equations in infinite m dimensional lattices with nonlinearities being superlinear at infinity by using variational methods. Our result extends some existing results in the literature.

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1 Introduction and main results

The discrete nonlinear Schrödinger equation is one of the most important discrete models, which plays an important role in many fields; for example, in biomolecular chains [1], nonlinear optics [2], Bose-Einstein condensates [3], and so on. In recent decades, a lot of results have been achieved in the study of homoclinic solutions for periodic discrete nonlinear Schrödinger equations; see [4–14], and so on. But we notice that there are only a few results of non-periodic discrete nonlinear Schrödinger equations, such as [15–23]. The authors of [16, 17, 19, 22] studied the case in infinite one dimensional lattices (*i.e.*, $n \in Z$), but the authors of [15, 18, 20, 21, 23] studied the case in infinite m dimensional lattices (*i.e.*, $n \in Z^m$).

Inspired by the above results, we will study homoclinic solutions of the following non-periodic discrete nonlinear equation in infinite m dimensional lattices by more general conditions than some existing results:

$$-\Delta u_n + v_n u_n - \omega u_n = f_n(u_n), \quad n \in Z^m, \tag{1.1}$$

where

$$\begin{aligned}\Delta u_n = {}& u_{(n_1+1,n_2,\ldots,n_m)} + u_{(n_1,n_2+1,\ldots,n_m)} + \cdots + u_{(n_1,n_2,\ldots,n_m+1)} - 2mu_{(n_1,n_2,\ldots,n_m)} \\ & + u_{(n_1-1,n_2,\ldots,n_m)} + u_{(n_1,n_2-1,\ldots,n_m)} + u_{(n_1,n_2,\ldots,n_m-1)}\end{aligned}$$

is the discrete Laplace operator in m dimensional space, $\omega \in R$, $V = \{v_n\}_{n\in Z^m}$, and $\{u_n\}_{n\in Z^m}$ are sequences of real numbers, and the nonlinearities f_n satisfy the condition:

$$f_n(e^{i\omega} s) = e^{i\omega} f_n(s), \quad \forall \omega \in R, \forall (n,s) \in Z^m \times R.$$

As usual, homoclinic solutions of equation (1.1) satisfy the following boundary condition:

$$\lim_{|n|=|n_1|+|n_2|+\cdots+|n_m|\to\infty} u_n = 0. \tag{1.2}$$

Here we are interested in the existence of infinitely many nontrivial homoclinic solutions for (1.1) ('u is nontrivial' means $u_n \not\equiv 0$). The problem (1.1) comes from the study of standing waves for the discrete nonlinear Schrödinger equation

$$i\dot{\psi}_n = -\Delta\psi_n + v_n\psi_n - f_n(\psi_n), \quad n \in Z^m. \tag{1.3}$$

By the definition of standing waves ($\psi_n = u_n e^{-i\omega t}$ with (1.2)), we see that (1.3) becomes (1.1). Therefore, the problem of the existence of standing waves of (1.3) has been reduced to that on the existence of homoclinic solutions of (1.1).

In order to overcome the difficulties caused by the unboundedness of Z^m and the lack of periodic conditions, we make some suitable assumptions and get the following result.

Theorem 1.1 *The problem* (1.1) *has infinitely many nontrivial homoclinic solutions if* $f_n(-s) = -f_n(s)$ *for all* $(n,s) \in Z^m \times R$ *and the following conditions hold*:

(V_1) $V = \{v_n\}_{n\in Z^m}$ *is bounded from below and satisfies*

$$\lim_{|n|\to+\infty} v_n = +\infty. \tag{1.4}$$

(F_1) $f_n \in C(R,R), f_n(s) = o(s)$ *as* $s \to 0$, *and there exist* $a_1 > 0$ *and* $\nu > 2$ *such that*

$$|f_n(s)| \le a_1(1 + |s|^{\nu-1}), \quad \forall (n,s) \in Z^m \times R.$$

(F_2) $\lim_{|s|\to+\infty} \frac{F_n(s)}{|s|^2} = +\infty, \forall n \in Z^m$, *where* $F_n(s) := \int_0^s f_n(t)\,dt, (n,s) \in Z^m \times R$.

(F_3) *There exist two positive constants* b *and* $\varrho > \max\{1, \nu - 2\}$ *such that*

$$\liminf_{|s|\to+\infty} \frac{f_n(s)s - 2F_n(s)}{|s|^\varrho} \ge b, \quad \forall n \in Z^m.$$

(F_4) $\frac{1}{2}f_n(s)s > F_n(s)$ *if* $s \ne 0$, $F_n(s) \ge 0$, $\forall (n,s) \in Z^m \times R$, *and*

$$\liminf_{|s|\to 0} \frac{f_n(s)s - 2F_n(s)}{|s|^\iota} \ge a_2 \quad \textit{for some } a_2 > 0 \textit{ and } \iota \in [1,\nu], \forall n \in Z^m.$$

To explain the rationality of the assumptions for the nonlinear terms f_n, we give the following example. It is easy to check that the functions given in the following example satisfy our assumptions.

Example 1.1 Let

$$F_n(s) = a_n\bigl(|s|^p + (p-2)|s|^{p-\varepsilon}\sin^2\bigl(|s|^\varepsilon/\varepsilon\bigr)\bigr), \quad s \in R,$$

where $a_n \ge C > 0$ for all $n \in Z^m$, $p > 2$, and $0 < \varepsilon < p - 2$. Note that

$$f_n(s)s - 2F_n(s) = (p-2)a_n\bigl[(p-2-\varepsilon)|s|^{p-\varepsilon}\sin^2\bigl(|s|^\varepsilon/\varepsilon\bigr) + \bigl(1 + \sin\bigl(2|s|^\varepsilon/\varepsilon\bigr)\bigr)|s|^p\bigr].$$

Remark 1.1 Our result extends some results [15, 18, 20, 21, 23] in infinite m dimensional lattices.

(1) The results [15, 18, 20, 21] are all about the positive definite case ($\omega < \inf\sigma(-\Delta + V)$), but the temporal frequency $\omega \in R$ in our paper.

(2) The authors of [15, 18, 20, 21] all used the conditions (V_1), (F_1), and (F_2). Besides, the authors of [15, 18] also used the following monotony condition:

$$\frac{f_n(s)}{s} \text{ is increasing for } s > 0 \text{ and decreasing for } s < 0. \tag{1.5}$$

The authors [20, 21] also used the following Ambrosetti-Rabinowitz superlinear condition: there exists $\nu > 2$ such that

$$0 < \nu F_n(s) \leq f_n(s)s, \quad \forall s \in R\backslash\{0\}. \tag{1.6}$$

But we use local conditions (F_3) and (F_4) to replace the conditions (1.5) and (1.6). The functions of Example 1.1 satisfy our conditions (F_1)-(F_4), but they do not satisfy (1.5) and (1.6), which shows that our conditions are weaker than the above conditions.

(3) The results in [23] also rely on the monotony condition (1.5).

In Section 2, we establish the variational framework of (1.1) and give some preliminary lemmas. In Section 3, we give the detailed proof of our main result.

2 Preliminary lemmas

Let

$$l^p \equiv l^p(Z^m) := \left\{u = \{u_n\} : n \in Z^m, u_n \in R, \|u\|_{l^p} = \left(\sum_{n\in Z^m} |u_n|^p\right)^{1/p} < \infty\right\},$$
$$p \in [1, +\infty),$$

be real sequence spaces. Clearly, the following elementary embedding relations hold:

$$l^p \subset l^q, \quad \|u\|_{l^q} \leq \|u\|_{l^p}, \quad 1 \leq p \leq q \leq \infty, \text{ where } \|u\|_{l^\infty} := \max_{n\in Z^m} |u_n|.$$

Let $L := -\triangle + V$ be defined by $Lu_n := -\triangle u_n + v_n u_n$ for $u \in l^2$. Let E be the form domain of L, *i.e.*, $E := \mathcal{D}(L^{1/2})$ (the domain of $L^{1/2}$). Under our assumptions, the operator L is an unbounded self-adjoint operator in l^2. Since the operator $-\triangle$ is bounded in l^2, it is easy to see that $E = \{u \in l^2 : V^{1/2}u \in l^2\}$, where $V^{1/2}u$ is defined by $V^{1/2}u_n := v_n^{1/2}u_n$ for $u \in l^2$. We define, respectively, on E the inner product and norm by

$$(u, v)_E := (u, v)_{l^2} + \left(L^{1/2}u, L^{1/2}v\right)_{l^2} \quad \text{and} \quad \|u\|_E = (u, u)_E^{1/2},$$

where $(u, v)_{l^2}$ is the inner product in l^2. Then E is a Hilbert space.

Lemma 2.1 ([21]) *If* (1.4) *holds, then we have*:

(1) *The embedding maps from E into l^p are compact*, $\forall p \in [2, \infty]$.

(2) *The spectrum* $\sigma(L)$ *is discrete and consists of simple eigenvalues accumulating to* $+\infty$.

By Lemma 2.1(2), we can assume that

$$\lambda_1 - \omega < \lambda_2 - \omega < \cdots < \lambda_k - \omega < \cdots \to +\infty$$

are all eigenvalues of $L - \omega$ and e_k is the associated normalized eigenfunction with the eigenvalue $\lambda_k - \omega$ for each k, *i.e.*, $(L-\omega)e_k = (\lambda_k - \omega)e_k$ and $\|e_k\|_{l^2} = 1, k = 1, 2, \ldots$. Moreover, $\{e_k : k = 1, 2, \ldots\}$ is an orthonormal basis of l^2. Let $\sharp(D)$ denote the number i with $i \in D$. Let

$$k_1 := \sharp\bigl(\{i : \lambda_i - \omega < 0\}\bigr), \qquad k_0 := \sharp\bigl(\{i : \lambda_i - \omega = 0\}\bigr), \qquad k_2 := k_0 + k_1, \tag{2.1}$$

and

$$E^- := \operatorname{span}\{e_1, \ldots, e_{k_1}\}, \qquad E^0 := \operatorname{span}\{e_{k_1+1}, \ldots, e_{k_2}\}, \qquad E^+ := \overline{\operatorname{span}\{e_{k_2+1}, \ldots\}},$$

where the closure is taken with respect to the norm $\|\cdot\|_E$. Then one has the orthogonal decomposition

$$E = E^- \oplus E^0 \oplus E^+$$

with respect to the inner product $(\cdot,\cdot)_E$. Now, we introduce, respectively, on E the following inner product and norm:

$$(u, v) := \bigl(u^0, v^0\bigr)_{l^2} + \bigl(L^{\frac{1}{2}}u, L^{\frac{1}{2}}v\bigr)_{l^2}, \qquad \|u\| = (u, u)^{\frac{1}{2}},$$

where $u, v \in E = E^- \oplus E^0 \oplus E^+$ with $u = u^- + u^0 + u^+$ and $v = v^- + v^0 + v^+$. Clearly, the norms $\|\cdot\|$ and $\|\cdot\|_E$ are equivalent, and the decomposition $E = E^- \oplus E^0 \oplus E^+$ is also orthogonal with respect to both inner products $(\cdot,\cdot)$ and $(\cdot,\cdot)_{l^2}$.

From the above arguments, we consider the functional Φ on E defined by

$$\begin{aligned}\Phi(u) &= \frac{1}{2}\bigl((L-\omega)u, u\bigr)_{l^2} - \sum_{n\in Z^m} F_n(u_n) \\ &= \frac{1}{2}\bigl\|u^+\bigr\|^2 - \frac{1}{2}\bigl\|u^-\bigr\|^2 - I(u),\end{aligned}$$

where $I(u) := \sum_{n\in Z^m} F_n(u_n)$. Under our assumptions, $I, \Phi \in C^1(E, R)$, and the derivatives are given by

$$\bigl\langle \Phi'(u), v\bigr\rangle = \bigl(u^+, v^+\bigr) - \bigl(u^-, v^-\bigr) - \bigl\langle I'(u), v\bigr\rangle, \qquad \bigl\langle I'(u), v\bigr\rangle = \sum_{n\in Z^m} f_n(u_n)v_n,$$

where $u, v \in E = E^- \oplus E^0 \oplus E^+$ with $u = u^- + u^0 + u^+$ and $v = v^- + v^0 + v^+$. The standard argument shows that nonzero critical points of Φ are nontrivial solutions of (1.1). We shall use the following critical point theorem to prove our main result.

Lemma 2.2 ([24]) *Let $E = \overline{\bigoplus_{j=1}^{\infty} X_j}$ $(\dim X_j < \infty, \forall j \in N)$ be a Banach space with the norm $\|\cdot\|$, $Y_k = \bigoplus_{j=1}^{k} X_j$, and $Z_k = \overline{\bigoplus_{j=k}^{\infty} X_j}$. Let the functional $\Phi_\lambda = A(u) - \lambda B(u) \in C^1 : E \to R$, $\lambda \in [1,2]$. Assume that Φ_λ satisfies*

- (F_1) *Φ_λ maps bounded sets to bounded sets for $\lambda \in [1,2]$, and $\Phi_\lambda(-u) = \Phi_\lambda(u)$, $\forall (\lambda, u) \in [1,2] \times E$.*
- (F_2) *$B(u) \geq 0$, $\forall u \in E$, $A(u) \to \infty$ or $B(u) \to \infty$ as $\|u\| \to \infty$.*
- (F_3) *There exist $r_k > \rho_k > 0$ such that*

$$\alpha_k(\lambda) := \inf_{u \in Z_k, \|u\| = \rho_k} \Phi_\lambda(u) > \beta_k(\lambda) := \max_{u \in Y_k, \|u\| = r_k} \Phi_\lambda(u), \quad \forall \lambda \in [1,2].$$

Then

$$\alpha_k(\lambda) \leq \zeta_k(\lambda) := \inf_{\gamma \in \Gamma_k} \max_{u \in B_k} \Phi_\lambda\big(\gamma(u)\big), \quad \forall \lambda \in [1,2],$$

where $B_k := \{u \in Y_k : \|u\| \leq r_k\}$ and $\Gamma_k := \{\gamma \in C(B_k, E) | \gamma$ is odd, $\gamma|_{\partial B_k} = id\}$. Moreover, for a.e. $\lambda \in [1,2]$, there exists a sequence $\{u^{km}(\lambda)\}_{m=1}^{\infty}$ such that

$$\sup_m \big\| u^{km}(\lambda) \big\| < \infty, \qquad \Phi'_\lambda\big(u^{km}(\lambda)\big) = 0 \quad \textit{and} \quad \Phi_\lambda\big(u^{km}(\lambda)\big) \to \zeta_k(\lambda) \quad \textit{as } m \to \infty.$$

Note that $\dim E^0$ and $\dim E^-$ are finite, we choose an orthonormal basis $\{e_j\}_{j=1}^{k_1}$ of E^-, an orthonormal basis $\{e_j\}_{j=k_1+1}^{k_2}$ of E^0, and an orthonormal basis $\{e_j\}_{j=k_2+1}^{\infty}$ of E^+, where k_1 and k_2 are defined in (2.1). Then $\{e_j\}_{j=1}^{\infty}$ is an orthonormal basis of E. Let $X_j := Re_j$, then $Y_k = \bigoplus_{m=1}^{k} X_m = \operatorname{span}\{e_1, \ldots, e_k\}$ and $Z_k = \overline{\bigoplus_{m=k}^{\infty} X_m} = \overline{\operatorname{span}\{e_k, \ldots\}}$ for all $k \in N$. In order to apply Lemma 2.2 to prove our main result, we define the functionals A, B, and Φ_λ on E by

$$A(u) = \frac{1}{2} \big\| u^+ \big\|^2, \qquad B(u) = \frac{1}{2} \big\| u^- \big\|^2 + \sum_{n \in Z^m} F_n(u_n)$$

and

$$\Phi_\lambda(u) = A(u) - \lambda B(u) = \frac{1}{2} \big\| u^+ \big\|^2 - \lambda \left(\frac{1}{2} \big\| u^- \big\|^2 + \sum_{n \in Z^m} F_n(u_n) \right), \quad \forall u \in E, \forall \lambda \in [1,2].$$

Clearly, $\Phi_\lambda \in C^1(E, R)$, $\forall \lambda \in [1,2]$.

Lemma 2.3 *If (F_4) holds, then (F_2) in Lemma 2.2 holds.*

Proof Obviously, $B(u) \geq 0$ for all $u \in E$ by (F_4) and the definition of $B(u)$. From the Fact 1 in the Appendix, we see that there is a constant $\epsilon > 0$ such that

$$\sharp\big(\big\{n \in Z^m : |u_n| \geq \epsilon \|u\|\big\}\big) \geq 1, \quad \forall u \in H \backslash \{0\}, \tag{2.2}$$

for any finite-dimensional subspace $H \subset E$. Let $\Lambda_u := \{n \in Z^m : |u_n| \geq \epsilon \|u\|\}$, $\forall u \in H \backslash \{0\}$. Then by (2.2),

$$\sharp(\Lambda_u) \geq 1, \quad \forall u \in H \backslash \{0\}. \tag{2.3}$$

(F_2) implies that there are $R_1, R_2 > 0$ such that

$$F_n(s) \geq R_1|s|^2, \quad \forall (n,s) \in Z^m \times R \text{ with } |s| \geq R_2. \tag{2.4}$$

For any $u \in H$ with $\|u\| \geq R_2/\epsilon$, we have

$$|u_n| \geq R_2, \quad \forall n \in \Lambda_u. \tag{2.5}$$

Note that $F_n(s) \geq 0$ for all $(n,s) \in Z^m \times R$, it follows from (2.3)-(2.5) and the definitions of $B(u)$ and Λ_u that, for any $u \in H$ with $\|u\| \geq R_2/\epsilon$,

$$\begin{aligned} B(u) &= \frac{1}{2}\|u^-\|^2 + \sum_{n\in Z^m} F_n(u_n) \\ &\geq \sum_{n\in\Lambda_u} F_n(u_n) \\ &\geq \sum_{n\in\Lambda_u} R_1|u_n|^2 \\ &\geq R_1\epsilon^2\|u\|^2 \cdot \sharp(\Lambda_u) \geq R_1\epsilon^2\|u\|^2. \end{aligned}$$

It implies

$$B(u) \to \infty \quad \text{as } \|u\| \to \infty \text{ on } E^- \oplus E^0,$$

which is due to $E^- \oplus E^0$ being of finite dimension. It follows from the fact $E = E^- \oplus E^0 \oplus E^+$ and the definitions of A and B that we have

$$A(u) \to \infty \quad \text{or} \quad B(u) \to \infty \quad \text{as } \|u\| \to \infty, \forall u \in E.$$

The proof is completed. □

Lemma 2.4 *If the assumptions in Theorem* 1.1 *are satisfied, then* (F_3) *in Lemma* 2.2 *holds*.

Proof (a) Note that (F_1) implies that for any $\varepsilon > 0$ there exists C_ε such that

$$\left|F_n(s)\right| \leq \varepsilon|s|^2 + C_\varepsilon|s|^\nu, \quad \forall (n,s) \in Z^m \times R.$$

It follows from the definition of Φ_λ that

$$\begin{aligned} \Phi_\lambda(u) &\geq \frac{1}{2}\|u\|^2 - 2\sum_{n\in Z^m} F_n(u_n) \\ &\geq \frac{1}{2}\|u\|^2 - 2\sum_{n\in Z^m}\left(\varepsilon\left|u_n\right|^2 + C_\varepsilon\left|u_n\right|^\nu\right), \quad \forall (\lambda,u) \in [1,2] \times E^+. \end{aligned} \tag{2.6}$$

Let

$$l_2(k) := \sup_{u\in Z_k\setminus\{0\}} \frac{\|u\|_{l^2}}{\|u\|}, \qquad l_\nu(k) := \sup_{u\in Z_k\setminus\{0\}} \frac{\|u\|_{l^\nu}}{\|u\|}, \quad \forall k \in N. \tag{2.7}$$

Note that

$$l_2(k) \to 0, \quad l_\nu(k) \to 0 \quad \text{as } k \to \infty, \tag{2.8}$$

which will be proved in the appendix. Obviously, $Z_k \subset E^+$ for all $k \geq k_2 + 1$ ($k_2 + 1$ is defined above Lemma 2.3), thus it follows from (2.6)-(2.7) that for any $k \geq k_2 + 1$ we have

$$\Phi_\lambda(u) \geq \frac{1}{2}\|u\|^2 - 2\varepsilon l_2^2(k)\|u\|^2 - 2C_\varepsilon l_\nu^\nu(k)\|u\|^\nu, \quad \forall(\lambda, u) \in [1,2] \times Z_k. \tag{2.9}$$

Let

$$\rho_k := \bigl(1 - 16\varepsilon l_2^2(k)\bigr)\bigl(16C_\varepsilon l_\nu^\nu(k)\bigr)^{\frac{1}{2-\nu}}. \tag{2.10}$$

By (2.8), there exists a large enough $k_3 > k_2 + 1$ such that

$$0 < 16\varepsilon l_2^2(k) < 1, \quad \forall k > k_3. \tag{2.11}$$

By (2.8), (2.10), (2.11), and $\nu > 2$, we have

$$\rho_k \to \infty \quad \text{as } k \to \infty. \tag{2.12}$$

By (2.9)-(2.11), we have

$$\alpha_k(\lambda) := \inf_{u \in Z_k, \|u\| = \rho_k} \Phi_\lambda(u) \geq \rho_k^2/4 > 0, \quad \forall k \geq k_3.$$

(b) Note that Y_k is of finite dimension, thus (2.2) implies that for any $k \in N$ there exists a constant $\epsilon_k > 0$ such that

$$\sharp\bigl(\{n \in Z^m : |u_n| \geq \epsilon_k \|u\|\}\bigr) \geq 1, \quad \forall u \in Y_k \setminus \{0\}. \tag{2.13}$$

By (F_2), for any $k \in N$, there exists a constant $S_k > 0$ such that

$$F_n(s) \geq \frac{|s|^2}{\epsilon_k^2}, \quad \forall (n,s) \in Z^m \times R \text{ with } |s| \geq S_k. \tag{2.14}$$

For any $k \in N$ and $u \in Y_k$ with $\|u\| \geq S_k/\epsilon_k$, by (2.13), (2.14), and the fact $F_n(s) \geq 0$, we have

$$\begin{aligned}
\Phi_\lambda(u) &\leq \frac{1}{2}\|u^+\|^2 - \sum_{n \in Z^m} F_n(u_n) \\
&\leq \frac{1}{2}\|u\|^2 - \sum_{n \in \{n \in Z^m : |u_n| \geq \epsilon_k \|u\|\}} \frac{|u_n|^2}{\epsilon_k^2} \\
&\leq \frac{1}{2}\|u\|^2 - \frac{\epsilon_k^2\|u\|^2}{\epsilon_k^2} \cdot \sharp\bigl(\{n \in Z^m : |u_n| \geq \epsilon \|u\|\}\bigr) \\
&\leq \frac{1}{2}\|u\|^2 - \|u\|^2 = -\frac{1}{2}\|u\|^2, \quad \forall \lambda \in [1,2].
\end{aligned} \tag{2.15}$$

Now for any $k \in N$, if we choose

$$r_k > \max\{\rho_k, S_k/\epsilon_k\},$$

then (2.15) implies that

$$\beta_k(\lambda) := \max_{u\in Y_k,\|u\|=r_k} \Phi_\lambda(u) \le -r_k^2/2 < 0, \quad \forall k \in N.$$

Therefore, the proof is finished. □

3 Proof of the main result

Proof of Theorem 1.1 It is easy to check that (F_1) of Lemma 2.2 holds. Besides, (F_2) and (F_3) hold for all $k \ge k_3$ by Lemmas 2.3 and 2.4. Thus Lemma 2.2 implies that for any $k \ge k_3$ and a.e. $\lambda \in [1,2]$ there exists a sequence $\{u_i^k(\lambda)\}_{i=1}^\infty \subset E$ such that

$$\sup_i \|u_i^k(\lambda)\| < \infty, \qquad \Phi'_\lambda\bigl(u_i^k(\lambda)\bigr) = 0 \quad \text{and} \quad \Phi_\lambda\bigl(u_i^k(\lambda)\bigr) \to \zeta_k(\lambda) \quad \text{as } i \to \infty, \tag{3.1}$$

where

$$\zeta_k(\lambda) := \inf_{\gamma\in\Gamma_k} \max_{u\in B_k} \Phi_\lambda\bigl(\gamma(u)\bigr), \quad \forall \lambda \in [1,2],$$

with $B_k := \{u \in Y_k : \|u\| \le r_k\}$ and $\Gamma_k := \{\gamma \in C(B_k, E) | \gamma \text{ is odd}, \gamma|_{\partial B_k} = id\}$. Furthermore, it follows from the proof of Lemma 2.4 that

$$\zeta_k(\lambda) \in [\overline{\alpha}_k, \overline{\zeta}_k], \quad \forall k \ge k_3, \tag{3.2}$$

where $\overline{\zeta}_k := \max_{u\in B_k} \Phi_1(u)$ and $\overline{\alpha}_k := \rho_k^2/4 \to \infty$ as $k \to \infty$ by (2.12). By (3.1), for each $k \ge k_3$, there exist $\lambda_j \to 1$ as $j \to \infty$ and $\{u_i^k(\lambda_j)\}_{i=1}^\infty \subset E$ such that

$$\sup_i \|u_i^k(\lambda_j)\| < \infty, \qquad \Phi'_{\lambda_j}\bigl(u_i^k(\lambda_j)\bigr) = 0 \quad \text{and} \quad \Phi_{\lambda_j}\bigl(u_i^k(\lambda_j)\bigr) \to \zeta_k(\lambda_j) \quad \text{as } i \to \infty. \tag{3.3}$$

Claim 1 $\{u_i^k(\lambda_j)\}_{i=1}^\infty$ *in* (3.3) *has a strong convergent subsequence.*

Proof Note that $\sup_i \|u_i^k(\lambda_j)\| < \infty$ for each $k \ge k_3$, without loss of generality, we may assume

$$\bigl(u_i^k(\lambda_j)\bigr)^- \to \bigl(u_j^k\bigr)^-, \qquad \bigl(u_i^k(\lambda_j)\bigr)^0 \to \bigl(u_j^k\bigr)^0 \quad \text{and} \quad \bigl(u_i^k(\lambda_j)\bigr)^+ \rightharpoonup \bigl(u_j^k\bigr)^+ \quad \text{as } i \to \infty, \forall j \in N, \tag{3.4}$$

for some $u_j^k = (u_j^k)^- + (u_j^k)^0 + (u_j^k)^+ \in E = E^- + E^0 + E^+$ since $\dim(E^- \oplus E^0) < \infty$. By virtue of the Riesz representation theorem, $\Phi'_{\lambda_j} : E \to E^*$ and $I' : E \to E^*$ can be viewed as Φ'_{λ_j} :

$E \to E$ and $I' : E \to E$, respectively, where E^* is the dual space of E. Note that (3.3) implies that for each $k \geq k_3$

$$0 = \Phi'_{\lambda_j}\big(u_i^k(\lambda_j)\big) = \big(u_i^k(\lambda_j)\big)^+ - \lambda_j\big[\big(u_i^k(\lambda_j)\big)^- + I'\big(u_i^k(\lambda_j)\big)\big], \quad \forall i,j \in N,$$

that is,

$$\big(u_i^k(\lambda_j)\big)^+ = \lambda_j\big[\big(u_i^k(\lambda_j)\big)^- + I'\big(u_i^k(\lambda_j)\big)\big], \quad \forall i,j \in N. \tag{3.5}$$

By the standard argument (see [25, 26]), we know $I' : E \to E^*$ is compact. Therefore, $I' : E \to E$ is also compact. It follows from (3.4) and (3.5) that the right-hand side of (3.5) converges strongly in E. Combining this with (3.4), we have

$$\lim_{i\to\infty} u_i^k(\lambda_j) = u_j^k \in E, \quad \forall j \in N \text{ and } k \geq k_3. \tag{3.6}$$

So Claim 1 is true. □

By (3.2), (3.3), and (3.6), we have

$$\Phi'_{\lambda_j}\big(u_j^k\big) = 0 \quad \text{and} \quad \Phi_{\lambda_j}\big(u_j^k\big) \in [\overline{\alpha}_k, \overline{\zeta}_k], \quad \forall j \in N \text{ and } k \geq k_3. \tag{3.7}$$

In fact, we can see $\{u_j^k\}_{j=1}^{\infty}$ is bounded in E, which will be proved in the appendix. Besides, by a similar proof to Claim 1, we can also see that $\{u_j^k\}_{j=1}^{\infty}$ possesses a strong convergent subsequence in E for all $k \geq k_3$. Without loss of generality, we may assume

$$u_j^k \to u^k \quad \text{as } j \to \infty, \forall k \geq k_3.$$

For each $k \geq k_3$, by (3.7), the limit u^k is just a critical point of $\Phi = \Phi_1$ with $\Phi(u^k) \in [\overline{\alpha}_k, \overline{\zeta}_k]$. Since $\overline{\alpha}_k \to \infty$ as $k \to \infty$ in (3.2), we get infinitely many nontrivial critical points of Φ. Therefore, we see that problem (1.1) possesses infinitely many nontrivial homoclinic solutions. The proof of Theorem 1.1 is completed. □

Appendix

Fact 1 *The result* (2.2) *holds.*

Proof If not, for any $j \in N$, there exists $u^j \in H\backslash\{0\}$ such that

$$\sharp\big(\big\{n \in Z^m : \big|u_n^j\big| \geq \big\|u^j\big\|/j\big\}\big) = 0.$$

Let $v^j := \frac{u^j}{\|u^j\|} \in H$, then $\|v^j\| = 1$ and

$$\sharp\big(\big\{n \in Z^m : \big|v_n^j\big| \geq 1/j\big\}\big) = 0, \quad \forall j \in N. \tag{A.1}$$

Note that since H is finite dimensional, passing to a subsequence if necessary, we may assume $v^j \to v$ in E for some $v \in H$. Evidently, $\|v\| = 1$. Note that any two norms on H are equivalent, thus by Lemma 2.1(1), we have

$$\|v^j - v\|_{l^2}^2 = \sum_{n\in Z^m} |v_n^j - v_n|^2 \to 0 \quad \text{as } j \to \infty. \tag{A.2}$$

The fact that $\|v\| = 1$ implies $\|v\|_{l^\infty} = \max_{n\in Z^m} |v_n| > 0$. By the definition of the norm $\|\cdot\|_{l^\infty}$, there exists a constant $\delta_0 > 0$ such that

$$\sharp\big(\{n \in Z^m : |v_n| \ge \delta_0\}\big) \ge 1. \tag{A.3}$$

For any $j \in N$, let

$$\Lambda_j := \{n \in Z^m : |v_n^j| < 1/j\} \quad \text{and} \quad \Lambda_j^c := Z^m \setminus \Lambda_j = \{n \in Z^m : |v_n^j| \ge 1/j\}.$$

Set $\Lambda_0 := \{n \in Z^m : |v_n| \ge \delta_0\}$. Then for j large enough, by (A.1) and (A.3), we have

$$\sharp(\Lambda_j \cap \Lambda_0) \ge \sharp(\Lambda_0) - \sharp(\Lambda_j^c) \ge 1 - 0 = 1.$$

It follows from the definitions of Λ_j and Λ_0 that for j large enough we have

$$\begin{aligned}\sum_{n\in Z^m} |v_n^j - v_n|^2 &\ge \sum_{n\in\Lambda_j\cap\Lambda_0} |v_n^j - v_n|^2 \\ &\ge \sum_{n\in\Lambda_j\cap\Lambda_0} |v_n|\big(|v_n| - 2|v_n^j|\big) \\ &\ge \delta_0(\delta_0 - 2/j) \cdot \sharp(\Lambda_j \cap \Lambda_0) \\ &\ge \delta_0^2/2 > 0.\end{aligned}$$

This is in contradiction to (A.2). Therefore, (2.2) holds. □

Fact 2 *The result* (2.8) *holds.*

Proof It is clear that $0 < l_\nu(k+1) \le l_\nu(k)$, so that $l_\nu(k) \to l \ge 0$ as $k \to \infty$. For every $k \ge 0$, there exists $u^k \in Z_k$ such that $\|u^k\| = 1$ and $\|u^k\|_{l^\nu} > l_\nu(k)/2$. By the definition of Z_k, $u^k \rightharpoonup 0$ in E, then $u^k \to 0$ in l^ν dues to Lemma 2.1(1). Therefore, we have $l = 0$, that is, $l_\nu(k) \to 0$. Similarly, $l_2(k) \to 0$. Therefore, (2.8) holds. □

Fact 3 $\{u_j^k\}_{j=1}^\infty$ *is bounded in E.*

Proof Note that (F_3) implies that there exists a constant $L_0 > 0$ such that

$$\frac{1}{2} f_n(s)s - F_n(s) \ge \frac{b}{4}|s|^\varrho, \quad \forall (n,s) \in Z^m \times R \text{ with } |s| \ge L_0. \tag{A.4}$$

For notational simplicity, we will set

$$u_j := u_j^k, \quad \forall j \in N \text{ and } k \ge k_3$$

throughout this paragraph. Note that (F_4) implies $\frac{1}{2}f_n(s)s - F_n(s) \geq 0$ for all $(n,s) \in Z^m \times R$, it follows from (3.7), (A.4), and the definition of Φ_λ that

$$\begin{aligned}
\Phi_{\lambda_j}(u_j) &= \Phi_{\lambda_j}(u_j) - \frac{1}{2}\langle \Phi'_{\lambda_j}(u_j), u_j\rangle \\
&= \lambda_j \sum_{n\in Z^m}\left[\frac{1}{2}f_n(u_{j,n})u_{j,n} - F_n(u_{j,n})\right] \\
&\geq \sum_{n\in\Pi_j}\left[\frac{1}{2}f_n(u_{j,n})u_{j,n} - F_n(u_{j,n})\right] \\
&\geq \frac{b}{4}\sum_{n\in\Pi_j}|u_{j,n}|^{\varrho}, \quad \forall j \in N,
\end{aligned} \tag{A.5}$$

where $\Pi_j := \{n \in Z^m : |u_{j,n}| \geq L_0\}$. It follows from (3.7) that

$$\sum_{n\in\Pi_j}|u_{j,n}|^{\varrho} \leq D_1, \quad \forall j \in N, \tag{A.6}$$

for some $D_1 > 0$. Note that (F_4) implies that there exists a constant $L_1 \in (0, L_0)$ such that

$$\frac{1}{2}f_n(s)s - F_n(s) \geq \frac{a_2}{4}|s|^{\iota}, \quad \forall (n,s) \in Z^m \times R \text{ with } |s| \leq L_1. \tag{A.7}$$

Similar to (A.5), by (A.7), (F_1), and the fact $\frac{1}{2}f_n(s)s - F_n(s) > 0$ if $s \neq 0$ (see (F_4)), we get

$$\begin{aligned}
\Phi_{\lambda_j}(u_j) &= \lambda_j \sum_{n\in Z^m}\left[\frac{1}{2}f_n(u_{j,n})u_{j,n} - F_n(u_{j,n})\right] \\
&\geq \sum_{n\in Z^m\setminus\Pi_j}\left[\frac{1}{2}f_n(u_{j,n})u_{j,n} - F_n(u_{j,n})\right] \\
&= \sum_{\{n\in Z^m:|u_{j,n}|\leq L_1\}}\left[\frac{1}{2}f_n(u_{j,n})u_{j,n} - F_n(u_{j,n})\right] \\
&\quad + \sum_{\{n\in Z^m:L_1\leq|u_{j,n}|<L_0\}}\left[\frac{1}{2}f_n(u_{j,n})u_{j,n} - F_n(u_{j,n})\right] \\
&\geq \sum_{\{n\in Z^m:|u_{j,n}|\leq L_1\}}\frac{a_2}{4}|u_{j,n}|^{\iota} + \sum_{\{n\in Z^m:L_1\leq|u_{j,n}|<L_0\}}D_2|u_{j,n}|^{\iota} \\
&\geq \sum_{\{n\in Z^m:|u_{j,n}|<L_0\}}D_3|u_{j,n}|^{\iota} = \sum_{n\in Z^m\setminus\Pi_j}D_3|u_{j,n}|^{\iota}, \quad \forall j \in N,
\end{aligned}$$

for some $D_2, D_3 > 0$. It follows from (3.7) that

$$\sum_{n\in Z^m\setminus\Pi_j}|u_{j,n}|^{\iota} \leq D_4, \quad \forall j \in N, \tag{A.8}$$

for some $D_4 > 0$. For any $j \in N$, let $\chi_j : Z^m \to R$ be the indicator of Π_j, that is,

$$\chi_{j,n} := \begin{cases} 1, & dn \in \Pi_j, \\ 0, & n \notin \Pi_j, \end{cases} \quad \forall j \in N.$$

Then by (A.6) and the definitions of Π_j and χ_j, we have

$$\|(1-\chi_j)u_j\|_{l^\infty} \le L_0 \quad \text{and} \quad \|\chi_j u_j\|_{l^\varrho}^\varrho = \sum_{n\in\Pi_j} |u_{j,n}|^\varrho \le D_1, \quad \forall j \in N.$$

It follows from the equivalence of any two norms on the finite-dimensional space $E^0 \oplus E^-$ and Hölder's inequality that

$$\begin{aligned}
\|u_j^- + u_j^0\|_{l^2}^2 &= (u_j^- + u_j^0, u_j)_{l^2} \\
&= (u_j^- + u_j^0, (1-\chi_j)u_j)_{l^2} + (u_j^- + u_j^0, \chi_j u_j)_{l^2} \\
&\le \|(1-\chi_j)u_j\|_{l^\infty} \cdot \|u_j^- + u_j^0\|_{l^1} + \|\chi_j u_j\|_{l^\varrho} \cdot \|u_j^- + u_j^0\|_{l^{\varrho'}} \\
&\le L_0 c_1 \|u_j^- + u_j^0\|_{l^2} + D_1^{1/\varrho} c_2 \|u_j^- + u_j^0\|_{l^2} \\
&= (L_0 c_1 + D_1^{1/\varrho} c_2) \|u_j^- + u_j^0\|_{l^2}, \quad \forall j \in N,
\end{aligned}$$

for some $c_1, c_2 > 0$, where $\varrho' := \frac{\varrho}{\varrho - 1}$. Consequently, we get

$$\|u_j^- + u_j^0\|_{l^2} \le L_0 c_1 + D_1^{1/\varrho} c_2, \quad \forall j \in N.$$

By the equivalence of norms $\|\cdot\|_{l^2}$ and $\|\cdot\|$ on $E^0 \oplus E^-$, we know there exists $c_3 > 0$ such that

$$\|u_j^- + u_j^0\| \le c_3, \quad \forall j \in N. \tag{A.9}$$

Therefore,

$$\|u_j^-\| \le c_3, \quad \forall j \in N. \tag{A.10}$$

By the definition of Φ_{λ_j}, we have

$$\|u_j^+\|^2 = 2\Phi_{\lambda_j}(u_j) + \lambda_j \|u_j^-\|^2 + 2\lambda_j \sum_{n\in Z^m} F_n(u_{j,n}), \quad \forall j \in N. \tag{A.11}$$

Note that (F_1) implies that for any $\varepsilon > 0$ there exists C_ε such that

$$|F_n(s)| \le \varepsilon |s|^2 + C_\varepsilon |s|^\nu, \quad \forall (n,s) \in Z^m \times R. \tag{A.12}$$

Therefore, by (3.7), (A.8)-(A.12), and the Sobolev embedding theorem we have

$$\begin{aligned}
\|u_j\|^2 &= \|u_j^- + u_j^0\|^2 + \|u_j^+\|^2 \\
&= \|u_j^- + u_j^0\|^2 + 2\Phi_{\lambda_j}(u_j) + \lambda_j \|u_j^-\|^2 + 2\lambda_j \sum_{n\in Z^m} F_n(u_{j,n}) \\
&\le c_4 + 2\lambda_j \sum_{n\in Z^m} (\varepsilon |u_{j,n}|^2 + C_\varepsilon |u_{j,n}|^\nu) \\
&= c_4 + 2\lambda_j \varepsilon \|u_j\|_{l^2}^2 + 2\lambda_j C_\varepsilon \left(\sum_{n\in Z^m \setminus \Pi_j} |u_{j,n}|^\nu + \sum_{n\in\Pi_j} |u_{j,n}|^\nu \right)
\end{aligned}$$

$$\leq c_4 + c_5\varepsilon\|u_j\|^2 + c_6 L_0^{\nu-\iota} \sum_{n\in Z^m\setminus\Pi_j} |u_{j,n}|^\iota + c_6 \sum_{n\in\Pi_j} |u_{j,n}|^\nu$$

$$\leq c_4 + c_5\varepsilon\|u_j\|^2 + c_6 L_0^{\nu-\iota} D_4 + c_6 \sum_{n\in\Pi_j} |u_{j,n}|^\nu, \quad \forall j \in N, \tag{A.13}$$

for some $c_4, c_5, c_6 > 0$, where $\iota \leq \nu$ is defined in (F_4). If $\nu - \varrho \geq 0$, by (A.6) and Lemma 2.1(1), we have

$$\sum_{n\in\Pi_j} |u_{j,n}|^\nu \leq \|u_j\|_{l^\infty}^{\nu-\varrho} \sum_{n\in\Pi_j} |u_{j,n}|^\varrho \leq D_1 c_7 \|u_j\|^{\nu-\varrho}, \quad \forall j \in N, \tag{A.14}$$

for some $c_7 > 0$. If $\nu - \varrho < 0$, by (A.6) and the definition of Π_j, we have

$$\sum_{n\in\Pi_j} |u_{j,n}|^\nu = \sum_{n\in\Pi_j} \frac{|u_{j,n}|^\varrho}{|u_{j,n}|^{\varrho-\nu}} \leq \frac{1}{L_0^{\varrho-\nu}} \sum_{n\in\Pi_j} |u_{j,n}|^\varrho \leq \frac{D_1}{L_0^{\varrho-\nu}}, \quad \forall j \in N. \tag{A.15}$$

Note that $\nu - \varrho < 2$ (see (F_3)) and $\varepsilon > 0$ is arbitrary, thus it follows from (A.13)-(A.15) that $\{u_j\}$ is bounded in E. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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