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Bilinear Bäcklund transformations and explicit solutions of a (3 + 1)-dimensional nonlinear equation

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Abstract

Binary Bell polynomials are applied to construct two kinds of bilinear derivative equations of a (3 + 1)-dimensional nonlinear equation. Based on one of the bilinear forms, we derive a Bäcklund transformation, the corresponding Lax pair, infinite conservation laws, and explicit solutions with an arbitrary function in *y*. In the meantime, from the other bilinear form, we get another bilinear Bäcklund transformation and exact solutions by utilizing the exchange formulas for Hirota's bilinear operators.

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Keywords: binary Bell polynomial; solitary wave solution; Bäcklund transformation

1 Introduction

Nonlinear evolution equations (NLEEs) have attracted intensive attention in the past few decades, since they can model many important phenomena and dynamic processes in physics, mechanics, chemistry, and biology [1, 2]. The study of solutions for NLEEs is very important in nonlinear physical phenomena and many effective methods have been used. Numerical methods include the decomposition method [3], the iteration method [4], and the spectral method [5, 6] are developed in recent years. Various effective analytic methods have been used to explore different kinds of solutions of NLEEs, such as the inverse scattering method [7], the Darboux transformation [8, 9], the Bäcklund transformation [10–12], the Hirota method [13, 14], the algebra-geometric method [15–17], the homotopy analysis method [18], and so on. Among the above methods, the Hirota method is a powerful and direct approach to construct exact solutions of NLEEs. Once a nonlinear equation is written in bilinear form, its multi-soliton solutions and rational solutions are usually obtained in a systematic way. Unfortunately, this method relies on particular skills, complex calculation, and suitable variable transformation. In recent years, Lambert, Gilson *et al.* proposed an effective method based on the use of the Bell polynomials to obtain bilinear Bäcklund transformation and Lax pair for soliton equations in a direct way [19–21]. Fan developed this method to find bilinear Bäcklund transformations, Lax pairs, infinite conservation laws of nonisospectral and variable-coefficient KdV, KP equations [22, 23]. Wang applied the binary Bell polynomials to construct bilinear forms, bilinear

Bäcklund transformations, Lax pairs of a generalized (2 + 1)-dimensional Korteweg-de Vries equation, and the modified generalized Vakhnenko equation [24, 25].

The Bäcklund transformation has been proven to be another powerful approach to obtain explicit solutions of NLEEs. It is well known that the Bäcklund transformation is a transformation that is related to a pair of solutions of a nonlinear equation. The exchange property of Hirota's bilinear operators is found to be an effective tool in deriving a Bäcklund transformation for a given equation [14]. Wu recently constructed a bilinear Bäcklund transformation for a (3 + 1)-dimensional soliton equation by using the Hirota bilinear method and the exchange property of the D-operators [26]. Ma extended this method to a (3 + 1)-dimensional generalized KP equation, and obtained its bilinear Bäcklund transformation and explicit solutions [27].

In this paper, we would like to study the following (3 + 1)-dimensional nonlinear equation:

$$u_{xxxy} - 3(u_x u_y)_x - 2u_{yt} + 3u_{yz} = 0. \tag{1.1}$$

If we take $z = t$, equation (1.1) is reduced to the Boiti-Leon-Manna-Pempinelli equation [28–32]

$$u_{xxxy} + u_{yt} - 3u_{xx}u_y - 3u_x u_{xy} = 0. \tag{1.2}$$

So, equation (1.1) is a generalization of the BLMP equation. The main goal of this paper is twofold. First of all, we will apply the binary Bell polynomials to construct two bilinear forms of equation (1.1). Further, from one of the bilinear forms, we will derive a Bäcklund transformation, the corresponding Lax pair, infinite conservation laws, and explicit solutions of equation (1.1). Second, based on the other bilinear form, we would like to construct another bilinear Bäcklund transformation and exact solutions of equation (1.1) by utilizing the exchange formulas for Hirota's bilinear operators.

This paper is organized as follows. In Section 2, we give a brief introduction of the binary Bell polynomials. In Section 3, we construct two bilinear forms of equation (1.1). Further, based on one of the bilinear forms, exact solutions are obtained by using the Hirota method. In Section 4, the bilinear Bäcklund transformation, the corresponding Lax pair, and infinite conservation laws are obtained by using the binary Bell polynomials. In Section 5, another bilinear Bäcklund transformation and traveling wave solutions of equation (1.1) are derived by using the exchange formulas of Hirota's bilinear operators. Finally, some conclusions are given in the last section.

2 Binary Bell polynomials

To begin with, we will briefly introduce some basic concepts and notations of the Bell polynomials. For details, refer to [19–21].

Let $f = f(x_1, x_2, \ldots, x_n)$ be a C^∞ function with n independent variables, the multi-dimensional Bell polynomials (Y-polynomials) are defined as follows:

$$\begin{aligned} Y_{n_1 x_1,\ldots,n_l x_l}(f) &= Y_{n_1,\ldots,n_l}(f_{r_1 x_1,\ldots,r_l x_l}) \\ &= \exp(-f)\partial_{x_1}^{n_1} \cdots \partial_{x_l}^{n_l} \exp(f), \end{aligned} \tag{2.1}$$

where

$$f_{r_1x_1,\ldots,r_lx_l} = \partial_{x_1}^{r_1}\cdots\partial_{x_l}^{r_l}f, \quad r_1 = 0,\ldots,n_1, r_l = 0,\ldots,n_l.$$

Based on the multi-dimensional Bell polynomials, the multi-dimensional binary Bell polynomials ($\mathcal{Y}$-polynomials) can be defined as follows:

$$\begin{aligned}\mathcal{Y}_{n_1x_1,\ldots,n_lx_l}(v,w) &= Y_{n_1x_1,\ldots,n_lx_l}(f)|_{f_{r_1x_1,\ldots,r_lx_l}} \\ &= \begin{cases} v_{r_1x_1,\ldots,r_lx_l}, & r_1+\cdots+r_l \text{ is odd,} \\ w_{r_1x_1,\ldots,r_lx_l}, & r_1+\cdots+r_l \text{ is even.}\end{cases}\end{aligned}$$

According to the above definitions, the first few lowest order binary Bell polynomials are

$$\begin{aligned}&y_x(v) = v_x, \qquad y_{2x}(v,w) = w_{xx} + v_x^2, \\ &y_{x,t}(v,w) = w_{x,t} + v_xv_t, \\ &y_{3x}(v,w) = v_{3x} + 3v_xw_{2x} + v_x^3, \\ &y_{4x}(v,w) = w_{4x} + 3w_{2x}^2 + 4v_xv_{3x} + 6v_x^2w_{2x} + v_x^4, \\ &\ldots.\end{aligned} \tag{2.2}$$

Proposition 1 *The link between binary Bell polynomials $\mathcal{Y}_{n_1x_1,\ldots,n_lx_l}(v,w)$ and the standard Hirota bilinear equation $D_{x_1}^{n_1}\cdots D_{x_l}^{n_l}F\cdot G$ can be given by the identity*

$$\mathcal{Y}_{n_1x_1,\ldots,n_lx_l}(v = \ln F/G, w = \ln FG) = (FG)^{-1}D_{x_1}^{n_1}\cdots D_{x_l}^{n_l}F\cdot G, \tag{2.3}$$

in which $n_1 + n_2 + \cdots + n_l \geq 1$, and operators $D_{x_1}\cdots D_{x_l}$ are classical Hirota's bilinear operators defined by

$$\begin{aligned}D_{x_1}^{n_1}\cdots D_{x_l}^{n_l}F\cdot G &= (\partial_{x_1} - \partial_{x_1'})^{n_1}\cdots(\partial_{x_l} - \partial_{x_l'})^{n_l}F(x_1,\ldots,x_l) \\ &\quad\times G(x_1',\ldots,x_l')|_{x_1'=x_1,\ldots,x_l'=x_l}.\end{aligned}$$

In the particular case when $F = G$, equation (2.3) *becomes*

$$\begin{aligned}G^{-2}D_{x_1}^{n_1}\cdots D_{x_l}^{n_l}G\cdot G &= \mathcal{Y}_{n_1x_1,\ldots,n_lx_l}(0, q = 2\ln G) \\ &= \begin{cases} 0, & n_1+\cdots+n_l \text{ is odd,} \\ P_{n_1x_1,\ldots,n_lx_l}(q), & n_1+\cdots+n_l \text{ is even,}\end{cases}\end{aligned} \tag{2.4}$$

which is also called the case of P-polynomials,

$$P_{n_1x_1,\ldots,n_lx_l}(q) = \mathcal{Y}_{n_1x_1,\ldots,n_lx_l}(0, q = 2\ln G).$$

The first few P-polynomials are

$$\begin{aligned}&P_{2x}(q) = q_{2x}, \qquad P_{x,t}(q) = q_{xt}, \qquad P_{3xy}(q) = q_{3xy} + 3q_{2x}q_{xy}, \\ &P_{6x}(q) = q_{6x} + 15q_{2x}q_{4x} + 15q_{2x}^3, \qquad \ldots.\end{aligned} \tag{2.5}$$

Equations (2.3) *and* (2.4) *play an important role in connecting bilinear equations with the corresponding NLEEs. Once an NLEE is written in the form of a combination of the $\mathcal{Y}$-polynomials, then its bilinear form can easily be obtained.*

Proposition 2 *The binary Bell polynomials $\mathcal{Y}_{n_1x_1,\ldots,n_lx_l}(v,w)$ can be separated into P-polynomials and Y-polynomials,*

$$\begin{aligned}
&(FG)^{-1}D_{x_1}^{n_1}\cdots D_{x_l}^{n_l}F\cdot G \\
&\quad= \mathcal{Y}_{n_1x_1,\ldots,n_lx_l}(v,w)|_{v=\ln F/G,w=\ln FG} \\
&\quad= \mathcal{Y}_{n_1x_1,\ldots,n_lx_l}(v,v+q)|_{v=\ln F/G,q=2\ln G} \\
&\quad= \sum_{n_1+\cdots+n_l=\text{even}}\sum_{r_1=0}^{n_1}\cdots\sum_{r_l=0}^{n_l}\prod_{i=1}^{l}\binom{n_i}{r_i}P_{r_1x_1,\ldots,r_lx_l}(q) \\
&\qquad\times Y_{(n_1-r_1)x_1,\ldots,(n_l-r_l)x_l}(v).
\end{aligned} \tag{2.6}$$

Under the Hopf-Cole transformation $v=\ln\psi$, that is, $\psi=F/G$, multi-dimensional binary Bell polynomials $\mathcal{Y}_{n_1x_1,\ldots,n_lx_l}(v)$ can be linearized into the following form:

$$\mathcal{Y}_{n_1x_1,\ldots,n_lx_l}|_{v=\ln\psi} = \psi_{n_1x_1,\ldots,n_lx_l}/\psi. \tag{2.7}$$

Equations (2.6) *and* (2.7) *provide a direct way to the associated Lax system of NLEEs.*

3 Bilinear forms and exact solutions of equation (1.1)

Consider the generalized-BLMP equation (1.1), setting

$$u=-q_x, \tag{3.1}$$

where q is the function of x, y, z, t. Substituting equation (3.1) into equation (1.1) and integrating with respect to x yields

$$q_{3xy}+3q_{xx}q_{xy}-2q_{yt}+3q_{yz}=0. \tag{3.2}$$

Now according to the P-polynomials (2.5), equation (3.2) can be written as

$$P_{3xy}-2P_{yt}+3P_{yz}=0. \tag{3.3}$$

Making a change of the dependent variable,

$$q=2\ln f \quad\Longleftrightarrow\quad u=-q_x=-2(\ln f)_x,$$

and noting equation (2.4), we can get the following bilinear form:

$$\left(D_x^3D_y-2D_yD_t+3D_yD_z\right)f\cdot f=0. \tag{3.4}$$

Assume that

$$u=-q_x-\Phi(y), \tag{3.5}$$

where $\Phi(y)$ is an arbitrary differential function of y. Substituting equation (3.5) into equation (1.1) and integrating with respect to x, the generalized-BLMP equation can be written as

$$q_{3xy} + 3q_{xx}q_{xy} + 3\varphi(y)q_{xx} - 2q_{yt} + 3q_{yz} = 0, \tag{3.6}$$

where $\varphi(y) = \Phi'(y)$. Equation (3.6) can be cast into a linear combination form of the P-polynomials,

$$P_{3xy} + 3\varphi(y)P_{xx} - 2P_{yt} + 3P_{yz} = 0. \tag{3.7}$$

Setting

$$q = 2\ln f \quad \Longleftrightarrow \quad u = -2(\ln f)_x - \Phi(y),$$

by using equation (2.4) which connects the $\mathcal{Y}$-polynomials with the Hirota operators, we can get another bilinear representation of equation (1.1),

$$\big(D_x^3 D_y + 3\varphi(y)D_x^2 - 2D_yD_t + 3D_yD_z\big)f \cdot f = 0. \tag{3.8}$$

Next, we would like to construct exact solutions for the generalized-BLMP equation (1.1) with the aid of the Hirota method. Noting that the bilinear form (3.8) contains an arbitrary function $\varphi(y) = \Phi'(y)$, we can get some particular solutions of equation (1.1) provided that $\Phi(y)$ is appropriately chosen. So we will compute soliton solutions for equation (1.1) via equation (3.8).

Expand f with respect to a formal parameter ϵ as follows:

$$f = 1 + f^{(1)}\epsilon + f^{(2)}\epsilon^2 + f^{(3)}\epsilon^3 + \cdots, \tag{3.9}$$

where $f^{(i)}$ $(i = 1, 2, \ldots)$ are the functions of x, y, z, and t. Substituting equation (3.9) into (3.8) and collecting the coefficients of each order of ϵ yields

$$f^{(1)}_{xxxy} + 3\varphi(y)f^{(1)}_{xx} - 2f^{(1)}_{yt} + 3f^{(1)}_{yz} = 0, \tag{3.10a}$$

$$\begin{aligned} &2\big(f^{(2)}_{xxxy} + 3\varphi(y)f^{(2)}_{xx} - 2f^{(2)}_{yt} + 3f^{(2)}_{yz}\big) \\ &\quad = -\big(D_x^3D_y + 3\varphi(y)D_x^2 - 2D_yD_t + 3D_yD_z\big)f^{(1)} \cdot f^{(1)}, \end{aligned} \tag{3.10b}$$

$$\begin{aligned} &f^{(3)}_{xxxy} + 3\varphi(y)f^{(3)}_{xx} - 2f^{(3)}_{yt} + 3f^{(3)}_{yz} \\ &\quad = -\big(D_x^3D_y + 3\varphi(y)D_x^2 - 2D_yD_t + 3D_yD_z\big)f^{(1)} \cdot f^{(2)}, \end{aligned} \tag{3.10c}$$

$$\ldots.$$

From equations (3.10a)-(3.10c), we can get

$$f^{(1)} = e^{kx - \frac{3\Phi(y)}{k} + mz + \frac{3}{2}mt}, \tag{3.11}$$

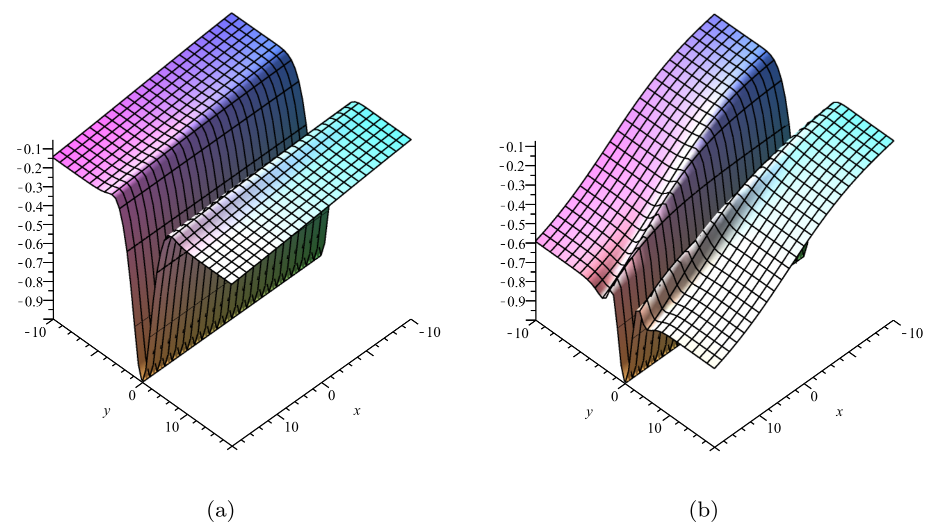

Figure 1 Solitary waves. (a) Bell-shaped solitary wave with $\Phi(y) = \mathrm{sech}(y)$, $t = 0$, $z = 0$, $k = 0.1$; **(b)** two bell-shaped solitary waves with $\Phi(y) = \mathrm{sech}(y)$, $t = 0$, $z = 0$, $k_1 = 0.1$, $k_2 = 0.3$.

and the one-soliton solution for equation (1.1) is denoted by

$$u = -2\partial_x\left[\ln\left(1 + e^{kx - \frac{3\Phi(y)}{k} + mz + \frac{3}{2}mt}\right)\right] - \Phi(y).$$

Noting that (3.10a) is a linear differential equation,

$$f^{(1)} = e^{\xi_1} + e^{\xi_2}, \quad \xi_j = k_j x - \frac{3\Phi(y)}{k_j} + m_j z + \frac{3}{2}m_j t \ (j = 1, 2) \tag{3.12}$$

is also a solution of equation (3.10a). Substituting equation (3.12) into equation (3.10b), one can readily obtain

$$f^{(2)} = e^{\xi_1 + \xi_2 + A_{12}}, \qquad e^{A_{12}} = \frac{(k_1 - k_2)^2(k_1^2 + k_2^2 - k_1 k_2)}{(k_1 + k_2)^2(k_1^2 + k_2^2 + k_1 k_2)}$$

and the two-soliton solution for equation (1.1) can be given as

$$u = -2\partial_x\left[\ln\left(1 + e^{\xi_1} + e^{\xi_2} + e^{\xi_1 + \xi_2 + A_{12}}\right)\right] - \Phi(y).$$

Two specific solutions of the above one-soliton and two-soliton solutions are plotted in Figures 1 and 2.

4 Bilinear Bäcklund transformation, associated Lax pair, and infinite conservation laws

In this section, we would like to construct the bilinear Bäcklund transformation, the Lax pair, and infinite conservation laws of equation (1.1) via the bilinear form (3.8).

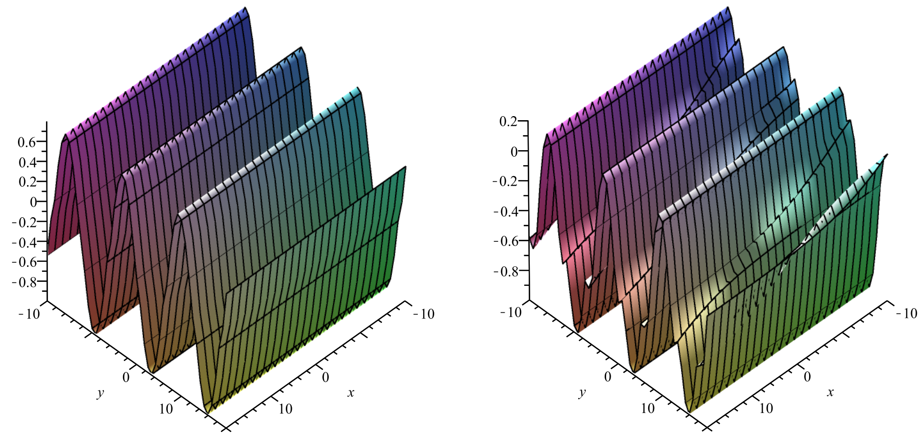

Figure 2 Solitary periodic waves. (a) Solitary periodic wave with $\Phi(y) = \sin(y)$, $t = 0$, $z = 0$, $k = 0.1$; **(b)** two solitary periodic waves with $\Phi(y) = \sin(y)$, $t = 0$, $z = 0$, $k_1 = 0.1$, $k_2 = 0.3$.

4.1 Bilinear Bäcklund transformation and associated Lax pair

Let $q' = 2\ln F$ and $q = 2\ln G$ be two different solutions of equation (3.6), we have the two-field condition

$$\begin{aligned} E(q') - E(q) = {} & (q'-q)_{3xy} + 3(q'_{xx}q'_{xy} - q_{xx}q_{xy}) + 3\varphi(y)(q'-q)_{2x} \\ & - 2(q'-q)_{yt} + 3(q'-q)_{yz}. \end{aligned} \tag{4.1}$$

If we set

$$v = (q'-q)/2 = \ln(F/G), \qquad w = (q'+q)/2 = \ln(FG), \tag{4.2}$$

then equation (4.1) can be rewritten as

$$\bigl(E(q') - E(q)\bigr)/2 = v_{3xy} + 3(w_{xx}v_{xy} + v_{2x}w_{xy}) + 3\varphi(y)v_{xx} - 2v_{yt} + 3v_{yz}. \tag{4.3}$$

Next, we need to impose a constraint to express equation (4.3) in the form of y-derivative of $\mathcal{Y}$-polynomials. A simple choice of such a constraint may be

$$\mathcal{Y}_{xy}(v,w) + \varphi(y) = 0, \tag{4.4}$$

from which we can get

$$v_x v_{xy} + v_y v_{xx} + w_{xxy} = 0. \tag{4.5}$$

Owing to (4.4) and (4.5), equation (4.3) can be written as

$$\bigl(E(q') - E(q)\bigr)/2 = \partial_y\bigl(\mathcal{Y}_{3x}(v,w) - 2\mathcal{Y}_t(v,w) + 3\mathcal{Y}_z(v,w)\bigr) = 0. \tag{4.6}$$

Thus, we deduce a coupled system of $\mathcal{Y}$-polynomials

$$\begin{aligned}&\mathcal{Y}_{xy}(v,w)+\varphi(y)=0,\\&\mathcal{Y}_{3x}(v,w)-2\mathcal{Y}_t(v,w)+3\mathcal{Y}_z(v,w)=0.\end{aligned}\tag{4.7}$$

By virtue of equation (2.3), one can immediately get the bilinear Bäcklund transformation of equation (1.1),

$$\begin{aligned}&\bigl(D_xD_y+\varphi(y)\bigr)F\cdot G=0,\\&\bigl(D_x^3-2D_t+3D_z\bigr)F\cdot G=0.\end{aligned}\tag{4.8}$$

For obtaining the corresponding Lax pair of system (4.7), one needs to make the Hopf-Cole transformation $v=\ln\psi$. It follows from equations (2.6) and (2.7) that

$$\begin{aligned}&\mathcal{Y}_t(v)=\psi_t/\psi,\qquad \mathcal{Y}_{xy}(v,w)=q_{xy}+\psi_{xy}/\psi,\\&\mathcal{Y}_z(v)=\psi_z/\psi,\qquad \mathcal{Y}_{3x}(v,w)=3q_{2x}\psi_x/\psi+\psi_{3x}/\psi.\end{aligned}\tag{4.9}$$

Then system (4.7) is linearized into a Lax pair

$$\begin{aligned}&L_1\psi=\psi_{xy}+\bigl(q_{xy}+\varphi(y)\bigr)\psi=0,\\&L_2\psi=\psi_{3x}+3q_{2x}\psi-2\psi_t+3\psi_z=0.\end{aligned}\tag{4.10}$$

It is easy to check that the integrability condition for (4.10)

$$[L_1,L_2]\psi=0\tag{4.11}$$

is satisfied if $u=-q_x-\Phi(y)$ is a solution of the generalized-BLMP equation (1.1).

4.2 Infinite conservation laws

To find infinite conservation laws for the generalized-BLMP equation (1.1), we would like to introduce a new potential function

$$\eta=\frac{q'_x-q_x}{2},$$

it follows from equation (4.2) that

$$v_x=\eta,\qquad w_x=q_x+\eta.\tag{4.12}$$

Substituting equation (4.12) into (4.4) yields

$$\eta_y+\eta\partial_x^{-1}\eta_y+q_{xy}+\varphi(y)=0.\tag{4.13}$$

Inserting the expansion

$$\eta=\varepsilon+\sum_{n=1}^{\infty}I^{(n)}(q,q_x,\ldots)\varepsilon^{-n}\tag{4.14}$$

into equation (4.13) and equating the coefficients for power of ε, we then get the recursion for $I^{(n)}$,

$$\begin{aligned} I^{(1)} &= -q_{2x} + \theta_1(x,z,t), \\ I^{(2)} &= q_{3x} + \theta_2(x,z,t), \\ I^{(n)} &= -I_x^{(n-1)} - \sum_{k=1}^{n-2} \partial_y^{-1}\partial_x\bigl(I^{(k)}\partial_x^{-1}I_y^{(n-1-k)}\bigr) \\ &\quad + \theta_n(x,z,t), \quad n \geq 3; \end{aligned} \tag{4.15}$$

here $\theta_n(x,z,t)$ $(n = 1,2,\ldots)$ are undetermined functions of x, z, t.

Rewrite equation (4.6) as the divergence-type form

$$\partial_t(-2v_y) + \partial_x(v_{xxy}) + \partial_y\bigl(3v_x w_{xx} + v_x^3\bigr) + \partial_z(3v_y) = 0. \tag{4.16}$$

Taking advantage of (4.12) and substituting expansion (4.14) into (4.16) leads to

$$\begin{aligned} &\partial_t\left(-2\sum_{n=1}^{\infty}\partial_x^{-1}I_y^{(n)}\varepsilon^{-n}\right) + \partial_x\left(\sum_{n=1}^{\infty}I_{xy}^{(n)}\varepsilon^{-n}\right) \\ &\quad + \partial_y\left[3\left(\varepsilon + \sum_{n=1}^{\infty}I^{(n)}\varepsilon^{-n}\right)\left(q_{xx} + \sum_{n=1}^{\infty}I_x^{(n)}\varepsilon^{-n}\right) + \left(\varepsilon + \sum_{n=1}^{\infty}I^{(n)}\varepsilon^{-n}\right)^3\right] \\ &\quad + \partial_z\left(3\sum_{n=1}^{\infty}\partial_x^{-1}I_y^{(n)}\varepsilon^{-n}\right) = 0. \end{aligned} \tag{4.17}$$

Comparing the coefficients of $\varepsilon^{-1}, \varepsilon^{-2}, \ldots$ yields the following infinite sequence of conservation laws:

$$F_t^{(n)} + G_x^{(n)} + H_y^{(n)} + J_z^{(n)}, \quad n = 1,2,\ldots. \tag{4.18}$$

In equation (4.18), $F^{(n)}$, $G^{(n)}$, $J^{(n)}$ are given by

$$F^{(n)} = -2\partial_x^{-1}I_y^{(n)}, \qquad G^{(n)} = I_{xy}^{(n)}, \qquad J^{(n)} = 3\partial_x^{-1}I_y^{(n)}, \quad n = 1,2,\ldots, \tag{4.19}$$

and $H^{(n)}$ are given by

$$\begin{aligned} H^{(1)} &= 3\bigl[I^{(1)}q_{xx} + I_x^{(2)} + \bigl(I^{(1)}\bigr)^2 + I^{(3)}\bigr], \\ H^{(2)} &= 3\bigl[I^{(2)}q_{xx} + I_x^{(3)} + I^{(1)}I_x^{(1)} + 2I^{(1)}I^{(2)} + I^{(4)}\bigr], \\ H^{(n)} &= 3\left(I_x^{(n+1)} + I^{(n)}q_{xx} + \sum_{k=1}^{n-1}I^{(k)}I_x^{(n-k)} + I^{(n+2)} + \sum_{k=1}^{n}I^{(k)}I^{(n+1-k)}\right), \\ &\quad + \sum_{k=1}^{n-1}\sum_{j=1}^{n-k-1}I^{(k)}I^{(j)}I^{(n-k-j)}, \quad n = 3,4,\ldots. \end{aligned} \tag{4.20}$$

Moreover, from the coefficients of ε^0 and ε, we can get

$$I^{(1)} = -q_{xx},$$
$$I_y^{(2)} = -I_{xy}^{(1)},$$

thus we have $\theta_1(x,z,t) = 0$ and $\theta_2(x,z,t)$ is an arbitrary function of x, z, t. From (4.19) and (4.20) we readily see that $\theta_n(x,z,t)$ $(n \geq 3)$ are also arbitrary functions of x, z, t.

5 Another bilinear Bäcklund transformation and traveling wave solutions

This section is contributed to construct another bilinear Bäcklund transformation and traveling wave solutions for equation (1.1) based on the bilinear form (3.4). Assume that f' is another solution of the generalized-BLMP equation (1.1):

$$\bigl(D_x^3 D_y - 2D_y D_t + 3D_y D_z\bigr) f' \cdot f' = 0. \tag{5.1}$$

Let us define a function

$$\begin{aligned} P = {} & \bigl[\bigl(D_x^3 D_y - 2D_y D_t + 3D_y D_z\bigr) f' \cdot f'\bigr] f^2 \\ & - \bigl[\bigl(D_x^3 D_y - 2D_y D_t + 3D_y D_z\bigr) f \cdot f\bigr] f'^2 = 0. \end{aligned} \tag{5.2}$$

Obviously, if $P = 0$, then f satisfies equation (3.4) if and only if f' satisfies the same equation. Therefore, if we can obtain from $P = 0$ a system of bilinear equations,

$$\begin{aligned} & B_1(D_t, D_x, D_y, D_z) f' \cdot f = 0, \\ & B_2(D_t, D_x, D_y, D_z) f' \cdot f = 0, \\ & \ldots, \\ & B_k(D_t, D_x, D_y, D_z) f' \cdot f = 0, \end{aligned} \tag{5.3}$$

where $B_1, B_2, \ldots, B_k$ are undetermined functions, then system (5.3) provides a bilinear Bäcklund transformation for the generalized-BLMP equation (1.1). To this end, we would like to apply the following three exchange formulas for Hirota's bilinear operators:

$$(D_t D_x a \cdot a) b^2 - (D_t D_x b \cdot b) a^2 = 2D_x (D_t a \cdot b) \cdot ba, \tag{5.4}$$

$$(D_t D_y a \cdot a) b^2 - (D_t D_y b \cdot b) a^2 = 2D_y (D_t a \cdot b) \cdot ba, \tag{5.5}$$

$$\begin{aligned} & 2\bigl(D_x^3 D_y a \cdot a\bigr) b^2 - 2\bigl(D_x^3 D_y b \cdot b\bigr) a^2 \\ & \quad = D_x \bigl[\bigl(3D_x^2 D_y a \cdot b\bigr) \cdot ba + \bigl(3D_x^2 a \cdot b\bigr) \cdot (D_y b \cdot a) + (6D_x D_y a \cdot b) \cdot (D_x b \cdot a)\bigr] \\ & \qquad + D_y \bigl[\bigl(D_x^3 a \cdot b\bigr) \cdot ba + \bigl(3D_x^2 a \cdot b\bigr) \cdot (D_x b \cdot a)\bigr]. \end{aligned} \tag{5.6}$$

Equations (5.4) and (5.5) can be found in [11], and equation (5.6) is obtained by Wu in [26]. From equations (5.4) and (5.5), we can get

$$\bigl(D_z^2 a \cdot a\bigr) b^2 - \bigl(D_z^2 b \cdot b\bigr) a^2 = 2D_z (D_z a \cdot b) \cdot ba, \tag{5.7}$$

$$D_r (D_s a \cdot b) \cdot ba = D_s (D_r a \cdot b) \cdot ba. \tag{5.8}$$

Then, by using equations (5.4)-(5.8), we can obtain

$$\begin{aligned}
2P &= \left[2\left(D_x^3 D_y f' \cdot f'\right)f^2 - 2\left(D_x^3 D_y f \cdot f\right)f'^2\right] - 4\left[\left(D_y D_t f' \cdot f'\right)f^2 \right.\\
&\quad \left. - \left(D_y D_t f \cdot f\right)f'^2\right] + 6\left[\left(D_y D_z f' \cdot f'\right)f^2 - \left(D_y D_z f \cdot f\right)f'^2\right]\\
&= D_x\left(3D_x^2 D_y f' \cdot f\right) \cdot ff' + D_x\left(3D_x^2 f' \cdot f\right) \cdot \left(D_y f \cdot f'\right)\\
&\quad + D_x\left(6D_x D_y f' \cdot f\right) \cdot \left(D_x f \cdot f'\right) + D_y\left[\left(D_x^3 - 8D_t + 12D_z\right)f' \cdot f\right] \cdot ff'\\
&\quad + D_y\left(3D_x^2 f' \cdot f\right) \cdot \left(D_x f \cdot f'\right).
\end{aligned} \tag{5.9}$$

By introducing seven new parameters λ_i ($i = 1, \ldots, 7$), equation (5.9) can be written as

$$\begin{aligned}
2P &= D_x\left(3D_x^2 D_y f' \cdot f + \lambda_1 D_y f' \cdot f + \lambda_2 f' f\right) \cdot ff'\\
&\quad + D_x\left(3D_x^2 f' \cdot f + \lambda_3 D_y f' \cdot f + \lambda_4 f' f\right) \cdot \left(D_y f \cdot f'\right)\\
&\quad + D_x\left(6D_x D_y f' \cdot f + \lambda_5 D_x f' \cdot f\right) \cdot \left(D_x f \cdot f'\right)\\
&\quad + D_y\left[\left(D_x^3 - 8D_t + 12D_z - \lambda_1 D_x + \lambda_6\right)f' \cdot f\right] \cdot ff'\\
&\quad + D_y\left(3D_x^2 f' \cdot f + \lambda_7 D_x f' \cdot f - \lambda_4 f' f\right) \cdot \left(D_x f \cdot f'\right)\\
&= D_x\left(B_1 f' \cdot f\right) \cdot ff' + D_x\left(B_2 f' \cdot f\right) \cdot \left(D_y f \cdot f'\right) + D_x\left(B_3 f' \cdot f\right) \cdot \left(D_x f \cdot f'\right)\\
&\quad + D_y\left(B_4 f' \cdot f\right) \cdot ff' + D_y\left(B_5 f' \cdot f\right) \cdot \left(D_x f \cdot f'\right).
\end{aligned} \tag{5.10}$$

Taking advantage of equation (5.8) and $D_r g \cdot g = 0$, we immediately see that the coefficients of $\lambda_1, \lambda_2, \ldots, \lambda_7$ in equation (5.10) are all zeros. Therefore, we obtain the following bilinear Bäcklund transformation for the generalized-BLMP equation (1.1):

$$\begin{cases}
B_1 f' \cdot f \equiv (3D_x^2 D_y + \lambda_1 D_y + \lambda_2) f' \cdot f = 0,\\
B_2 f' \cdot f \equiv (3D_x^2 + \lambda_3 D_y + \lambda_4) f' \cdot f = 0,\\
B_3 f' \cdot f \equiv (6D_x D_y + \lambda_5 D_x) f' \cdot f = 0,\\
B_4 f' \cdot f \equiv (D_x^3 - 8D_t + 12D_z - \lambda_1 D_x + \lambda_6) f' \cdot f = 0,\\
B_5 f' \cdot f \equiv (3D_x^2 + \lambda_7 D_x - \lambda_4) f' \cdot f = 0.
\end{cases} \tag{5.11}$$

In the following, we would like to derive explicit solutions the generalized-BLMP equation (1.1) by using the bilinear Bäcklund transformation (5.11). To begin with, we start with a simple solution $f = 1$, from which we can get the original solution $u = -2(\ln f)_x = 0$. Substituting $f = 1$ into equation (5.11), one can readily obtain

$$\begin{cases}
3f'_{xxy} + \lambda_1 f'_y + \lambda_2 f' = 0,\\
3f'_{xx} + \lambda_3 f'_y + \lambda_4 f' = 0,\\
6f'_{xy} + \lambda_5 f'_x = 0,\\
f'_{xxx} - 8f'_t + 12f'_z - \lambda_1 f'_x + \lambda_6 f' = 0,\\
3f'_{xx} + \lambda_7 f'_x - \lambda_4 f' = 0.
\end{cases} \tag{5.12}$$

Case 1. By setting

$$\lambda_2 = 0, \qquad \lambda_4 = 0, \qquad \lambda_6 = 0,$$

$$\lambda_1 = -3k^2, \qquad \lambda_3 = -\frac{3k^2}{l}, \qquad \lambda_5 = -6l, \qquad \lambda_7 = -3k,$$

we get a class of exponential wave solutions of equation (5.1):

$$f' = 1 + \varepsilon e^{kx+ly+mz-\omega t+\xi^0}, \tag{5.13}$$

where k, m, ε, ξ^0 are arbitrary constants and $l \neq 0$, $\omega = -\frac{3}{2}m - \frac{1}{2}k^3$. Hence, $u = -2(\ln f')_x$ solves the generalized-BLMP equation (1.1).

Case 2. Let

$$\lambda_i = 0 \quad (1 \leq i \leq 7),$$

k, l, m are arbitrary constants and $\omega = -\frac{3}{2}m$, it is easy to check that

$$f' = kx + ly + mz - \omega t \tag{5.14}$$

satisfies the bilinear generalized-BLMP equation (3.4), and so

$$u = -2\left(\ln f'\right)_x = \frac{-2k}{kx + ly + mz - \omega t} \tag{5.15}$$

gives a class of rational solutions for the generalized-BLMP equation (1.1).

6 Conclusions

In this paper, bilinear Bäcklund transformations and explicit solutions of a (3 + 1)-dimensional nonlinear equation are investigated. By virtue of the Bell-polynomial approach, two bilinear forms of equation (1.1) are derived and two kinds of bilinear Bäcklund transformations are constructed. Furthermore, explicit solutions for equation (1.1) are also obtained. It is interesting to note that one can obtain different bilinear forms and bilinear Bäcklund transformations via different approaches. We think that there is still much to do to explore more methods of constructing bilinear forms and bilinear Bäcklund transformations for NLEEs.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally in this article. All authors read and approved the final manuscript.

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