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Solvability of boundary value problems for a class of partial difference equations on the combinatorial domain

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Abstract

By modifying our recent method of half-lines we show how the following boundary value problem for partial difference equations can be solved in closed form:

$$d_{n,k} = d_{n-1,k-1} + f(k)d_{n-1,k}, \quad 1 \le k < n,$$
$$d_{n,0} = u_n, \qquad d_{n,n} = v_n, \quad n \in \mathbb{N},$$

where $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ are given sequences of complex numbers, and f is a complex-valued function on $\mathbb{N}$.

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1 Introduction

Let $\mathbb{N}$ denote the set of natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $k, l \in \mathbb{N}_0$ be such that $k < l$, then the notation $j = \overline{k,l}$ means $k \le j \le l$. In the rest of this section we give some motivation for the study, as well as notions that will be used in the rest of the paper.

Let

$$x^{(k)} = x(x-1)\cdots(x-k+1),$$

where $k \in \mathbb{N}$, and for $k = 0$ let $x^{(0)} := 1$.

Let Δ be the standard (forward) difference operator defined by

$$\Delta f(x) = f(x+1) - f(x),$$

where f is a function. Then, for $k \in \mathbb{N}$, we have

$$\Delta x^{(k)} = kx(x-1)\cdots(x-k+2) = kx^{(k-1)}, \tag{1}$$

while for $k = 0$ a direct calculation shows that $\Delta x^{(0)} = 0$.

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The set of polynomials $\{x^{(k)} : k \in \mathbb{N}_0\}$, is linearly independent. Indeed, assume

$$\sum_{j=0}^{n} a_j x^{(j)} \equiv 0, \tag{2}$$

for some real numbers a_j, $j = \overline{0,n}$, and for a fixed but arbitrary $n \in \mathbb{N}_0$. Then by acting with operator Δ on both sides of equality (2) n times, and using (1), we get $n!a_n = \sum_{j=0}^{n} a_j \Delta^{(n)} x^{(j)} \equiv 0$, from which it follows that $a_n = 0$. Hence $\sum_{j=0}^{n-1} a_j x^{(j)} \equiv \sum_{j=0}^{n} a_j x^{(j)} \equiv 0$. By repeated use of the argument we get $a_{n-1} = \cdots = a_1 = a_0 = 0$, from which the claim follows. From this it easily follows that the set $\{x^{(k)} : k \in \mathbb{N}_0\}$ is a basis in the linear space of all polynomials, so that

$$x^n = \sum_{k=0}^{n} s_k^n x^{(k)}, \tag{3}$$

for some numbers s_k^n, $n, k \in \mathbb{N}_0$.

The relations $x = x^{(1)}$, $x^2 = x^{(2)} + x^{(1)}$ and $x^3 = x^{(3)} + 3x^{(2)} + x^{(1)}$, suggest that $s_0^n = 0$, $n \in \mathbb{N}$ (note that $s_0^0 = s_0^0 x^{(0)} = x^0 = 1$). Indeed, assume that we have proved $s_0^j = 0$, $j = \overline{1,n}$ for some $n \in \mathbb{N}$. Then, since $x^{n+1} = x^n x$ and $xx^{(k)} = x^{(k+1)} + kx^{(k)}$, $k \in \mathbb{N}_0$, by using the hypothesis and (3), we get

$$\begin{aligned} x^{n+1} &= \sum_{k=0}^{n+1} s_k^{n+1} x^{(k)} \\ &= x \sum_{k=1}^{n} s_k^n x^{(k)} \\ &= \sum_{k=1}^{n} s_k^n \bigl(x^{(k+1)} + kx^{(k)}\bigr) \\ &= s_n^n x^{(n+1)} + \sum_{k=2}^{n} \bigl(s_{k-1}^n + ks_k^n\bigr) x^{(k)} + s_1^n x^{(1)}. \end{aligned} \tag{4}$$

Comparing the coefficients in (4) we get $s_0^{n+1} = 0$, from which the statement follows. Beside this, the following recurrent relation holds:

$$s_k^{n+1} = s_{k-1}^n + ks_k^n, \tag{5}$$

when $2 \le k \le n$, and that $s_{n+1}^{n+1} = s_n^n$. From this equality and since $s_0^0 = s_1^1 = 1$ we get $s_n^n = 1$ for $n \in \mathbb{N}_0$. On the other hand, from $s_0^j = 0$ for $j = \overline{1,n}$ and (5) we get $s_1^{n+1} = s_1^n$, which along with $s_1^1 = 1$ implies that $s_1^n = 1$ for $n \in \mathbb{N}$. The numbers s_k^n, $n, k \in \mathbb{N}$ are called *Stirling's numbers of the second kind*, and the above described procedure is one of the ways how these numbers can be obtained (see, for example, [1, 2]). Another classical approach in getting these numbers is combinatorial. Namely, s_k^n represents the number of ways to partition a set of n elements into k nonempty subsets (for details see, for example, [3]). Stirling numbers can be calculated explicitly. In [3] are given several explicit formulas and probably the nicest one is the following:

$$s_k^n = \frac{1}{k!}\sum_{j=1}^{k} C_j^k (-1)^{k-j} j^n. \tag{6}$$

where $C_j^k = k!/(j!(k-j)!)$ are the *binomial coefficients*.

There are several ways to prove equation (6). A combinatorial-analytic proof was given in [3]. A relatively simple analytic proof is based on the following well-known formula:

$$\Delta^k f(x) = \sum_{j=0}^{k} (-1)^{k-j} C_j^k f(x+j). \tag{7}$$

Namely, if $P_n(x) = \sum_{j=0}^{n} a_j x^{(j)}$ by calculating $\Delta^k P_n$ at $x = 0$ (using (1)), we get

$$\Delta^k P_n(0) = \sum_{j=k}^{n} a_j j(j-1)\cdots(j-k+1)x^{(j-k)}|_{x=0} = k! a_k,$$

$k = \overline{0,n}$ (see, for example, [1, 2]). From this and by using (7) with $f(x) = x^n$, we get

$$\begin{aligned} s_k^n &= \frac{\Delta^k x^n}{k!}\bigg|_{x=0} = \frac{1}{k!}\sum_{j=0}^{k} (-1)^{k-j} C_j^k (x+j)^n \bigg|_{x=0} \\ &= \frac{1}{k!}\sum_{j=1}^{k} C_j^k (-1)^{k-j} j^n. \end{aligned}$$

For more on the Stirling numbers and related topics, see, for example, [1, 3, 4].

Old topic of solving difference equations and systems of difference equations with their applications (see, for example, [1, 2, 4–8]), has re-attracted some recent attention. The publication of our note [9], in which we explained how a nonlinear difference equation can be solved in an elegant way, by transforming it to a linear, has attracted some attention. The results and methods in [9] were later extended for the case of higher-order difference equations in [10], and for some related systems in [11] and [12]. The main idea from [9] have been used and developed a lot in many other papers (see, for example, [13–19] and numerous references therein). One of the joint features in these papers is that the equations/systems therein are somehow transformed to (solvable) linear ones. A frequent situation is that the following difference equation:

$$a_{n+1} = b_n a_n + c_n, \quad n \in \mathbb{N}, \tag{8}$$

where $(b_n)_{n\in\mathbb{N}}$, $(c_n)_{n\in\mathbb{N}}$, and a_0 are real or complex numbers, decides the solvability. Some methods for solving equation (8) can be found in [2, 6] (see, also the introduction in [17], the references therein, as well as numerous special cases of the equation appearing therein). For some results on the solvability of systems of difference equations see, for example, [12, 14, 16, 18–20]. Some related, but considerably different methods (see, for example, [21, 22] and the related references therein) are used for solving product-type equations/systems.

Statement of the problem. Now note that (5) can be written as

$$d_{n,k} = d_{n-1,k-1} + kd_{n-1,k}. \tag{9}$$

This means that the Stirling numbers are a solution to a two-dimensional recurrent relation with given boundary conditions s_n^n and s_0^n, $n \in \mathbb{N}_0$. The fact that equation (9) has a closed form formula for the solution with the conditions $d_{n,0} = 0$, $d_{n,n} = 1$, $n \in \mathbb{N}$, suggests that (5) can be solved on the following domain $A = \{0 \le k \le n; n, k \in \mathbb{N}_0\}$, which we call the *combinatorial domain*. Some classical methods for solving partial difference equations can be found, for example, in [1] and [8], whereas some recent ones can be found, for example, in [23–28], but an analysis shows that they are not suitable for getting a formula for solutions to (9) for a special shape of domain A.

Motivated by [29] and numerous recent applications of equation (8) (for example, those in [9, 10, 13–17, 19, 20]), here we show that there is a closed form formula for solutions to an extension of equation (9) on domain $A \setminus \{(0,0)\}$ in terms of given boundary values $d_{n,0}$ and $d_{n,n}$, $n \in \mathbb{N}$.

2 Main results

We show how general solution to (9) on set $A \setminus \{(0,0)\}$ can be found by using a method in [29], which we call *the method of half-lines*. Namely, the domain is divided into some half-lines, and (9) is regarded on each line as an equation of type (8). It is solved, and based on the obtained formulas one gets the general solution. However, the method cannot be applied directly, so it needs some modifications.

Assume first that $n = k + 1$. Then equation (9) is reduced to

$$d_{k+1,k} = d_{k,k-1} + kd_{k,k}, \quad k \in \mathbb{N}. \tag{10}$$

Using the change of variables $x_k = d_{k+1,k}$ equation (10) can be seen as a special case of equation (8), since $d_{k,k}$ can be regarded as an 'independent' variable due to the fact that the multi-index (k,k) belongs to the boundary of $A \setminus \{(0,0)\}$. In fact, since in (10) the corresponding coefficients b_k are equal to one, here we use the telescoping method of summation (this is a specificity for the equation but the method can be applied for other values of the coefficients too) and get

$$d_{k+1,k} = d_{1,0} + \sum_{j=1}^{k} jd_{j,j}, \quad k \in \mathbb{N}_0. \tag{11}$$

Now assume that we have proved

$$d_{k+l,k} = d_{l,0} + \sum_{i_l=1}^{k} i_l \left(d_{l-1,0} + \sum_{i_{l-1}=1}^{i_l} i_{l-1} \left(\cdots i_2 \left(d_{1,0} + \sum_{i_1=1}^{i_2} i_1 d_{i_1,i_1} \right) \cdots \right)\right), \tag{12}$$

for every $k \in \mathbb{N}_0$ and some $l \in \mathbb{N}$.

Then from (9) with $n = k + l + 1$ we have

$$d_{k+l+1,k} = d_{k+l,k-1} + kd_{k+l,k}, \quad k \in \mathbb{N}, \tag{13}$$

from which, as in the case $n = k + 1$, we get

$$d_{k+l+1,k} = d_{l+1,0} + \sum_{i_{l+1}=1}^{k} i_{l+1} d_{i_{l+1}+l,i_{l+1}}, \tag{14}$$

for every $k \in \mathbb{N}$.

By using the inductive hypothesis in (14) we obtain

$$d_{k+l+1,k} = d_{l+1,0} + \sum_{i_{l+1}=1}^{k} i_{l+1}\left(d_{l,0} + \sum_{i_l=1}^{i_{l+1}} i_l\left(\cdots i_2\left(d_{1,0} + \sum_{i_1=1}^{i_2} i_1 d_{i_1,i_1}\right)\cdots\right)\right),$$

from which along with the induction we see that (12) holds for every $l \in \mathbb{N}$ and $k \in \mathbb{N}_0$.

Note that equation (12) can be written in the following form:

$$d_{k+l,k} = d_{l,0} + d_{l-1,0}\sum_{i_l=1}^{k} i_l + \cdots + d_{1,0}\sum_{i_l=1}^{k} i_l \cdots \sum_{i_2=1}^{i_3} i_2 + \sum_{i_l=1}^{k} i_l \cdots \sum_{i_2=1}^{i_3} i_2 \sum_{i_1=1}^{i_2} i_1 d_{i_1,i_1}, \tag{15}$$

for $l \in \mathbb{N}$, $k \in \mathbb{N}_0$. Since k and l are arbitrary equation (15) is, in fact, the general solution to (9) on domain $A \setminus \{(0,0)\}$.

Now note that if $d_{j,0} = 0$ and $d_{j,j} = 1$, $j \in \mathbb{N}$, then such a solution is noting but the Stirling sequence s_k^{k+l}, while on the other hand from (15), we get

$$s_k^{k+l} = \sum_{i_l=1}^{k} i_l \cdots \sum_{i_2=1}^{i_3} i_2 \sum_{i_1=1}^{i_2} i_1, \tag{16}$$

for every $l \in \mathbb{N}$ and $k \in \mathbb{N}_0$.

Using (16) in (15), we get

$$d_{k+l,k} = d_{l,0} + d_{l-1,0}s_k^{k+1} + \cdots + d_{1,0}s_k^{k+l-1} + \sum_{i_l=1}^{k} i_l \cdots \sum_{i_2=1}^{i_3} i_2 \sum_{i_1=1}^{i_2} i_1 d_{i_1,i_1}, \tag{17}$$

for $l \in \mathbb{N}$ and $k \in \mathbb{N}_0$.

Now we are in a position to state and prove our first result based on the above consideration.

Theorem 1 *Let $(u_j)_{j\in\mathbb{N}}$, $(v_j)_{j\in\mathbb{N}}$ be given sequences of complex numbers. Then the solution to the following boundary value problem*:

$$\begin{aligned} &d_{n,k} = d_{n-1,k-1} + k d_{n-1,k}, \quad 1 \le k < n, \\ &d_{n,0} = u_n, \qquad d_{n,n} = v_n, \quad n \in \mathbb{N}, \end{aligned} \tag{18}$$

is given by

$$d_{n,k} = \sum_{i=1}^{n-k} s_k^{k+i-1} u_{n-k-i+1} + \sum_{i_{n-k}=1}^{k} i_{n-k} \cdots \sum_{i_2=1}^{i_3} i_2 \sum_{i_1=1}^{i_2} i_1 v_{i_1}. \tag{19}$$

Proof By choosing $l = n - k$ in (17), using the conditions in (18), and some calculations, formula (19) is obtained. □

Remark 1 Note that general solution to equation (9) can be written in the form

$$d_{n,k} = \sum_{i=1}^{n-k} s_k^{k+i-1} d_{n-k-i+1,0} + \sum_{i_{n-k}=1}^{k} i_{n-k} \cdots \sum_{i_2=1}^{i_3} i_2 \sum_{i_1=1}^{i_2} i_1 d_{i_1,i_1}, \tag{20}$$

for k and n such that $0 \le k \le n$, $k, n \in \mathbb{N}_0$ and $(k,n) \neq (0,0)$.

Corollary 1 *If $d_{n,k}$ is a solution to equation* (9) *on domain $A \setminus \{(0,0)\}$ such that the following conditions hold*:

$$d_{n,n} = c \in \mathbb{C}, \quad n \in \mathbb{N}, \tag{21}$$

then it is given by

$$d_{n,k} = \sum_{i=1}^{n-k} s_k^{k+i-1} d_{n-k-i+1,0} + c s_k^n. \tag{22}$$

Proof By using condition (21) and (16) with $l = n - k$, in formula (19), we obtain (22). □

Remark 2 Note that under the conditions of Corollary 1 equation (22) gives a closed form formula for such solutions due to equation (6).

Now note that the above described procedure can be applied to every recurrent relation in the following form:

$$d_{n,k} = d_{n-1,k-1} + f(k) d_{n-1,k}, \tag{23}$$

where coefficients $f(k)$, $k \in \mathbb{N}$, are complex numbers. The following theorem holds (the proof is omitted for the similarity with the given one in the case $f(k) = k$).

Theorem 2 *Let $(u_j)_{j\in\mathbb{N}}$, $(v_j)_{j\in\mathbb{N}}$ be given sequences of complex numbers. Then the solution to the following boundary value problem*:

$$d_{n,k} = d_{n-1,k-1} + f(k) d_{n-1,k}, \quad 1 \le k < n,$$
$$d_{n,0} = u_n, \qquad d_{n,n} = v_n, \quad n \in \mathbb{N},$$

where f is a complex-valued function on $\mathbb{N}$, is given by

$$d_{n,k} = \sum_{i=1}^{n-k} u_{n-k-i+1} S_{k,i-1}(f) + \sum_{i_{n-k}=1}^{k} f(i_{n-k}) \cdots \sum_{i_2=1}^{i_3} f(i_2) \sum_{i_1=1}^{i_2} f(i_1) v_{i_1}, \tag{24}$$

where

$$S_{k,l}(f) = \sum_{i_l=1}^{k} f(i_l) \cdots \sum_{i_2=1}^{i_3} f(i_2) \sum_{i_1=1}^{i_2} f(i_1), \quad k, l \in \mathbb{N}, \tag{25}$$

and where we regard that

$$S_{k,0}(f) = 1, \quad k \in \mathbb{N}.$$

Theorem 2 shows that the calculation of iterated sums plays an important role in solving equation (23). For the equation treated in [29] we have $f(k) = 1$ for every $k \in \mathbb{N}$, so that the iterated sums in (25) becomes

$$\sum_{i_l=1}^{k} \cdots \sum_{i_{s+1}=1}^{i_{s+2}} \sum_{i_s=1}^{i_{s+1}} 1, \quad s = \overline{1,l}.$$

This value of the sum is known and is equal to C_{l-s+1}^{k+l-s} (see, for example, [2]), which is one of the main reasons why in this case we have a closed form formula for its general solution. The process of calculating the sum was essentially explained in [29].

Now we will apply the method described above to the following partial difference equation:

$$d_{n,k} = d_{n-1,k-1} + z^k d_{n-1,k}, \tag{26}$$

where z is a complex number different from 1.

By doing this we see that the following relation, corresponding to equation (12), holds:

$$d_{k+l,k} = d_{l,0} + \sum_{i_l=1}^{k} z^{i_l} \left(d_{l-1,0} + \sum_{i_{l-1}=1}^{i_l} z^{i_{l-1}} \left(\cdots z^{i_2} \left(d_{1,0} + \sum_{i_1=1}^{i_2} z^{i_1} d_{i_1,i_1} \right) \cdots \right) \right),$$

for every $l \in \mathbb{N}$ and $k \in \mathbb{N}_0$, which can be written in the following, somewhat better, form:

$$\begin{aligned} d_{k+l,k} = {} & d_{l,0} + d_{l-1,0} \sum_{i_l=1}^{k} z^{i_l} + d_{l-2,0} \sum_{i_l=1}^{k} z^{i_l} \sum_{i_{l-1}=1}^{i_l} z^{i_{l-1}} + \cdots + d_{1,0} \sum_{i_l=1}^{k} z^{i_l} \cdots \sum_{i_2=1}^{i_3} z^{i_2} \\ & + \sum_{i_l=1}^{k} z^{i_l} \cdots \sum_{i_2=1}^{i_3} z^{i_2} \sum_{i_1=1}^{i_2} z^{i_1} d_{i_1,i_1}. \end{aligned} \tag{27}$$

Now we prove the following lemma.

Lemma 1 *Assume that $k, l \in \mathbb{N}$ and that $z \in \mathbb{C} \setminus \{1\}$. Then the following formula holds*:

$$\sum_{i_l=1}^{k} z^{i_l} \cdots \sum_{i_2=1}^{i_3} z^{i_2} \sum_{i_1=1}^{i_2} z^{i_1} = z^l \prod_{j=1}^{l} \frac{z^{k+j-1} - 1}{z^j - 1}. \tag{28}$$

Proof We will prove it by induction. Assume first that $l = 1$, then since $z \neq 1$, we have

$$\sum_{i_1=1}^{k} z^{i_1} = z \sum_{j=0}^{k-1} z^j = z \frac{z^k - 1}{z - 1}, \tag{29}$$

which shows that (28) holds in this case.

In order to present the main point in the proof, that is, the arranging a sum to use the telescoping method, we will also give the proof in the case $l = 2$. By using (29) with $l = i_2$, we have

$$\begin{aligned}\sum_{i_2=1}^{k} z^{i_2} \sum_{i_1=1}^{i_2} z^{i_1} &= \frac{z^2}{z-1} \sum_{i_2=1}^{k} z^{i_2-1}\big(z^{i_2}-1\big)\\ &= \frac{z^2}{z-1} \sum_{i_2=1}^{k} \frac{(z^{i_2+1}-1)(z^{i_2}-1)-(z^{i_2}-1)(z^{i_2-1}-1)}{z^2-1}\\ &= z^2 \prod_{j=1}^{2} \frac{z^{k+j-1}-1}{z^j-1}.\end{aligned}$$

Now assume that (28) holds for some $l \in \mathbb{N}$ and every $k \in \mathbb{N}_0$. Then, by using the inductive hypothesis, we have

$$\begin{aligned}&\sum_{i_{l+1}=1}^{k} z^{i_{l+1}} \sum_{i_l=1}^{i_{l+1}} z^{i_l} \cdots \sum_{i_1=1}^{i_2} z^{i_1}\\ &\quad= z^{l+1} \sum_{i_{l+1}=1}^{k} z^{i_{l+1}-1} \prod_{j=1}^{l} \frac{z^{i_{l+1}+j-1}-1}{z^j-1}\\ &\quad= \frac{z^{l+1}}{\prod_{j=1}^{l}(z^j-1)} \sum_{i_{l+1}=1}^{k} \frac{\prod_{j=0}^{l}(z^{i_{l+1}+j}-1)-\prod_{j=0}^{l}(z^{i_{l+1}+j-1}-1)}{z^{l+1}-1}\\ &\quad= z^{l+1} \prod_{j=1}^{l+1} \frac{z^{k+j-1}-1}{z^j-1},\end{aligned}$$

for every $k \in \mathbb{N}$, from which along with (29) and the induction formula (28) follows. □

Let

$$P_{k,l}(z) := z^l \prod_{j=1}^{l} \frac{z^{k+j-1}-1}{z^j-1}, \tag{30}$$

where $k, l \in \mathbb{N}_0$, and where we use the following standard convention:

$$\prod_{j=k}^{k-1} a_j = 1, \quad k \in \mathbb{N}_0.$$

By using (28) into (27), and a change in the order of the summation, we get

$$d_{k+l,k} = \sum_{j=1}^{l} d_{j,0} P_{k,l-j}(z) + \sum_{i_1=1}^{k} z^{i_1} d_{i_1,i_1} \sum_{i_2=i_1}^{k} z^{i_2} \cdots \sum_{i_l=i_{l-1}}^{k} z^{i_l}. \tag{31}$$

Now by using Lemma 1 and some calculation, we have

$$\begin{aligned}
\sum_{i_2=i_1}^{k} z^{i_2} \cdots \sum_{i_l=i_{l-1}}^{k} z^{i_l} &= \sum_{i_l=i_1}^{k} z^{i_l} \cdots \sum_{i_2=i_1}^{i_3} z^{i_2} \\
&= z^{(i_1-1)(l-1)} \sum_{i_l=i_1}^{k} z^{i_l-i_1+1} \cdots \sum_{i_2=i_1}^{i_3} z^{i_2-i_1+1} \\
&= z^{(i_1-1)(l-1)} \sum_{j_l=1}^{k-i_1+1} z^{j_l} \cdots \sum_{j_2=1}^{i_3-i_1+1} z^{j_2} \\
&= z^{i_1(l-1)} \prod_{j=1}^{l-1} \frac{z^{k-i_1+j}-1}{z^j-1}.
\end{aligned} \tag{32}$$

By using (32) in (31) we get

$$d_{k+l,k} = \sum_{j=1}^{l} d_{j,0} P_{k,l-j}(z) + \sum_{i_1=1}^{k} d_{i_1,i_1} z^{i_1 l-l+1} P_{k-i_1+1,l-1}(z), \tag{33}$$

for every $l \in \mathbb{N}$ and $k \in \mathbb{N}_0$.

Hence, the following result holds.

Theorem 3 *Let $(u_j)_{j\in\mathbb{N}}$, $(v_j)_{j\in\mathbb{N}}$ be given sequences of complex numbers. Then the solution to the following boundary value problem*:

$$\begin{aligned}
&d_{n,k} = d_{n-1,k-1} + z^k d_{n-1,k}, \quad 1 \le k < n, \\
&d_{n,0} = u_n, \quad d_{n,n} = v_n, \qquad n \in \mathbb{N},
\end{aligned} \tag{34}$$

is given by

$$d_{n,k} = \sum_{j=1}^{n-k} u_j P_{k,n-k-j}(z) + \sum_{i=1}^{k} v_i z^{(i-1)(n-k)+1} P_{k-i+1,n-k-1}(z). \tag{35}$$

Proof By choosing $l = n - k$ in (33), using the conditions in (34), and some calculations, equation (35) is easily obtained. □

Competing interests
The author declares that he has no competing interests.

Author's contributions
The author contributed alone to the writing of this paper. He read and approved the manuscript.

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