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Interval oscillation criteria for second-order nonlinear forced differential equations involving variable exponent

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Abstract
In this paper, we establish some interval oscillation criteria for a class of second-order nonlinear forced differential equations with variable exponent growth conditions. Our results not only give the sufficient conditions for the oscillation of equations with variable exponent growth conditions, but also they extend some existing results in the literature for equations with a Riemann-Stieltjes integral. Two examples are also considered to illustrate the main results.

Keywords: oscillation; Riemann-Stieltjes integral; second-order nonlinear equation; variable exponent

1 Introduction

In this paper, we will establish some interval oscillation criteria for the following equation:

$$\bigl(p(t)u'(t)\bigr)' + q(t)u(t) + \int_a^b g(t,s)\bigl|u(t)\bigr|^{\gamma(t,s)+1-\beta(t)} \operatorname{sgn} u(t)\,\mathrm{d}\xi(s) = e(t), \quad t \geq t_0, \tag{1}$$

where $p, q, \beta, e \in C[t_0, +\infty)$ with $p(t) > 0$, $\beta(t) > 0$, $a \in \mathbf{R}$, $b \in (a, +\infty)$, $g \in C([t_0, +\infty) \times [a, b])$, $\xi : [a, b] \to \mathbf{R}$ is strictly increasing, $\gamma \in C([t_0, +\infty) \times [a, b])$, and $\gamma(t, \cdot)$ is strictly increasing on $[a, b]$ such that

$$0 < \gamma(t,a) < \beta(t) < \gamma(t,b), \qquad \beta(t) \leq \gamma(t,a) + 1, \quad t \in [t_0, +\infty).$$

Here $\int_a^b f(s)\,\mathrm{d}\xi(s)$ denotes the Riemann-Stieltjes integral of the function f on $[a, b]$ with respect to ξ.

As usual, a nontrivial solution $u(t)$ of equation (1) is called oscillatory if it has arbitrary large zeroes, otherwise it is called nonoscillatory. Equation (1) is said to be oscillatory if all its solutions are oscillatory.

For the particular case when $\gamma(t,s) = \alpha(s)$, $a = 0$, and $\beta(t) \equiv 1$, equation (1) reduces to the following equation:

$$\bigl(p(t)u'(t)\bigr)' + q(t)u(t) + \int_0^b g(t,s)\bigl|u(t)\bigr|^{\alpha(s)} \operatorname{sgn} u(t)\,\mathrm{d}\xi(s) = e(t), \tag{2}$$

which have been observed in Sun and Kong [1].

For the particular case when $a = 1$, $b = l + m + 1$, where $l, m \in \mathbf{N}$ and for $s \in [0, l + m + 1)$, $\xi(s) = \sum_{j=1}^{l+m} \chi(s - j)$ with

$$\chi(s) = \begin{cases} 1, & s \geq 0, \\ 0, & s < 0, \end{cases}$$

$\gamma \in C([t_0, +\infty) \times [0, l + m + 1))$ such that

$$\gamma(t,j) = \alpha_j(t), \quad j = 1, 2, \ldots, m; \qquad \gamma(t, m + i) = \theta_i(t), \quad i = 1, 2, \ldots, l,$$

satisfying $0 < \alpha_1(t) < \cdots < \alpha_m(t) < \beta(t) < \theta_1(t) < \cdots < \theta_l(t)$, $\beta(t) \leq \alpha_1(t) + 1$, $t \in [t_0, +\infty)$,

$$g(t,j) = A_j(t) \in C[t_0, +\infty), \quad j = 1, 2, \ldots, m;$$
$$g(t, m + i) = B_i(t) \in C[t_0, +\infty), \quad i = 1, 2, \ldots, l.$$

Then equation (1) reduces to the following equation with variable exponent growth conditions:

$$\begin{aligned} &\bigl(p(t)u'(t)\bigr)' + q(t)u(t) + \sum_{j=1}^{m} A_j(t)\bigl|u(t)\bigr|^{\alpha_j(t)+1-\beta(t)} \operatorname{sgn} u(t) \\ &\quad + \sum_{i=1}^{l} B_i(t)\bigl|u(t)\bigr|^{\theta_i(t)+1-\beta(t)} \operatorname{sgn} u(t) = e(t). \end{aligned} \tag{3}$$

For the particular case when $p(t) \equiv 1$, $q(t) \equiv 0$, $m = 1$, $\alpha_1(t) \equiv \alpha \neq 1$, $\beta(t) \equiv 1$, $B_i(t) \equiv 0$, $i = 1, 2, \ldots, l$, equation (3) reduces to the well-known Emden-Fowler equation,

$$u''(t) + q(t)\bigl|u(t)\bigr|^{\alpha} \operatorname{sgn} u(t) = 0. \tag{4}$$

In the past 50 years, extensive work has been done and great progress has been made on oscillation of equation (4) and more general equations (see [1–24] and the references therein). On the other hand, with wide use in the nonlinear elasticity theory and electrorheological fluids (see [25, 26]), the differential equations and variational problems with variable exponent growth conditions have been investigated by many authors in recent years (see [27–37]). However, we notice that no criteria were found for equation (1) even for the special case of equation (3) to be oscillatory so far in the literature. The purpose of this paper is to establish some interval oscillation results for equation (1) which involves variable exponent growth conditions. Clearly, our work is of significance because equation (1) allows an infinite number of nonlinear terms and even a continuum of nonlinearities determined by the function ξ.

The organization of this article is as follows. After this introduction, in Section 2, we establish interval oscillation criteria of both the El-Sayed type and the Kong type for equation (1). In Section 3, we give two examples to illustrate our main results.

2 Main results

In the sequel, we denote by $L_\xi[a, b]$ the set of Riemann-Stieltjes integrable functions on $[a, b]$ with respect to ξ. We further assume that for any $t \in [t_0, +\infty)$, $\gamma(t, \cdot), 1/\gamma(t, \cdot) \in L_\xi[a, b]$.

Lemma 2.1 *Suppose that $\gamma \in C([t_0,+\infty) \times [a,b])$, and for any $t \in [t_0,+\infty)$, $\gamma(t,\cdot)$ is strictly increasing on $[a,b]$, $\beta \in C[t_0,+\infty)$ such that*

$$0 < \gamma(t,a) < \beta(t) < \gamma(t,b), \qquad \beta(t) \le \gamma(t,a)+1, \quad t \in [t_0,+\infty).$$

Let $h = \sup\{s \in (a,b) : \gamma(t,s) \le \beta(t), t \in [t_0,+\infty)\}$, and

$$m_1(t) := \int_h^b \frac{\beta^2(t)}{\gamma(t,s)} \left(\int_h^b d\xi(s)\right)^{-1} d\xi(s), \quad t \in [t_0,+\infty),$$
$$m_2(t) := \int_a^h \frac{\beta^2(t)}{\gamma(t,s)} \left(\int_a^h d\xi(s)\right)^{-1} d\xi(s), \quad t \in [t_0,+\infty).$$

Then, for any function θ satisfying $\theta(t) \in (m_1(t), m_2(t))$ for $t \in [t_0,+\infty)$, there exists $\eta : [t_0,+\infty) \times [a,b] \to (0,+\infty)$ satisfying for any $t \in [t_0,+\infty)$, $\eta(t,\cdot) \in L_\xi[a,b]$, such that

$$\int_a^b \gamma(t,s)\eta(t,s)\, d\xi(s) = \beta^2(t), \quad (t,s) \in [t_0,+\infty) \times [a,b], \tag{5}$$

and

$$\int_a^b \eta(t,s)\, d\xi(s) = \theta(t), \quad (t,s) \in [t_0,+\infty) \times [a,b]. \tag{6}$$

Proof Define

$$\eta_1(t,s) = \begin{cases} \frac{\beta^2(t)}{\gamma(t,s)}(\int_h^b d\xi(s))^{-1}, & (t,s) \in [t_0,+\infty) \times [h,b], \\ 0, & (t,s) \in [t_0,+\infty) \times [a,h), \end{cases} \tag{7}$$

and

$$\eta_2(t,s) = \begin{cases} 0, & (t,s) \in [t_0,+\infty) \times [h,b], \\ \frac{\beta^2(t)}{\gamma(t,s)}(\int_a^h d\xi(s))^{-1}, & (t,s) \in [t_0,+\infty) \times [a,h). \end{cases} \tag{8}$$

Note that, for any $t \in [t_0,+\infty)$, $1/\gamma(t,\cdot) \in L_\xi[a,b]$. Thus, for any $t \in [t_0,+\infty)$, $\eta_i(t,\cdot) \in L_\xi[a,b]$, and

$$\int_a^b \gamma(t,s)\eta_i(t,s)\, d\xi(s) = \beta^2(t), \quad i = 1,2. \tag{9}$$

Moreover, by the choice of h, we can easily get, for any $t \in [t_0,+\infty)$,

$$\begin{aligned} m_1(t) &= \int_h^b \frac{\beta^2(t)}{\gamma(t,s)} \left(\int_h^b d\xi(s)\right)^{-1} d\xi(s) \\ &= \beta(t)\left(\int_h^b d\xi(s)\right)^{-1} \int_h^b \frac{\beta(t)}{\gamma(t,s)}\, d\xi(s) \\ &< \beta(t)\left(\int_h^b d\xi(s)\right)^{-1} \int_h^b d\xi(s) \\ &= \beta(t), \end{aligned} \tag{10}$$

$$\begin{aligned}m_2(t) &= \int_a^h \frac{\beta^2(t)}{\gamma(t,s)}\left(\int_a^h \mathrm{d}\xi(s)\right)^{-1} \mathrm{d}\xi(s)\\ &= \beta(t)\left(\int_a^h \mathrm{d}\xi(s)\right)^{-1}\int_a^h \frac{\beta(t)}{\gamma(t,s)}\,\mathrm{d}\xi(s)\\ &> \beta(t)\left(\int_a^h \mathrm{d}\xi(s)\right)^{-1}\int_a^h \mathrm{d}\xi(s)\\ &= \beta(t).\end{aligned}\tag{11}$$

Therefore, for any $\theta(t) \in (m_1(t), m_2(t))$, $t \in [t_0, +\infty)$, there exists a function $p^* : [t_0, +\infty) \to (0,1)$ such that

$$\left[1 - p^*(t)\right]m_1(t) + p^*(t)m_2(t) = \theta(t).\tag{12}$$

Let

$$\eta(t,s) = \left[1 - p^*(t)\right]\eta_1(t,s) + p^*(t)\eta_2(t,s), \quad \text{for } (t,s) \in [t_0, +\infty) \times [a,b],\tag{13}$$

then $\eta(t,s) > 0$ for $(t,s) \in [t_0, +\infty) \times [a,b]$, and for any $t \in [t_0, +\infty)$, $\eta(t,\cdot) \in L_\xi[a,b]$. From (10)-(13), we get

$$\begin{aligned}\int_a^b \eta(t,s)\,\mathrm{d}\xi(s) &= \left[1 - p^*(t)\right]\int_a^b \eta_1(t,s)\,\mathrm{d}\xi(s) + p^*(t)\int_a^b \eta_2(t,s)\,\mathrm{d}\xi(s)\\ &= \left[1 - p^*(t)\right]m_1(t) + p^*(t)m_2(t)\\ &= \theta(t).\end{aligned}$$

Also from (9) and (13), we have

$$\begin{aligned}\int_a^b \gamma(t,s)\eta(t,s)\,\mathrm{d}\xi(s) &= \left[1 - p^*(t)\right]\int_a^b \gamma(t,s)\eta_1(t,s)\,\mathrm{d}\xi(s)\\ &\quad + p^*(t)\int_a^b \gamma(t,s)\eta_2(t,s)\,\mathrm{d}\xi(s)\\ &= \left[1 - p^*(t)\right]\beta^2(t) + p^*(t)\beta^2(t)\\ &= \beta^2(t).\end{aligned}$$

This completes the proof of Lemma 2.1. □

Remark 2.1 We will see from the proof of Lemma 2.1 that the function η can be constructed explicitly for any nondecreasing function ξ.

Remark 2.2 If we take $\gamma(t,s) = \alpha(s)$, $a = 0$, and $\beta(t) \equiv 1$, then Lemma 2.1 reduces to Lemma 2.1 in [1].

Lemma 2.2 *Let functions* $\theta : [t_0, +\infty) \to (0, +\infty)$, $w : [t_0, +\infty) \times [a,b] \to [0, +\infty)$, $\eta : [t_0, +\infty) \times [a,b] \to (0, +\infty)$ *satisfy for any* $t \in [t_0, +\infty)$, $\omega(t,\cdot) \in L_\xi[a,b]$, $\eta(t,\cdot) \in L_\xi[a,b]$, *and*

$$\int_a^b \eta(t,s)\,\mathrm{d}\xi(s) = \theta(t), \quad (t,s) \in [t_0,+\infty) \times [a,b]. \tag{14}$$

Then, for any $t \in [t_0,+\infty)$,

$$\int_a^b \eta(t,s)w(t,s)\,\mathrm{d}\xi(s) \geq \exp\biggl(\frac{\int_a^b \eta(t,s)\ln[\theta(t)w(t,s)]\,\mathrm{d}\xi(s)}{\theta(t)}\biggr), \tag{15}$$

where we use the convention that $\ln 0 = -\infty$ *and* $e^{-\infty} = 0$.

Proof Without loss of generality we assume that, for any $t \in [t_0,+\infty)$,

$$\int_a^b \eta(t,s)w(t,s)\,\mathrm{d}\xi(s) > 0.$$

For otherwise

$$\int_a^b \eta(t,s)w(t,s)\,\mathrm{d}\xi(s) = 0,$$
$$\frac{\int_a^b \eta(t,s)\ln[\theta(t)w(t,s)]\,\mathrm{d}\xi(s)}{\theta(t)} = -\infty,$$

and hence (15) is obviously satisfied. It is easy to check that $\ln t \leq t-1$ for $t \geq 0$, and then, for any $(t,s) \in [t_0,+\infty) \times [a,b]$,

$$\begin{aligned}
&\ln\bigl[\theta(t)w(t,s)\bigr] - \ln\biggl(\int_a^b \eta(t,s)w(t,s)\,\mathrm{d}\xi(s)\biggr) \\
&\quad = \ln\biggl(\frac{\theta(t)w(t,s)}{\int_a^b \eta(t,s)w(t,s)\,\mathrm{d}\xi(s)}\biggr) \leq \frac{\theta(t)w(t,s)}{\int_a^b \eta(t,s)w(t,s)\,\mathrm{d}\xi(s)} - 1.
\end{aligned} \tag{16}$$

Multiplying (16) by $\eta(t,s)$, we obtain

$$\begin{aligned}
&\eta(t,s)\biggl[\ln\bigl[\theta(t)w(t,s)\bigr] - \ln\biggl(\int_a^b \eta(t,s)w(t,s)\,\mathrm{d}\xi(s)\biggr)\biggr] \\
&\quad \leq \eta(t,s)\biggl(\frac{\theta(t)w(t,s)}{\int_a^b \eta(t,s)w(t,s)\,\mathrm{d}\xi(s)} - 1\biggr),
\end{aligned} \tag{17}$$

by integrating the inequality (17) over $\mathrm{d}\xi(s)$ and applying (14), we get

$$\begin{aligned}
&\int_a^b \eta(t,s)\biggl[\ln\bigl[\theta(t)w(t,s)\bigr] - \ln\biggl(\int_a^b \eta(t,s)w(t,s)\,\mathrm{d}\xi(s)\biggr)\biggr]\,\mathrm{d}\xi(s) \\
&\quad \leq \int_a^b \eta(t,s)\biggl(\frac{\theta(t)w(t,s)}{\int_a^b \eta(t,s)w(t,s)\,\mathrm{d}\xi(s)} - 1\biggr)\,\mathrm{d}\xi(s) \\
&\quad = \theta(t) - \int_a^b \eta(t,s)\,\mathrm{d}\xi(s) = 0,
\end{aligned} \tag{18}$$

that is

$$\begin{aligned}\int_a^b \eta(t,s)\ln\bigl[\theta(t)w(t,s)\bigr]\,\mathrm{d}\xi(s) &\le \int_a^b \eta(t,s)\ln\biggl(\int_a^b \eta(t,s)w(t,s)\,\mathrm{d}\xi(s)\biggr)\,\mathrm{d}\xi(s)\\ &= \ln\biggl(\int_a^b \eta(t,s)w(t,s)\,\mathrm{d}\xi(s)\biggr)\int_a^b \eta(t,s)\,\mathrm{d}\xi(s)\\ &= \ln\biggl(\int_a^b \eta(t,s)w(t,s)\,\mathrm{d}\xi(s)\biggr)\theta(t). \end{aligned} \tag{19}$$

Dividing (19) by $\theta(t)$, we have

$$\frac{\int_a^b \eta(t,s)\ln[\theta(t)w(t,s)]\,\mathrm{d}\xi(s)}{\theta(t)} \le \ln\biggl(\int_a^b \eta(t,s)w(t,s)\,\mathrm{d}\xi(s)\biggr), \tag{20}$$

which implies (15). This completes the proof of Lemma 2.2. □

Following El-Sayed [38], for $c,d \in [t_0,+\infty)$ with $c < d$, we define the function class $\mathcal{V}(c,d) := \{v \in C^1[c,d] : v(c) = 0 = v(d), v \not\equiv 0\}$. Our first result provides an oscillation criterion for equation (1) of the El-Sayed type.

Theorem 2.1 *Suppose that for any $T > t_0$, there exist $T \le a_1 < b_1 \le a_2 < b_2$ such that, for $i = 1,2$,*

$$g(t,s) \ge 0, \quad \textit{for } (t,s) \in [a_i,b_i] \times [a,b] \quad \textit{and} \quad (-1)^i e(t) \ge 0, \quad \textit{for } t \in [a_i,b_i]. \tag{21}$$

We further assume that, for $i = 1,2$, there exist functions $v_i \in \mathcal{V}(a_i,b_i)$ and θ satisfying $\theta(t) \in (m_1(t),\beta(t)]$ for $t \in [t_0,+\infty)$, and a continuous function $\eta : [t_0,+\infty) \times [a,b] \to (0,+\infty)$ satisfying (5) *and* (6), *where $m_1(t)$ is defined as in Lemma* 2.1 *such that*

$$\int_{a_i}^{b_i} \bigl[Q(t)v_i^2(t) - p(t)v_i'^2(t)\bigr]\,\mathrm{d}t \ge 0, \tag{22}$$

where

$$\begin{aligned} Q(t) = q(t) &+ \biggl(\frac{(\beta^2(t)-\theta(t)\beta(t)+\theta(t))|e(t)|}{\beta^2(t)-\theta(t)\beta(t)}\biggr)^{\frac{\beta^2(t)-\theta(t)\beta(t)}{\beta^2(t)-\theta(t)\beta(t)+\theta(t)}}\\ &\cdot \exp\biggl(\frac{\theta(t)}{\beta^2(t)-\theta(t)\beta(t)+\theta(t)}\biggl[\ln\bigl(\beta^2(t)-\theta(t)\beta(t)+\theta(t)\bigr)\\ &+ \frac{\int_a^b \eta(t,s)\ln\frac{g(t,s)}{\eta(t,s)}\,\mathrm{d}\xi(s)}{\theta(t)}\biggr]\biggr). \end{aligned} \tag{23}$$

Here we use the convention that $\ln 0 = -\infty$ *and* $e^{-\infty} = 0$, *and* $0^0 = 1$. *Then equation* (1) *is oscillatory.*

Proof Assume, for the sake of contradiction, that equation (1) has an extendible solution $u(t)$ which is eventually positive or negative. Without loss of generality, we may assume that $u(t) > 0$ for all $t \geq t_0$. When $u(t)$ is eventually negative, the proof is carried out in the same way using the interval the interval $[a_2, b_2]$ instead of $[a_1, b_1]$. Define

$$\omega(t) = -\frac{p(t)u'(t)}{u(t)}, \quad t \in [a_1, b_1].$$

Then, for $t \geq t_0$, ω satisfies

$$\omega'(t) = q(t) + \int_a^b g(t,s)\big[u(t)\big]^{\gamma(t,s)-\beta(t)}\,\mathrm{d}\xi(s) - \frac{e(t)}{u(t)} + \frac{\omega^2(t)}{p(t)}. \tag{24}$$

(I) We first consider the case when $\theta(t) \equiv \beta(t)$.

From (21) and (24) we have, for $t \in [a_1, b_1]$,

$$\omega'(t) \geq q(t) + \int_a^b g(t,s)\big[u(t)\big]^{\gamma(t,s)-\beta(t)}\,\mathrm{d}\xi(s) + \frac{\omega^2(t)}{p(t)}. \tag{25}$$

Since $\eta(t,s)$ satisfying (5) and (6) with $\theta(t) \equiv \beta(t)$, it follows that

$$\int_a^b \eta(t,s)\big[\gamma(t,s) - \beta(t)\big]\,\mathrm{d}\xi(s) \equiv 0, \quad \text{for any } t \in [t_0, +\infty). \tag{26}$$

Therefore, by (26) and Lemma 2.2, we get, for $t \in [a_1, b_1]$,

$$\begin{aligned}
&\int_a^b g(t,s)\big[u(t)\big]^{\gamma(t,s)-\beta(t)}\,\mathrm{d}\xi(s)\\
&\quad= \int_a^b \eta(t,s)\eta^{-1}(t,s)g(t,s)\big[u(t)\big]^{\gamma(t,s)-\beta(t)}\,\mathrm{d}\xi(s)\\
&\quad\geq \exp\biggl(\frac{1}{\beta(t)}\int_a^b \eta(t,s)\ln\bigl[\beta(t)\eta^{-1}(t,s)g(t,s)\big[u(t)\big]^{\gamma(t,s)-\beta(t)}\bigr]\,\mathrm{d}\xi(s)\biggr)\\
&\quad= \exp\biggl(\frac{1}{\beta(t)}\int_a^b \eta(t,s)\ln\frac{\beta(t)g(t,s)}{\eta(t,s)}\,\mathrm{d}\xi(s)\\
&\qquad+ \frac{\ln u(t)}{\beta(t)}\int_a^b \bigl[\eta(t,s)\bigl(\gamma(t,s) - \beta(t)\bigr)\bigr]\,\mathrm{d}\xi(s)\biggr)\\
&\quad= \exp\biggl(\frac{1}{\beta(t)}\int_a^b \eta(t,s)\ln\frac{\beta(t)g(t,s)}{\eta(t,s)}\,\mathrm{d}\xi(s)\biggr)\\
&\quad= \exp\biggl(\ln\beta(t) + \frac{1}{\beta(t)}\int_a^b \eta(t,s)\ln\frac{g(t,s)}{\eta(t,s)}\,\mathrm{d}\xi(s)\biggr).
\end{aligned} \tag{27}$$

Substituting (27) into (25),

$$\begin{aligned}
\omega'(t) &\geq q(t) + \exp\biggl(\ln\beta(t) + \frac{1}{\beta(t)}\int_a^b \eta(t,s)\ln\frac{g(t,s)}{\eta(t,s)}\,\mathrm{d}\xi(s)\biggr) + \frac{\omega^2(t)}{p(t)}\\
&= Q(t) + \frac{\omega^2(t)}{p(t)}, \quad t \in [a_1, b_1],
\end{aligned} \tag{28}$$

where $Q(t)$ is defined by (23) with $\theta(t) \equiv \beta(t)$. Multiplying both sides of (28) by $v_1^2(t)$, integrating every term from a_1 to b_1, and using integration by parts, we find

$$\int_{a_1}^{b_1} \left[Q(t)v_1^2(t) - p(t)v_1'^2(t)\right] \mathrm{d}t + \int_{a_1}^{b_1} \left[\frac{\omega(t)v_1(t)}{p^{1/2}(t)} + p^{1/2}(t)v_1'(t)\right]^2 \mathrm{d}t \le 0. \tag{29}$$

From (22), we see that

$$\frac{\omega(t)v_1(t)}{p^{1/2}(t)} + p^{1/2}(t)v_1'(t) \equiv 0, \quad t \in [a_1, b_1],$$

which implies from the definition of w that $v_1'(t)/v_1(t) \equiv u'(t)/u(t)$ and hence $v_1(t) \equiv u(t)$, $t \in [a_1, b_1]$ for some constant $c \neq 0$. This contradicts the assumption that $v_1(a_1) = v_1(b_1) = 0$ and $u(t)$is positive on $[a_1, b_1]$.

(II) Next we consider the case when $\theta(t) \in (m_1(t), \beta(t))$. From (6), we have

$$\begin{aligned}
&\int_a^b g(t,s)\left[u(t)\right]^{\gamma(t,s)-\beta(t)} \mathrm{d}\xi(s) - \frac{e(t)}{u(t)} \\
&\quad = \int_a^b \left[g(t,s)\left[u(t)\right]^{\gamma(t,s)-\beta(t)} - \frac{e(t)}{u(t)}\frac{\eta(t,s)}{\theta(t)}\right] \mathrm{d}\xi(s) \\
&\quad = \int_a^b \left[g(t,s)\left[u(t)\right]^{\gamma(t,s)-\beta(t)} + \frac{|e(t)|}{u(t)}\frac{\eta(t,s)}{\theta(t)}\right] \mathrm{d}\xi(s) \\
&\quad = \int_a^b \frac{\eta(t,s)}{\theta(t)}\left[\frac{\theta(t)}{\eta(t,s)}g(t,s)\left[u(t)\right]^{\gamma(t,s)-\beta(t)} + \frac{|e(t)|}{u(t)}\right] \mathrm{d}\xi(s).
\end{aligned} \tag{30}$$

If we let

$$p = \frac{\theta(t)}{\beta^2(t) - \theta(t)\beta(t) + \theta(t)}, \qquad q = \frac{\beta^2(t) - \theta(t)\beta(t)}{\beta^2(t) - \theta(t)\beta(t) + \theta(t)}, \tag{31}$$

$$A = \frac{\beta^2(t) - \theta(t)\beta(t) + \theta(t)}{\eta(t,s)} g(t,s)\left[u(t)\right]^{\gamma(t,s)-\beta(t)}, \qquad B = \frac{1}{q}\frac{|e(t)|}{u(t)}, \tag{32}$$

then from the Young inequality ($pA + qB \ge A^p B^q$, where $p + q = 1$, $p, q > 0$, $A \ge 0$, $B \ge 0$), we get

$$\begin{aligned}
&\frac{\theta(t)}{\eta(t,s)} g(t,s)\left[u(t)\right]^{\gamma(t,s)-\beta(t)} + \frac{|e(t)|}{u(t)} \\
&\quad \ge \left(\frac{\beta^2(t) - \theta(t)\beta(t) + \theta(t)}{\eta(t,s)} g(t,s)\left[u(t)\right]^{\gamma(t,s)-\beta(t)}\right)^p \left(\frac{1}{q}\frac{|e(t)|}{u(t)}\right)^q \\
&\quad = \left(\frac{\beta^2(t) - \theta(t)\beta(t) + \theta(t)}{\eta(t,s)} g(t,s)\right)^p \left(\frac{|e(t)|}{q}\right)^q \left[u(t)\right]^{(\gamma(t,s)-\beta(t))p-q} \\
&\quad = \left(\frac{\beta^2(t) - \theta(t)\beta(t) + \theta(t)}{\eta(t,s)} g(t,s)\right)^p \left(\frac{|e(t)|}{q}\right)^q \left[u(t)\right]^{\frac{\gamma(t,s)\theta(t)-\beta^2(t)}{\beta^2(t)-\theta(t)\beta(t)+\theta(t)}}.
\end{aligned} \tag{33}$$

By (5) and (6), we get

$$\int_a^b \eta(t,s)\left[\gamma(t,s)\theta(t) - \beta^2(t)\right] \mathrm{d}\xi(s) \equiv 0, \quad \text{for any } t \in [t_0, +\infty). \tag{34}$$

From (6), (30), (31), (33), (34), and Lemma 2.2, we see that, for $t \in [a_1, b_1]$,

$$\begin{aligned}
&\int_a^b g(t,s)\big[u(t)\big]^{\gamma(t,s)-\beta(t)}\,\mathrm{d}\xi(s) - \frac{e(t)}{u(t)} \\
&\quad\geq \int_a^b \frac{\eta(t,s)}{\theta(t)}\left(\frac{\beta^2(t)-\theta(t)\beta(t)+\theta(t)}{\eta(t,s)}g(t,s)\right)^p\left(\frac{|e(t)|}{q}\right)^q\big[u(t)\big]^{\frac{\gamma(t,s)\theta(t)-\beta^2(t)}{\beta^2(t)-\theta(t)\beta(t)+\theta(t)}}\,\mathrm{d}\xi(s) \\
&\quad\geq \exp\left(\frac{\int_a^b \eta(t,s)\ln[(\frac{\beta^2(t)-\theta(t)\beta(t)+\theta(t)}{\eta(t,s)}g(t,s))^p(\frac{|e(t)|}{q})^q[u(t)]^{\frac{\gamma(t,s)\theta(t)-\beta^2(t)}{\beta^2(t)-\theta(t)\beta(t)+\theta(t)}}]\,\mathrm{d}\xi(s)}{\theta(t)}\right) \\
&\quad= \exp\left(\frac{\int_a^b \eta(t,s)\ln[(\frac{\beta^2(t)-\theta(t)\beta(t)+\theta(t)}{\eta(t,s)}g(t,s))^p(\frac{|e(t)|}{q})^q]\,\mathrm{d}\xi(s)}{\theta(t)}\right) \\
&\qquad\cdot\exp\left(\frac{\int_a^b \eta(t,s)[\frac{\gamma(t,s)\theta(t)-\beta^2(t)}{\beta^2(t)-\theta(t)\beta(t)+\theta(t)}]\ln u(t)\,\mathrm{d}\xi(s)}{\theta(t)}\right) \\
&\quad= \exp\left(\frac{\int_a^b \eta(t,s)\ln[(\frac{\beta^2(t)-\theta(t)\beta(t)+\theta(t)}{\eta(t,s)}g(t,s))^p(\frac{|e(t)|}{q})^q]\,\mathrm{d}\xi(s)}{\theta(t)}\right) \\
&\qquad\cdot\exp\left(\frac{\frac{\ln u(t)}{\beta^2(t)-\theta(t)\beta(t)+\theta(t)}\int_a^b \eta(t,s)[\gamma(t,s)\theta(t)-\beta^2(t)]\,\mathrm{d}\xi(s)}{\theta(t)}\right) \\
&\quad= \exp\left(\frac{p\int_a^b \eta(t,s)\ln[\frac{\beta^2(t)-\theta(t)\beta(t)+\theta(t)}{\eta(t,s)}g(t,s)]\,\mathrm{d}\xi(s)}{\theta(t)} + \frac{\ln(\frac{|e(t)|}{q})^q\int_a^b \eta(t,s)\,\mathrm{d}\xi(s)}{\theta(t)}\right) \\
&\quad= \exp\left(\frac{p\int_a^b \eta(t,s)[\ln(\beta^2(t)-\theta(t)\beta(t)+\theta(t))+\ln\frac{g(t,s)}{\eta(t,s)}]\,\mathrm{d}\xi(s)}{\theta(t)} + \ln\left(\frac{|e(t)|}{q}\right)^q\right) \\
&\quad= \left(\frac{|e(t)|}{q}\right)^q\exp\left(\frac{p\ln(\beta^2(t)-\theta(t)\beta(t)+\theta(t))\int_a^b \eta(t,s)\,\mathrm{d}\xi(s) + p\int_a^b \eta(t,s)\ln\frac{g(t,s)}{\eta(t,s)}\,\mathrm{d}\xi(s)}{\theta(t)}\right) \\
&\quad= \left(\frac{(\beta^2(t)-\theta(t)\beta(t)+\theta(t))|e(t)|}{\beta^2(t)-\theta(t)\beta(t)}\right)^{\frac{\beta^2(t)-\theta(t)\beta(t)}{\beta^2(t)-\theta(t)\beta(t)+\theta(t)}} \\
&\qquad\cdot\exp\left(\frac{\theta(t)}{\beta^2(t)-\theta(t)\beta(t)+\theta(t)}\left[\ln\big(\beta^2(t)-\theta(t)\beta(t)+\theta(t)\big) + \frac{\int_a^b \eta(t,s)\ln\frac{g(t,s)}{\eta(t,s)}\,\mathrm{d}\xi(s)}{\theta(t)}\right]\right).
\end{aligned}$$

Then from (24) and the above inequality, we have

$$\begin{aligned}
\omega'(t) &\geq q(t) + \left(\frac{(\beta^2(t)-\theta(t)\beta(t)+\theta(t))|e(t)|}{\beta^2(t)-\theta(t)\beta(t)}\right)^{\frac{\beta^2(t)-\theta(t)\beta(t)}{\beta^2(t)-\theta(t)\beta(t)+\theta(t)}} \\
&\qquad\cdot\exp\left(\frac{\theta(t)}{\beta^2(t)-\theta(t)\beta(t)+\theta(t)}\left[\ln\big(\beta^2(t)-\theta(t)\beta(t)+\theta(t)\big)\right.\right. \\
&\qquad\left.\left.+\frac{\int_a^b \eta(t,s)\ln\frac{g(t,s)}{\eta(t,s)}\,\mathrm{d}\xi(s)}{\theta(t)}\right]\right) + \frac{\omega^2(t)}{p(t)} \\
&= Q(t) + \frac{\omega^2(t)}{p(t)},
\end{aligned} \tag{35}$$

where $Q(t)$ is defined by (23) with $\theta(t) \in (m_1(t), \beta(t))$. The rest of the proof is similar to that of part (I) and hence is omitted. This completes the proof of Theorem 2.1. □

Following Philos [39] and Kong [40], we say that a function $H = H(t,s)$ belongs to a function class $\mathbf{H}$, denoted by $H \in \mathbf{H}$, if $H \in C(D,[0,\infty))$, where $D = \{(t,s) : t \geq s \geq t_0\}$, and H satisfies

$$H(t,t) = 0, \quad \text{for } t \geq t_0 \quad \text{and} \quad H(t,s) > 0, \quad \text{for } t > s \geq t_0, \tag{36}$$

and has continuous partial derivatives $\partial H/\partial t$ and $\partial H/\partial s$ on D such that

$$\frac{\partial H}{\partial t} = h_1(t,s)\sqrt{H(t,s)} \quad \text{and} \quad \frac{\partial H}{\partial s} = -h_2(t,s)\sqrt{H(t,s)}, \tag{37}$$

where $h_1, h_2 \in L_{\mathrm{loc}}(D,\mathbf{R})$.

Next, we use the function class $\mathbf{H}$ to establish an oscillation criterion for equation (1) of the Kong type.

Theorem 2.2 *Suppose that for any $T > 0$, there exist nontrivial subintervals $[a_1,b_1]$ and $[a_2,b_2]$ of $[T,+\infty)$ such that* (21) *holds for $i = 1,2$. We further assume that, for $i = 1,2$, there exist a constant $c_i \in (a_i,b_i)$ and functions $H \in \mathbf{H}$ and θ satisfying $\theta(t) \in (m_1(t),\beta(t)]$ for $t \in [t_0,+\infty)$, and a continuous function $\eta : [t_0,+\infty) \times [a,b] \to (0,+\infty)$ satisfying* (5) *and* (6), *where $m_1(t)$ is defined as in Lemma* 2.1 *such that*

$$\begin{aligned}&\frac{1}{H(c_i,a_i)}\int_{a_i}^{c_i}\left[Q(t)H(t,a_i) - \frac{p(t)h_1^2(t,a_i)}{4}\right]\mathrm{d}t\\&\quad + \frac{1}{H(b_i,c_i)}\int_{c_i}^{b_i}\left[Q(t)H(b_i,t) - \frac{p(t)h_2^2(b_i,t)}{4}\right]\mathrm{d}t \geq 0,\end{aligned} \tag{38}$$

where $Q(t)$ is defined by (23). *Then equation* (1) *is oscillatory.*

Proof Proceeding as in the proof of Theorem 2.1, we get

$$\omega'(t) \geq Q(t) + \frac{\omega^2(t)}{p(t)}, \quad t \in [a_1,b_1]; \tag{39}$$

see (28) and (35) for the cases when $\theta(t) \equiv \beta(t)$ and $\theta(t) \in (m_1(t),\beta(t))$, respectively. Let $c_i \in (a_i,b_i)$ be such that (38) holds. Multiplying both sides of (39) by $H(t,a_1)$, integrating it from a_1 to c_1, and using integration by parts we have

$$H(c_1,a_1)\omega(c_1) \geq \int_{a_1}^{c_1}\left[Q(t)H(t,a_1) + \omega(t)\frac{\partial H}{\partial t}(t,a_1) + \frac{H(t,a_1)\omega^2(t)}{p(t)}\right]\mathrm{d}t. \tag{40}$$

It follows from (36), (37), and (40) that

$$\begin{aligned}H(c_1,a_1)\omega(c_1) \geq{}& \int_{a_1}^{c_1}\left[Q(t)H(t,a_1) - \frac{p(t)h_1^2(t,a_1)}{4}\right]\mathrm{d}t\\&+ \int_{a_1}^{c_1}\left[\frac{\sqrt{p(t)}h_1(t,a_1)}{2} + \sqrt{\frac{H(t,a_1)}{p(t)}}\omega(t)\right]^2\mathrm{d}t.\end{aligned} \tag{41}$$

Similarly, multiplying both sides of (39) by $H(b_1,t)$ and integrating it from c_1 to b_1, we get

$$\begin{aligned}-H(b_1,c_1)\omega(c_1) \geq \int_{c_1}^{b_1}\left[Q(t)H(b_1,t)-\frac{p(t)h_2^2(b_1,t)}{4}\right]\mathrm{d}t \\ +\int_{c_1}^{b_1}\left[\frac{\sqrt{p(t)}h_2(b_1,t)}{2}+\sqrt{\frac{H(b_1,t)}{p(t)}}\omega(t)\right]^2\mathrm{d}t.\end{aligned} \tag{42}$$

By dividing (41) and (42) by $H(c_1,a_1)$ and $H(b_1,c_1)$, respectively, and then adding them together, from (38) we have

$$\frac{1}{H(c_1,a_1)}\int_{a_1}^{c_1}\left[\frac{\sqrt{p(t)}h_1(t,a_1)}{2}+\sqrt{\frac{H(t,a_1)}{p(t)}}\omega(t)\right]^2\mathrm{d}t=0 \tag{43}$$

and

$$\frac{1}{H(b_1,c_1)}\int_{c_1}^{b_1}\left[\frac{\sqrt{p(t)}h_2(b_1,t)}{2}+\sqrt{\frac{H(b_1,t)}{p(t)}}\omega(t)\right]^2\mathrm{d}t=0. \tag{44}$$

We can reach a contradiction from either of the above. For instance, (43) implies that

$$\frac{\sqrt{p(t)}h_1(t,a_1)}{2}+\sqrt{\frac{H(t,a_1)}{p(t)}}\omega(t)\equiv 0, \quad t\in[a_1,c_1].$$

It follows from the definition of w and (37) that

$$\frac{u'(t)}{u(t)}=\frac{h_1(t,a_1)}{2\sqrt{H(t,a_1)}}=\frac{\frac{\partial H}{\partial t}(t,a_1)}{2H(t,a_1)},$$

and hence $u(t)\equiv c\sqrt{H(t,a_1)}$ on $[a_1,c_1]$ for some constant $c\neq 0$. This contradicts the assumption that $H(a_1,a_1)=0$ and $u(a_1)>0$. This completes the proof of Theorem 2.2. □

3 Examples

In this section, we will work out two numerical examples to illustrate our main results. Here we use the convention that $\ln 0=-\infty$ and $e^{-\infty}=0$.

Example 3.1 We consider the following equation:

$$u''(t)+q(t)u(t)+\int_a^b g(t,s)\left|u(t)\right|^{\gamma(t,s)+1-\beta(t)}\operatorname{sgn}u(t)\,\mathrm{d}\xi(s)=e(t), \quad t\geq 0, \tag{45}$$

where $q(t)=\lambda\sin 4t$, $a=0$, $b=1$, $\gamma(t,s)=2se^{-t}$, $g(t,s)=\cos t$, $\beta(t)=e^{-t}$, $\xi(s)=s$, $e(t)=-f(t)\cos 2t$, and $\lambda>0$ is a constant and $f(t)\in C[0,\infty)$ is any nonnegative function. For any $T\in\mathbf{R}$, we choose $k\in\mathbf{Z}$ large enough for $2k\pi\geq T$ and let $a_1=2k\pi$, $a_2=b_1=2k\pi+\frac{\pi}{4}$, and $b_2=2k\pi+\frac{\pi}{2}$. Then $m_1(t)=\ln 2e^{-t}$ and (21) holds. Set

$$\theta(t)=\delta e^{-t}, \quad \delta\in(\ln 2,1], \qquad \eta(t,s)=\frac{\delta}{2\delta-1}s^{\frac{2-2\delta}{2\delta-1}}e^{-t}.$$

It is easy to verify that (5) and (6) are valid, and for $t \in [a_i, b_i]$, $i = 1, 2$, Let $v(t) = \sin 4t$. Note that, for $i = 1, 2$,

$$\int_{a_i}^{b_i} Q(t)v^2(t)\,\mathrm{d}t = \int_0^{\frac{\pi}{4}} F(\lambda, \delta, t)\sin^2 4t\,\mathrm{d}t,$$

where

$$\begin{aligned} F(\lambda, \delta, t) &= \lambda \sin 4t + \left[\left(1 + \frac{\delta e^t}{1-\delta}\right) f(t + 2k\pi)\cos(2t)\right]^{\frac{1-\delta}{1-\delta+\delta e^t}} \\ &\quad \cdot \exp\left(\frac{\delta e^t}{1-\delta+\delta e^t}\left[\ln\left(e^{-2t} - \delta e^{-2t} + \delta e^{-t}\right)\right.\right. \\ &\quad \left.\left. + \ln\cos t - \frac{e^t}{\delta}\int_0^1 \eta(t,s)\ln\eta(t,s)\,\mathrm{d}s\right]\right) \end{aligned}$$

and

$$\int_{a_i}^{b_i} v'^2(t)\,\mathrm{d}t = \int_0^{\frac{\pi}{4}} 16\cos^2 4t\,\mathrm{d}t = 2\pi.$$

Thus, by Theorem 2.1 we see that equation (45) is oscillatory for $\int_0^{\frac{\pi}{4}} F(\lambda, \delta, t)\sin^2 4t\,\mathrm{d}t \geq 2\pi$.

Example 3.2 We consider the following equation:

$$u''(t) + q(t)u(t) + \int_a^b g(t,s)\left|u(t)\right|^{\gamma(t,s)+1-\beta(t)} \operatorname{sgn} u(t)\,\mathrm{d}\xi(s) = e(t), \quad t \geq 0, \tag{46}$$

where $q(t) = \lambda \sin t$, $a = 0$, $b = 1$, $\gamma(t,s) = 2s(\cos\frac{t}{2})$, $g(t,s) = \cos t$, $\beta(t) = \cos\frac{t}{2}$, $\xi(s) = s$, $\lambda > 0$ is a constant. For any $T \in \mathbf{R}$, we choose $k \in \mathbf{Z}$ large enough for $2k\pi \geq T$ and let $a_1 = 2k\pi$, $a_2 = b_1 = 2k\pi + \frac{\pi}{4}$, $b_2 = 2k\pi + \frac{\pi}{2}$, $c_1 = 2k\pi + \frac{\pi}{8}$, and $c_2 = 2k\pi + \frac{3\pi}{8}$. Assume that $e(t) \in C[0,\infty)$ is any function satisfying $(-1)^i e(t) \geq 0$ on $[a_i, b_i]$ for $i = 1, 2$. Then (21) holds. Set

$$\theta(t) = \delta\cos\frac{t}{2}, \quad \delta \in (\ln 2, 1], \qquad \eta(t,s) = \frac{\delta}{2\delta - 1}s^{\frac{2-2\delta}{2\delta-1}}\cos\frac{t}{2}.$$

It is easy to verify that (5) and (6) are valid, and for $t \in [a_i, b_i]$, $i = 1, 2$,

$$\begin{aligned} Q(t) &= \lambda \sin t + \left[\left(1 + \frac{\delta}{\cos\frac{t}{2} - \delta\cos\frac{t}{2}}\right)\left|e(t)\right|\right]^{\frac{\cos\frac{t}{2} - \delta\cos\frac{t}{2}}{\cos\frac{t}{2} - \delta\cos\frac{t}{2} + \delta}} \\ &\quad \cdot \exp\left(\frac{\delta}{\cos\frac{t}{2} - \delta\cos\frac{t}{2} + \delta}\left[\ln\left(\cos^2\frac{t}{2} - \delta\cos^2\frac{t}{2} + \delta\cos\frac{t}{2}\right)\right.\right. \\ &\quad \left.\left. + \ln\cos t - \frac{\sec\frac{t}{2}}{\delta}\int_0^1 \eta(t,s)\ln\eta(t,s)\,\mathrm{d}s\right]\right). \end{aligned}$$

We choose $H(t,s) = (t-s)^2$, then by Theorem 2.2 we see that equation (46) is oscillatory if

$$\int_{2k\pi}^{2k\pi+\frac{\pi}{8}} Q(t)(t - 2k\pi)^2\,\mathrm{d}t + \int_{2k\pi+\frac{\pi}{8}}^{2k\pi+\frac{\pi}{4}} Q(t)(2k\pi + \pi/4 - t)^2\,\mathrm{d}t \geq \frac{\pi}{4}$$

and

$$\int_{2k\pi+\frac{\pi}{4}}^{2k\pi+\frac{3\pi}{8}} Q(t)(t-2k\pi-\pi/4)^2\,\mathrm{d}t + \int_{2k\pi+\frac{3\pi}{8}}^{2k\pi+\frac{\pi}{2}} Q(t)(2k\pi+\pi/2-t)^2\,\mathrm{d}t \geq \frac{\pi}{4}.$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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