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New concepts of Hahn calculus and impulsive Hahn difference equations

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Abstract
In this paper, we introduce new concepts of Hahn difference operator, the q_k, ω_k-Hahn difference operator. We aim to establish a calculus of differences based on the q_k, ω_k-Hahn difference operator. We construct a right inverse of the q_k, ω_k-Hahn operator and study some of its properties. As applications, we establish existence and uniqueness results for first- and second-order impulsive q_k, ω_k-Hahn difference equations.

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1 Introduction and preliminaries

Many physical phenomena are described by equations involving nondifferentiable functions, *e.g.*, generic trajectories of quantum mechanics [1]. Several different approaches to deal with nondifferentiable functions are followed in the literature, including the time scale approach, the fractional approach, and the quantum approach.

Quantum difference operators are receiving an increase of interest due to their applications see, *e.g.*, [2–10]. Roughly speaking, a quantum calculus substitutes the classical derivative by a difference operator, which allows one to deal with sets of nondifferentiable functions.

In [11], Hahn introduced the quantum difference operator $D_{q,\omega}$, where $q \in (0,1)$ and $\omega > 0$ are fixed. The Hahn operator unifies (in the limit) the two best-known and most-used quantum difference operators: the Jackson q-difference derivative D_q, where $q \in (0,1)$ (*cf.* [6, 12, 13]); and the forward difference D_ω where $\omega > 0$ (*cf.* [14–16]). The Hahn difference operator is a successful tool for constructing families of orthogonal polynomials and investigating some approximation problems (*cf.* [17–19]).

The aim of this paper is to introduce new concepts of Hahn's difference operator, the q_k, ω_k*-Hahn difference operator*, to establish a calculus based on this operator and to construct the associated integral. The steps are parallel to [20]. While some properties are straightforward extensions of classical results, some others need special treatments. As applications of the q_k, ω_k-Hahn difference operator we establish existence and uniqueness results for first- and second-order impulsive fractional differential equations.

Impulsive differential equations serve as basic models to study the dynamics of processes that are subject to sudden changes in their states. Recent development in this field has been motivated by many applied problems, such as control theory, population dynamics, and medicine. For some recent works on the theory of impulsive differential equations, we refer the interested reader to the monographs [21–23]. Impulsive quantum difference equations have been established by Tariboon and Ntouyas in [24] by improving the classical quantum calculus which does not work when there exists at least one impulsive point appearing between two different points in the definition of q-derivative. For recent results on the topics of initial and boundary value problems of impulsive quantum difference equations, we refer the reader to [7].

We organize this paper as follows. In Section 2, some basic formulas of Hahn's difference operator and the associated Jackson-Nörlund integral calculus are briefly reviewed. Our results are formulated and proved in Section 3. Applications to impulsive fractional difference equations are given in Section 4.

2 Preliminaries

Let $q \in (0,1)$ and $\omega > 0$. Define

$$\omega_0 := \frac{\omega}{1-q} \tag{2.1}$$

and let I be a real interval containing ω_0.

Definition 2.1 (Hahn's difference operator [11]) Let $f : I \to \mathbb{R}$. The Hahn difference operator of f is defined by

$$D_{q,\omega}f(t) = \begin{cases} \frac{f(t)-f(qt+\omega)}{t(1-q)-\omega}, & t \neq \omega_0, \\ f'(\omega_0), & t = \omega_0, \end{cases} \tag{2.2}$$

provided that f is differentiable at ω_0.

The function f is called q,ω-differentiable on I, if $D_{q,\omega}f(t)$ exists for all $t \in I$.

Note that when $q \to 1$ we obtain the forward ω-difference operator

$$D_{1,\omega}f(t) = \frac{f(t+\omega)-f(t)}{\omega}, \tag{2.3}$$

and when $\omega = 0$ we obtain the Jackson q-difference operator

$$D_{q,0}f(t) = \begin{cases} \frac{f(t)-f(qt)}{t(1-q)}, & t \neq 0, \\ f'(0), & t = 0, \end{cases} \tag{2.4}$$

provided that $f'(0)$ exists. Here f is supposed to be defined on a q-geometric set $A \subset \mathbb{R}$, for which $qt \in A$ whenever $t \in A$.

Hence, we can state that the $D_{q,\omega}$ operator generalizes (in the limit) the forward ω-difference and the Jackson q-difference operators [6, 25].

Notice also that, under appropriate conditions,

$$\lim_{q\to 1,\omega\to 0} D_{q,\omega}f(t) = f'(t).$$

The Hahn difference operator has the following properties.

Lemma 2.2 ([20]) *Let $f,g : I \to \mathbb{R}$ be q,ω-differentiable at $t \in I$. Then the following statements are true*:

(i) $D_{q,\omega}(f+g)(t) = D_{q,\omega}f(t) + D_{q,\omega}g(t)$,
(ii) $D_{q,\omega}fg(t) = g(t)D_{q,\omega}f(t) + f(qt+\omega)D_{q,\omega}g(t)$,
(iii) $D_{q,\omega}cf(t) = cD_{q,\omega}f(t)$, *for any constant* $c \in \mathbb{R}$,
(iv) $D_{q,\omega}(\frac{f}{g})(t) = \frac{g(t)D_{q,\omega}f(t)-f(t)D_{q,\omega}g(t)}{g(t)g(qt+\omega)}$, *for* $g(t)g(qt+\omega) \neq 0$,
(v) $f(tq+\omega) = f(t) + ((qt+\omega)-t)D_{q,\omega}f(t)$, $t \in I$.

Let $h(t) = qt+\omega$, $t \in I$. Note that h is a contraction, $h(I) \subseteq I$, $h(t) < t$ for $t > \omega_0$, $h(t) > t$ for $t < \omega_0$, and $h(\omega_0) = \omega_0$.

We use the standard notation of the q-number as $[\alpha]_q = \frac{1-q^\alpha}{1-q}$ for $\alpha \in \mathbb{R}$.

Lemma 2.3 ([20]) *Let $k \in \mathbb{N}$ and $t \in I$. Then*

$$h^k(t) = \underbrace{h \circ h \circ \cdots \circ h(t)}_{i\text{-times}} = q^k t + \omega[k]_q, \quad t \in I. \tag{2.5}$$

Next, we define the notion of a q,ω-integral, known as the Jackson-Nörlund integral.

Definition 2.4 ([20]) Let $f : I \to \mathbb{R}$ be a function and $a, b, \omega_0 \in I$. The q,ω-integral of f from a to b is defined by

$$\int_a^b f(s)\,d_{q,\omega}s = \int_{\omega_0}^b f(s)\,d_{q,\omega}s - \int_{\omega_0}^a f(s)\,d_{q,\omega}s, \tag{2.6}$$

where

$$\int_{\omega_0}^t f(s)\,d_{q,\omega}s = \bigl(t(1-q)-\omega\bigr)\sum_{k=0}^{\infty} q^k f\bigl(tq^k + \omega[k]_q\bigr), \quad t \in I, \tag{2.7}$$

provided that the series converges at $t = a$ and $t = b$.

The function f is q,ω-integrable over I if it is q,ω-integrable over $[a,b]$, for all $a, b \in I$.

Note that in the integral formulas (2.6) and (2.7), when $\omega \to 0$, we obtain the Jackson q-integral

$$\int_a^b f(s)\,d_q s = \int_0^b f(s)\,d_q s - \int_0^a f(s)\,d_q s,$$

where

$$\int_0^t f(s)\,d_q s = t(1-q)\sum_{k=0}^{\infty} q^k f\bigl(tq^k\bigr), \quad t \in I$$

(see, *e.g.*, [26]); while if $q \to 1$ we obtain the Nörlund sum,

$$\int_a^b f(s)\Delta_\omega s = \int_{+\infty}^b f(s)\Delta_\omega s - \int_{+\infty}^a f(s)\Delta_\omega s,$$

where

$$\int_{+\infty}^{t} f(s)\Delta_\omega s = -\omega \sum_{k=1}^{+\infty} f(t+k\omega)$$

(see, *e.g.*, [15, 27, 28]).

The following properties of Jackson-Nörlund integration can be found in [20].

Lemma 2.5 *Let $f,g: I \to \mathbb{R}$ be q,ω-integrable on I, $K \in \mathbb{R}$, and $a,b,c \in I$. Then the following formulas hold*:

(i) $\int_a^a f(t)\,d_{q,\omega}t = 0$,
(ii) $\int_a^b Kf(t)\,d_{q,\omega}t = K\int_a^b f(t)\,d_{q,\omega}t$,
(iii) $\int_a^b f(t)\,d_{q,\omega}t = -\int_b^a f(t)\,d_{q,\omega}t$,
(iv) $\int_a^b f(t)\,d_{q,\omega}t = \int_c^b f(t)\,d_{q,\omega}t + \int_a^c f(t)\,d_{q,\omega}t$,
(v) $\int_a^b (f(t)+g(t))\,d_{q,\omega}t = \int_a^b f(t)\,d_{q,\omega}t + \int_a^b g(t)\,d_{q,\omega}t$,
(vi) $\int_a^b f(t)D_{q,\omega}g(t)\,d_{q,\omega}t = [f(t)g(t)]_a^b - \int_a^b D_{q,\omega}f(t)g(qt+\omega)\,d_{q,\omega}t$.

Property (vi) of the above lemma is known as *q,ω-integration by parts*.

The next result is the *fundamental theorem of Hahn calculus*.

Lemma 2.6 ([20]) *Let $f: I \to \mathbb{R}$ be continuous at ω_0 and define $F(t) := \int_{\omega_0}^t f(s)\,d_{q,\omega}s$. Then F is continuous at ω_0. In addition, $D_{q,\omega}F(t)$ exists for every $t \in I$ and*

$$D_{q,\omega}F(t) = f(t). \tag{2.8}$$

On the other hand,

$$\int_a^b D_{q,\omega}f(s)\,d_{q,\omega}s = f(b) - f(a) \quad \text{for all } a,b \in I. \tag{2.9}$$

Existence and uniqueness results for first-order abstract Hahn difference equations were studied in [29], by using the method of successive approximation.

3 New concepts of Hahn calculus

Let there be a dense interval $J_k = [t_k, t_{k+1}] \subseteq \mathbb{R}$ and given constants $0 < q_k < 1$, $\omega_k > 0$ and

$$\theta_k = \frac{\omega_k}{1-q_k} + t_k. \tag{3.1}$$

Note that if $t_k = 0$, $q_k = q$, and $\omega_k = \omega$, then $\theta_k = \omega_0$, where ω_0 is defined in (2.1).

Definition 3.1 Let f be a function defined on J_k. The *q_k,ω_k-Hahn difference operator* is given by

$${}_{t_k}D_{q_k,\omega_k}f(t) = \frac{f(t) - f(q_k t + (1-q_k)t_k + \omega_k)}{(1-q_k)(t-t_k) - \omega_k}, \quad t \neq \theta_k, \tag{3.2}$$

and ${}_{t_k}D_{q_k,\omega_k}f(\theta_k) = f'(\theta_k)$ provided that f is differentiable at θ_k.

We say that f is q_k,ω_k-differentiable on J_k provided ${}_{t_k}D_{q_k,\omega_k}f(t)$ exists for all $t \in J_k$. Note that if $\omega_k = 0$ in (3.2), then ${}_{t_k}D_{q_k,0}f = {}_{t_k}D_{q_k}f$, where ${}_{t_k}D_{q_k}$ is the q_k-derivative of the function $f(t)$ which was first established in [24] by

$$ {}_{t_k}D_{q_k}f(t) = \frac{f(t) - f(q_k t + (1-q_k)t_k)}{(1-q_k)(t-t_k)}. \tag{3.3}$$

It is easy to see that if $t_k = 0$ and $q_k = q$, then (3.3) is reduced to the Jackson q-difference operator in (2.4).

Example 3.2 Let $f(t) = t^2$ for $t \in J_k = [2,16]$ and constants $q_k = 1/2$, $\omega_k = 3$. Then $\theta_k = 8$ and the q_k,ω_k-Hahn derivative on J_k is given by

$$\begin{aligned} {}_2D_{\frac{1}{2},3}f(t) &= \frac{t^2 - (\frac{1}{2}t + 4)^2}{(\frac{1}{2})(t-2) - 3} \\ &= \frac{3t^2 - 16t - 64}{2(t-8)}, \quad t \neq 8, \end{aligned}$$

and ${}_2D_{\frac{1}{2},3}f(8) = 64$.

It is easy to prove the following results.

Theorem 3.3 *Let $f,g : J_k \to \mathbb{R}$ be q_k,ω_k-differentiable at $t \in J_k$. Then the following formulas hold*:

(i) ${}_{t_k}D_{q_k,\omega_k}(f+g)(t) = {}_{t_k}D_{q_k,\omega_k}f(t) + {}_{t_k}D_{q_k,\omega_k}g(t)$,

(ii) ${}_{t_k}D_{q_k,\omega_k}fg(t) = g(t){}_{t_k}D_{q_k,\omega_k}f(t) + f(q_k t + (1-q_k)t_k + \omega_k){}_{t_k}D_{q_k,\omega_k}g(t)$,

(iii) ${}_{t_k}D_{q_k,\omega_k}cf(t) = c{}_{t_k}D_{q_k,\omega_k}f(t)$, *for any constant* $c \in \mathbb{R}$,

(iv) ${}_{t_k}D_{q_k,\omega_k}(\frac{f}{g})(t) = \frac{g(t){}_{t_k}D_{q_k,\omega_k}f(t) - f(t){}_{t_k}D_{q_k,\omega_k}g(t)}{g(t)g(q_k t + (1-q_k)t_k + \omega_k)}$, *for* $g(t)g(q_k t + (1-q_k)t_k + \omega_k) \neq 0$.

Next, we define the higher-order q_k,ω_k-derivative of functions.

Definition 3.4 Let f be a function defined on J_k. We define the second-order q_k,ω_k-derivative ${}_{t_k}D^2_{q_k,\omega_k}f$ provided ${}_{t_k}D_{q_k,\omega_k}f$ is q_k,ω_k-differentiable on J_k with ${}_{t_k}D^2_{q_k,\omega_k}f = {}_{t_k}D_{q_k,\omega_k}({}_{t_k}D_{q_k,\omega_k}f) : J_k \to \mathbb{R}$. In addition, we define the higher-order q_k,ω_k-derivative ${}_{t_k}D^n_{q_k,\omega_k}f : J_k \to \mathbb{R}$, with ${}_{t_k}D^n_{q_k,\omega_k}f = {}_{t_k}D_{q_k,\omega_k}({}_{t_k}D^{n-1}_{q_k,\omega_k}f)$ and ${}_{t_k}D^0_{q_k,\omega_k}f = f$.

The new definition of q_k,ω_k-integral is given as follows.

Definition 3.5 Assume $f : J_k \to \mathbb{R}$ is a function and $a,b \in J_k$. We define the q_k,ω_k-integral of f from a to b by

$$\int_a^b f(s)_{t_k}\,d_{q_k,\omega_k}s := \int_{\theta_k}^b f(s)_{t_k}\,d_{q_k,\omega_k}s - \int_{\theta_k}^a f(s)_{t_k}\,d_{q_k,\omega_k}s, \tag{3.4}$$

where

$$\int_{\theta_k}^t f(s)_{t_k}\,d_{q_k,\omega_k}s = \big[(t-t_k)(1-q_k) - \omega_k\big]\sum_{i=0}^{\infty} q_k^i f\big(q_k^i t + \big(1-q_k^i\big)t_k + \omega_k[i]_{q_k}\big) \tag{3.5}$$

for $t \in J_k$, provided that the series converge at $t = a$ and $t = b$. The function f is called q_k, ω_k-integrable on J_k and we say that f is q_k, ω_k-integrable over $[a, b]$ for all $a, b \in J_k$.

Note that if $t_k = 0$, $q_k = q$, and $\omega_k = \omega$, then (3.4) and (3.5) are reduced to (2.6) and (2.7), respectively.

As customary, the following properties should be to stated. However, the proof is easy and we omit it.

Theorem 3.6 *Let $f, g : J_k \to \mathbb{R}$ be q_k, ω_k-integrable on J_k, $K \in \mathbb{R}$, and $a, b, c \in J_k$. Then the following formulas hold*:

(i) $\int_a^a f(s)_{t_k}\, d_{q_k,\omega_k}s = 0$,

(ii) $\int_a^b Kf(s)_{t_k}\, d_{q_k,\omega_k}s = K\int_a^b f(s)_{t_k}\, d_{q_k,\omega_k}s$,

(iii) $\int_a^b f(s)_{t_k}\, d_{q_k,\omega_k}s = -\int_b^a f(s)_{t_k}\, d_{q_k,\omega_k}s$,

(iv) $\int_a^b f(s)_{t_k}\, d_{q_k,\omega_k}s = \int_c^b f(s)_{t_k}\, d_{q_k,\omega_k}s + \int_a^c f(s)_{t_k}\, d_{q_k,\omega_k}s$,

(v) $\int_a^b (f(s) + g(s))_{t_k}\, d_{q_k,\omega_k}s = \int_a^b f(s)_{t_k}\, d_{q_k,\omega_k}s + \int_a^b g(s)_{t_k}\, d_{q_k,\omega_k}s$.

Lemma 3.7 *Let h be the transformation*

$$h(t) := q_k t + (1 - q_k)t_k + \omega_k, \quad t \in J_k, \tag{3.6}$$

and $\theta_k \in J_k$ is defined by (3.1). *Then the ith-order iteration of h is given by*

$$h^i(t) = \underbrace{h \circ h \circ \cdots \circ h(t)}_{i\text{-times}} = q_k^i t + \left(1 - q_k^i\right)t_k + \omega_k[i]_{q_k}, \quad t \in J_k. \tag{3.7}$$

In addition, the sequence $\{h^i(t)\}_{i=1}^{\infty}$ is an increasing (*a decreasing*) *sequence in i when $t < \theta_k$* ($\theta_k < t$) *with*

$$\lim_{i\to\infty} h^i(t) = \theta_k, \quad t \in J_k. \tag{3.8}$$

Proof By directly computation, it is easy to show that (3.7) holds. For $t \in J_k$ and $i \in \mathbb{N}$, we have

$$\begin{aligned} h^{i+1}(t) - h^i(t) &= q_k^i(1 - q_k)(t_k - t) + \omega_k\left([i + 1]_{q_k} - [i]_{q_k}\right) \\ &= q_k^i(1 - q_k)(\theta_k - t). \end{aligned}$$

If $t < \theta_k$ or $\theta_k < t$, then we see that the sequence $\{h^i(t)\}_{i=1}^{\infty}$ is increasing or decreasing, respectively. Therefore, equation (3.8) is true for all $t \in J_k$. □

Now, we will state and prove the fundamental theorem of q_k, ω_k-Hahn calculus.

Theorem 3.8 *Suppose that the function $f : J_k \to \mathbb{R}$ is continuous at $\theta_k \in J_k$. We define*

$$F(t) := \int_{\theta_k}^{t} f(s)_{t_k}\, d_{q_k,\omega_k}s, \quad t \in J_k. \tag{3.9}$$

Then we have, for $t, a, b \in J_k$,

(i) ${}_{t_k}D_{q_k,\omega_k}F(t) = f(t)$,
(ii) $\int_{\theta_k}^{t} {}_{t_k}D_{q_k,\omega_k}f(s)_{t_k}\,d_{q_k,\omega_k}s = f(t) - f(\theta_k)$,
(iii) $\int_{a}^{b} {}_{t_k}D_{q_k,\omega_k}f(s)_{t_k}\,d_{q_k,\omega_k}s = f(b) - f(a)$.

Proof From (3.9), we observe that

$$\begin{aligned}
&F\big(q_k t + (1-q_k)t_k + \omega_k\big) \\
&\quad = \big[\big(\big(q_k t + (1-q_k)t_k + \omega_k\big) - t_k\big)(1-q_k) - \omega_k\big] \\
&\qquad \times \sum_{i=0}^{\infty} q_k^i f\big(\big(q_k t + (1-q_k)t_k + \omega_k\big)q_k^i + \big(1-q_k^i\big)t_k + \omega_k[i]_{q_k}\big) \\
&\quad = \big[\big(q_k(t-t_k) + \omega_k\big)(1-q_k) - \omega_k\big] \sum_{i=0}^{\infty} q_k^i f\big(q_k^{i+1}t + \big(1-q_k^{i+1}\big)t_k + \omega_k[i+1]_{q_k}\big).
\end{aligned}$$

Then, by (3.2), we have

$$\begin{aligned}
{}_{t_k}D_{q_k,\omega_k}F(t) &= \frac{F(t) - F(q_k t + (1-q_k)t_k + \omega_k)}{(1-q_k)(t-t_k) - \omega_k} \\
&= \sum_{i=0}^{\infty} q_k^i \bigg[f\big(q_k^i t + \big(1-q_k^i\big)t_k + \omega_k[i]_{q_k}\big) \\
&\quad - \frac{(q_k(t-t_k) + \omega_k)(1-q) - \omega_k}{(1-q_k)(t-t_k) - \omega_k} f\big(q_k^{i+1}t + \big(1-q_k^{i+1}\big)t_k + \omega_k[i+1]_{q_k}\big)\bigg] \\
&= \sum_{i=0}^{\infty} q_k^i \big[f\big(q_k^i t + \big(1-q_k^i\big)t_k + \omega_k[i]_{q_k}\big) \\
&\quad - q_k f\big(q_k^{i+1}t + \big(1-q_k^{i+1}\big)t_k + \omega_k[i+1]_{q_k}\big)\big] \\
&= f(t).
\end{aligned}$$

This shows that (i) holds.

To prove (ii), by Definitions 3.1, 3.5, and Lemma 3.7, we get

$$\begin{aligned}
&\int_{\theta_k}^{t} {}_{t_k}D_{q_k,\omega_k}f(s)_{t_k}\,d_{q_k,\omega_k}s \\
&\quad = \big[(t-t_k)(1-q_k) - \omega_k\big] \sum_{i=0}^{\infty} q_k^i ({}_{t_k}D_{q_k,\omega_k}f)\big(q_k^i t + \big(1-q_k^i\big)t_k + \omega_k[i]_{q_k}\big) \\
&\quad = \big[(t-t_k)(1-q_k) - \omega_k\big] \sum_{i=0}^{\infty} q_k^i \\
&\qquad \times \frac{f(q_k^i t + (1-q_k^i)t_k + \omega_k[i]_{q_k}) - f(q_k(q_k^i t + (1-q_k^i)t_k + \omega_k[i]_{q_k}) + (1-q_k)t_k + \omega_k)}{(1-q_k)(q_k^i t + (1-q_k^i)t_k + \omega_k[i]_{q_k} - t_k) - \omega_k} \\
&\quad = \sum_{i=0}^{\infty} \big(f\big(q_k^i t + \big(1-q_k^i\big)t_k + \omega_k[i]_{q_k}\big) - f\big(q_k^{i+1}t + \big(1-q_k^{i+1}\big)t_k + \omega_k[i+1]_{q_k}\big)\big) \\
&\quad = f(t) - f(\theta_k).
\end{aligned}$$

Now, we show that (iii) holds. From (ii) for any $a, b \in J_k$, we obtain

$$\begin{aligned}\int_a^b {}_{t_k}D_{q_k,\omega_k}f(s)_{t_k}\,d_{q_k,\omega_k}s &= \int_{\theta_k}^b {}_{t_k}D_{q_k,\omega_k}f(s)_{t_k}\,d_{q_k,\omega_k}s - \int_{\theta_k}^a {}_{t_k}D_{q_k,\omega_k}f(s)_{t_k}\,d_{q_k,\omega_k}s \\ &= f(b) - f(a).\end{aligned}$$

This completes the proof. □

Lemma 3.9 *Let $f, g : J_k \to \mathbb{R}$ be q_k, ω_k-integrable on J_k. Then the following integration by parts formula holds*:

$$\begin{aligned}&\int_a^b g(s)_{t_k}D_{q_k,\omega_k}f(s)_{t_k}\,d_{q_k,\omega_k}s \\ &\quad = \big[f(s)g(s)\big]_a^b - \int_a^b f\big(q_k s + (1-q_k)t_k + \omega_k\big)_{t_k}D_{q_k,\omega_k}g(s)_{t_k}\,d_{q_k,\omega_k}s.\end{aligned}$$

Proof By Theorem 3.8 we have

$$\int_a^b {}_{t_k}D_{q_k,\omega_k}\big[f(s)g(s)\big]_{t_k}\,d_{q_k,\omega_k}s = (fg)(b) - (fg)(a).$$

On the other hand, by (ii) of Theorem 3.3 and (v) of Theorem 3.6,

$$\begin{aligned}&\int_a^b {}_{t_k}D_{q_k,\omega_k}\big[f(s)g(s)\big]_{t_k}\,d_{q_k,\omega_k}s \\ &\quad = \int_a^b g(s)_{t_k}D_{q_k,\omega_k}f(s)_{t_k}\,d_{q_k,\omega_k}s \\ &\qquad + \int_a^b f\big(q_k s + (1-q_k)t_k + \omega_k\big)_{t_k}D_{q_k,\omega_k}g(s)_{t_k}\,d_{q_k,\omega_k}s.\end{aligned}$$

Combining these two equalities we get the desired formula. □

Lemma 3.10 *Let $\theta_k \in J_k$, $\alpha \in \mathbb{R}$, and $\beta \in \mathbb{R} \setminus \{-1\}$. Then for $t \in J_k$ the following formulas hold*:

(i) ${}_{t_k}D_{q_k}(t-\theta_k)^\alpha = [\alpha]_{q_k}(t-\theta_k)^{\alpha-1}$,

(ii) $\int_{\theta_k}^t (s-\theta_k)^\beta {}_{t_k}\,d_{q_k,\omega_k}s = (\frac{1-q_k}{1-q_k^{\beta+1}})(t-\theta_k)^{\beta+1}$.

Proof From Definition 3.1, for $t \neq \theta_k$, we have

$$\begin{aligned}{}_{t_k}D_{q_k,\omega_k}(t-\theta_k)^\alpha &= \frac{(t-\theta_k)^\alpha - (q_k t + (1-q_k)t_k + \omega_k - \theta_k)^\alpha}{(1-q_k)(t-t_k) - \omega_k} \\ &= \frac{(t-\theta_k)^\alpha - q_k^\alpha(t-\theta_k)^\alpha}{(1-q_k)(t-\theta_k)} \\ &= [\alpha]_{q_k}(t-\theta_k)^{\alpha-1}.\end{aligned}$$

For $t = \theta_k$, we obtain ${}_{t_k}D_{q_k,\omega_k}0 = 0$. Therefore the formula (i) holds.

Now, we are going to prove (ii). For $\beta \in \mathbb{R} \setminus \{-1\}$, Definition 3.5 implies

$$\begin{aligned}\int_{\theta_k}^{t} (s-\theta_k)^{\beta}{}_{t_k}\, d_{q_k,\omega_k} s &= \big[(t-t_k)(1-q_k)-\omega_k\big] \\ &\quad \times \sum_{i=0}^{\infty} q_k^i \big(q_k^i t + \big(1-q_k^i\big)t_k + \omega_k[i]_{q_k} - \theta_k\big)^{\beta} \\ &= (1-q_k)[t-\theta_k] \sum_{i=0}^{\infty} q_k^i \big(q_k^i (t-\theta_k)\big)^{\beta} \\ &= \left(\frac{1-q_k}{1-q_k^{\beta+1}}\right)(t-\theta_k)^{\beta+1}.\end{aligned}$$

The proof is completed. □

Corollary 3.11 *For $a, b \in J_k$, the following formula holds*:

$$\int_{a}^{b} (s-\theta_k)^{\beta}{}_{t_k}\, d_{q_k,\omega_k} s = \left(\frac{1-q_k}{1-q_k^{\beta+1}}\right)\big[(b-\theta_k)^{\beta+1} - (a-\theta_k)^{\beta+1}\big]. \tag{3.10}$$

Example 3.12 From Corollary 3.11 for $a, b \in J_k$, we have the following cases:

(i) If $\beta = 0$, then $\int_a^b 1_{t_k}\, d_{q_k,\omega_k} s = b-a$.

(ii) If $\beta = 1$, then $\int_a^b (s-\theta_k)_{t_k}\, d_{q_k,\omega_k} s = \frac{(b-a)}{1+q_k}[b+a-2\theta_k]$.

(iii) $\int_{t_k}^b (s-t_k)_{t_k}\, d_{q_k,\omega_k} s = \frac{(b-t_k)^2-\omega_k(b-t_k)}{1+q_k}$.

(i) and (ii) are obvious. To prove (iii), from (i) and (ii) we obtain

$$\begin{aligned}\int_{t_k}^{b} (s-t_k)_{t_k}\, d_{q_k,\omega_k} s &= \int_{t_k}^{b} (s-\theta_k)_{t_k}\, d_{q_k,\omega_k} s - (t_k-\theta_k)\int_{t_k}^{b} {}_{t_k}\, d_{q_k,\omega_k} s \\ &= \frac{(b-t_k)}{1+q_k}[b+t_k-2\theta_k] - (t_k-\theta_k)(b-t_k) \\ &= \frac{(b-t_k)}{1+q_k}\left[b-t_k-\frac{2\omega_k}{1-q_k}\right] + \frac{\omega_k}{1-q_k}(b-t_k) \\ &= \frac{(b-t_k)^2-\omega_k(b-t_k)}{1+q_k}.\end{aligned}$$

Theorem 3.13 *Let f be the q_k, ω_k-integrable function on J_k. Then we have*

$$\int_{\theta_k}^{t} \int_{\theta_k}^{s} f(r)_{t_k}\, d_{q_k,\omega_k} r_{t_k}\, d_{q_k,\omega_k} s = \int_{\theta_k}^{t} \int_{q_k r+(1-q_k)t_k+\omega_k}^{t} f(r)_{t_k}\, d_{q_k,\omega_k} s_{t_k}\, d_{q_k,\omega_k} r. \tag{3.11}$$

Proof By Definition 3.5, we have

$$\begin{aligned}&\int_{\theta_k}^{t} \int_{\theta_k}^{s} f(r)_{t_k}\, d_{q_k,\omega_k} r_{t_k}\, d_{q_k,\omega_k} s \\ &\quad = \int_{\theta_k}^{t} \big[(s-t_k)(1-q_k)-\omega_k\big] \sum_{i=0}^{\infty} q_k^i f\big(q_k^i s + \big(1-q_k^i\big)t_k + \omega_k[i]_{q_k}\big)_{t_k}\, d_{q_k,\omega_k} s\end{aligned}$$

$$
\begin{aligned}
&= (1-q_k)\sum_{i=0}^{\infty} q_k^i \int_{\theta_k}^{t} (s-\theta_k) f\big(q_k^i s + \big(1-q_k^i\big)t_k + \omega_k[i]_{q_k}\big)_{t_k}\, d_{q_k,\omega_k} s \\
&= (1-q_k)^2 (t-\theta_k)\sum_{i=0}^{\infty} q_k^i \Bigg(\sum_{j=0}^{\infty} q_k^j \big(q_k^j t + \big(1-q_k^j\big)t_k + \omega_k[j]_{q_k} - \theta_k\big) \\
&\qquad \times f\big(q_k^i\big(q_k^j t + \big(1-q_k^j\big)t_k + \omega_k[j]_{q_k}\big) + \big(1-q_k^i\big)t_k + \omega_k[i]_{q_k}\big)\Bigg) \\
&= (1-q_k)^2 (t-\theta_k)^2 \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} q_k^{i+2j} f\big(q_k^{i+j} t + \big(1-q_k^{i+j}\big)t_k + \omega_k[i+j]_{q_k}\big).
\end{aligned}
$$

Indeed,

$$
\begin{aligned}
&\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} q_k^{i+2j} f\big(q_k^{i+j} t + \big(1-q_k^{i+j}\big)t_k + \omega_k[i+j]_{q_k}\big) \\
&= \sum_{i=0}^{\infty}\big[q_k^i f\big(q_k^i t + \big(1-q_k^i\big)t_k + \omega[i]_{q_k}\big) + q_k^{i+2} f\big(q_k^{i+1} t + \big(1-q_k^{i+1}\big)t_k + \omega[i+1]_{q_k}\big) \\
&\quad + q_k^{i+4} f\big(q_k^{i+2} t + \big(1-q_k^{i+2}\big)t_k + \omega[i+2]_{q_k}\big) \\
&\quad + q_k^{i+6} f\big(q_k^{i+3} t + \big(1-q_k^{i+3}\big)t_k + \omega[i+3]_{q_k}\big) + \cdots\big] \\
&= f(t) + q_k^2 f\big(q_k t + (1-q_k)t_k + \omega_k[1]_{q_k}\big) + q_k^4 f\big(q_k^2 t + \big(1-q_k^2\big)t_k + \omega_k[2]_{q_k}\big) \\
&\quad + q^6 f\big(q_k^3 t + \big(1-q_k^3\big)t_k + \omega_k[3]_{q_k}\big) + \cdots + q_k f\big(q_k t + (1-q_k)t_k + \omega_k[1]_{q_k}\big) \\
&\quad + q_k^3 f\big(q_k^2 t + \big(1-q_k^2\big)t_k + \omega_k[2]_{q_k}\big) + q^5 f\big(q_k^3 t + \big(1-q_k^3\big)t_k + \omega_k[3]_{q_k}\big) + \cdots \\
&\quad + q_k^2 f\big(q_k^2 t + \big(1-q_k^2\big)t_k + \omega_k[2]_{q_k}\big) + q_k^4 f\big(q_k^3 t + \big(1-q_k^3\big)t_k + \omega_k[3]_{q_k}\big) \\
&\quad + q_k^6 f\big(q_k^4 t + \big(1-q_k^4\big)t_k + \omega_k[4]_{q_k}\big) + \cdots + q_k^3 f\big(q_k^3 t + \big(1-q_k^3\big)t_k + \omega_k[3]_{q_k}\big) + \cdots \\
&= f(t) + q_k(1+q_k) f\big(q_k t + (1-q_k)t_k + \omega_k[1]_{q_k}\big) \\
&\quad + q_k^2\big(1+q_k+q_k^2\big) f\big(q_k^2 t + \big(1-q_k^2\big)t_k + \omega_k[2]_{q_k}\big) \\
&\quad + q_k^3\big(1+q_k+q_k^2+q_k^3\big) f\big(q_k^3 t + \big(1-q_k^3\big)t_k + \omega_k[3]_{q_k}\big) + \cdots \\
&= \sum_{n=0}^{\infty} q_k^n \left(\frac{1-q_k^{n+1}}{1-q_k}\right) f\big(q_k^n t + \big(1-q_k^n\big)t_k + \omega_k[n]_{q_k}\big).
\end{aligned}
$$

Hence, we obtain

$$
\begin{aligned}
&\int_{\theta_k}^{t}\int_{\theta_k}^{s} f(r)_{t_k}\, d_{q_k,\omega_k} r_{t_k}\, d_{q_k,\omega_k} s \\
&= (1-q_k)(t-\theta_k)\sum_{i=0}^{\infty} q_k^n\big(1-q_k^{n+1}\big)(t-\theta_k) f\big(q_k^n t + \big(1-q_k^n\big)t_k + \omega_k[n]_{q_k}\big) \\
&= \int_{\theta_k}^{t}\big(t - q_k r - (1-q_k)t_k - \omega_k\big) f(r)_{t_k}\, d_{q_k,\omega_k} s_{t_k}\, d_{q_k,\omega_k} r \\
&= \int_{\theta_k}^{t}\int_{q_k r + (1-q_k)t_k + \omega_k}^{t} f(r)_{t_k}\, d_{q_k,\omega_k} s_{t_k}\, d_{q_k,\omega_k} r.
\end{aligned}
$$

This completes the proof. □

4 Impulsive q_k, ω_k-Hahn difference equations

In this section, we use our results on q_k, ω_k-Hahn calculus to establish existence and uniqueness results for impulsive q_k, ω_k-Hahn difference equations of the first and second order. Let $J_0 = [t_0, t_1]$, $J_k = (t_k, t_{k+1}]$ for $k = 1, 2, \ldots, m$ be subintervals of $J = [0, T]$ such that $\theta_k \in J_k$ for $k = 0, 1, 2, \ldots, m$. Let $PC(J, \mathbb{R}) = \{x : J \to \mathbb{R} : x(t)$ is continuous everywhere except for some t_k at which $x(t_k^+)$ and $x(t_k^-)$ exist and $x(t_k^-) = x(t_k)$, $k = 1, 2, \ldots, m\}$. $PC(J, \mathbb{R})$ is a Banach space with the norm $\|x\|_{PC} = \sup\{|x(t)| : t \in J\}$.

4.1 First-order impulsive q_k, ω_k-Hahn difference equations

In this subsection, we study the existence and uniqueness of solutions for the following initial value problem for first-order impulsive q_k, ω_k-Hahn difference equation

$$\begin{cases} {}_{t_k}D_{q_k,\omega_k} x(t) = f(t, x(t)), & t \in J, t \neq t_k, \\ \Delta x(t_k) = \varphi_k(x(t_k)), & k = 1, 2, \ldots, m, \\ x(0) = \alpha, \end{cases} \tag{4.1}$$

where $\alpha \in \mathbb{R}$, $0 = t_0 < t_1 < t_2 < \cdots < t_k < \cdots < t_m < t_{m+1} = T$, $f : J \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $\varphi_k \in C(\mathbb{R}, \mathbb{R})$, $\Delta x(t_k) = x(t_k^+) - x(t_k)$, $k = 1, 2, \ldots, m$, and quantum numbers $0 < q_k < 1$, $\omega_k > 0$ such that $\theta_k \in J_k$ for $k = 0, 1, 2, \ldots, m$.

Lemma 4.1 *Let $x \in PC(J, \mathbb{R})$ satisfying* (4.1). *The impulsive q_k, ω_k-Hahn difference initial value problem* (4.1) *is equivalent to the integral equation*

$$\begin{aligned} x(t) = \alpha &+ \sum_{t_0 < t_k < t} \int_{t_{k-1}}^{t_k} f\big(s, x(s)\big)_{t_{k-1}} \, d_{q_{k-1},\omega_{k-1}} s + \sum_{t_0 < t_k < t} \varphi_k\big(x(t_k)\big) \\ &+ \int_{t_k}^{t} f\big(s, x(s)\big)_{t_k} \, d_{q_k,\omega_k} s, \end{aligned} \tag{4.2}$$

with $\sum_{t_0 < t_0} = 0$.

Proof For $t \in J_0$, applying q_0, ω_0-integral from t_0 to t in the first equation of (4.1) and using Theorem 3.8(iii), we obtain

$$x(t) = \alpha + \int_{t_0}^{t} f\big(s, x(s)\big)_{t_0} \, d_{q_0,\omega_0} s.$$

Since $\theta_0 \in J_0$, we have $t_1 \geq \theta_0$ and also, for $t = t_1$,

$$x(t_1) = \alpha + \int_{t_0}^{t_1} f\big(s, x(s)\big)_{t_0} \, d_{q_0,\omega_0} s.$$

For $t \in J_1$, taking the q_1, ω_1-integral to the first equation of (4.1) with $k = 1$ and applying Theorem 3.8(iii) again, we have

$$x(t) = x\big(t_1^+\big) + \int_{t_1}^{t} f\big(s, x(s)\big)_{t_1} \, d_{q_1,\omega_1} s.$$

From the impulsive condition $x(t_1^+) = x(t_1) + \varphi_1(x(t_1))$, we get

$$x(t) = \alpha + \int_{t_0}^{t_1} f\bigl(s,x(s)\bigr)_{t_0}\, d_{q_0,\omega_0}s + \int_{t_1}^{t} f\bigl(s,x(s)\bigr)_{t_1}\, d_{q_1,\omega_1}s + \varphi_1\bigl(x(t_1)\bigr).$$

For $t \in J_2$, the q_2,ω_2-integration and impulsive condition imply

$$\begin{aligned} x(t) &= x\bigl(t_2^+\bigr) + \int_{t_2}^{t} f\bigl(s,x(s)\bigr)_{t_2}\, d_{q_2,\omega_2}s \\ &= \alpha + \int_{t_0}^{t_1} f\bigl(s,x(s)\bigr)_{t_0}\, d_{q_0,\omega_0}s + \int_{t_1}^{t_2} f\bigl(s,x(s)\bigr)_{t_1}\, d_{q_1,\omega_1}s + \int_{t_2}^{t} f\bigl(s,x(s)\bigr)_{t_2}\, d_{q_2,\omega_2}s \\ &\quad + \varphi_1\bigl(x(t_1)\bigr) + \varphi_2\bigl(x(t_2)\bigr). \end{aligned}$$

From the above process, for any $t \in J_k$, $k = 0,1,\ldots,m$, we obtain the desired result in (4.2).

Conversely, for any $t \in J_k$, $k = 0,1,\ldots,m$, applying q_k,ω_k-derivative to (4.2) and using Theorem 3.8(i), we have

$${}_{t_k}D_{q_k,\omega_k}x(t) = f\bigl(t,x(t)\bigr).$$

By direct computation, we have $\Delta x(t_k) = \varphi_k(x(t_k))$ and also $x(0) = \alpha$. The proof is completed. □

Now, we are in a position to prove an existence and uniqueness result for the problem (4.1), via Banach contraction mapping principle.

Theorem 4.2 *Suppose that the following assumptions are fulfilled*:

(H_1) *the continuous function* $f : J \times \mathbb{R} \to \mathbb{R}$ *satisfies*

$$\bigl|f(t,x) - f(t,y)\bigr| \le L_1|x-y|, \quad L_1 > 0, \forall t \in J, x,y \in \mathbb{R};$$

(H_2) *the continuous functions* $\varphi_k : \mathbb{R} \to \mathbb{R}$, $k = 1,2,\ldots,m$ *satisfy*

$$\bigl|\varphi_k(x) - \varphi_k(y)\bigr| \le L_2|x-y|, \quad L_2 > 0, \forall x,y \in \mathbb{R}.$$

If

$$L_1T + mL_2 < 1, \tag{4.3}$$

then the impulsive q_k,ω_k*-Hahn difference initial value problem* (4.1) *has a unique solution on* J.

Proof Let us define an operator $\mathcal{A} : PC(J,\mathbb{R}) \to PC(J,\mathbb{R})$ by

$$\begin{aligned} \mathcal{A}x(t) &= \alpha + \sum_{t_0<t_k<t} \int_{t_{k-1}}^{t_k} f\bigl(s,x(s)\bigr)_{t_{k-1}}\, d_{q_{k-1},\omega_{k-1}}s + \sum_{t_0<t_k<t} \varphi_k\bigl(x(t_k)\bigr) \\ &\quad + \int_{t_k}^{t} f\bigl(s,x(s)\bigr)_{t_k}\, d_{q_k,\omega_k}s, \end{aligned}$$

with $\sum_{t_0<t_0} = 0$. Let $\sup_{t\in J}|f(t,0)| = M_1$ and $\max\{|\varphi_k(0)| : k = 1,2,\dots,m\} = M_2$. Choosing a positive constant r such that

$$r \geq \frac{|\alpha| + M_1T + mM_2}{1-(L_1T+mL_2)},$$

and setting a ball $B_r = \{x \in PC(J,\mathbb{R}) : \|x\| \leq r\}$, we will show that $\mathcal{A}B_r \subset B_r$. For any $x \in B_r$ and $t \in J$, we have

$$\begin{aligned}
\big|\mathcal{A}x(t)\big| &\leq |\alpha| + \sum_{t_0<t_k<t} \int_{t_{k-1}}^{t_k} \big|f\big(s,x(s)\big)\big|_{t_{k-1}}\, d_{q_{k-1},\omega_{k-1}}s + \sum_{t_0<t_k<t} \big|\varphi_k\big(x(t_k)\big)\big| \\
&\quad + \int_{t_k}^{t} \big|f\big(s,x(s)\big)\big|_{t_k}\, d_{q_k,\omega_k}s \\
&\leq |\alpha| + \sum_{t_0<t_k<T} \int_{t_{k-1}}^{t_k} \big(\big|f\big(s,x(s)\big)-f(t,0)\big| + \big|f(t,0)\big|\big)_{t_{k-1}}\, d_{q_{k-1},\omega_{k-1}}s \\
&\quad + \sum_{t_0<t_k<T} \big(\big|\varphi_k\big(x(t_k)\big) - \varphi_k(0)\big| + \big|\varphi_k(0)\big|\big) \\
&\quad + \int_{t_m}^{T} \big(\big|f\big(s,x(s)\big)-f(t,0)\big| + \big|f(t,0)\big|\big)_{t_m}\, d_{q_m,\omega_m}s \\
&\leq |\alpha| + (L_1r+M_1) \sum_{t_0<t_k<T} \int_{t_{k-1}}^{t_k} {}_{t_{k-1}} d_{q_{k-1},\omega_{k-1}}s \\
&\quad + m(L_2r+M_2) + (L_1r+M_1) \int_{t_m}^{T} {}_{t_m} d_{q_m,\omega_m}s \\
&= |\alpha| + M_1T + mM_2 + r(L_1T+mL_2) \leq r.
\end{aligned}$$

This means that $\|\mathcal{A}x\| \leq r$, which yields $\mathcal{A}B_r \subset B_r$.

For $x, y \in PC(J,\mathbb{R})$ and for each $t \in J$, we have

$$\begin{aligned}
\big|\mathcal{A}x(t) - \mathcal{A}y(t)\big| &\leq \sum_{t_0<t_k<t} \int_{t_{k-1}}^{t_k} \big|f\big(s,x(s)\big) - f\big(s,y(s)\big)\big|_{t_{k-1}}\, d_{q_{k-1},\omega_{k-1}}s \\
&\quad + \sum_{t_0<t_k<t} \big|\varphi_k\big(x(t_k)\big) - \varphi_k\big(y(t_k)\big)\big| \\
&\quad + \int_{t_k}^{t} \big|f\big(s,x(s)\big) - f\big(s,y(s)\big)\big|_{t_k}\, d_{q_k,\omega_k}s \\
&\leq L_1\|x-y\| \sum_{t_0<t_k<T} \int_{t_{k-1}}^{t_k} {}_{t_{k-1}} d_{q_{k-1},\omega_{k-1}}s \\
&\quad + mL_2\|x-y\| + L_1\|x-y\| \int_{t_m}^{T} {}_{t_m} d_{q_m,\omega_m}s \\
&= (L_1T+mL_2)\|x-y\|,
\end{aligned}$$

which leads to $\|\mathcal{A}x - \mathcal{A}y\| \leq (L_1T+mL_2)\|x-y\|$. As $L_1T + mL_2 < 1$, it follows from the Banach contraction mapping principle that $\mathcal{A}$ is a contraction. Hence, we deduce that $\mathcal{A}$ has a fixed point which is the unique solution of (4.1) on J. This completes the proof. □

Example 4.3 Consider the first-order impulsive q_k,ω_k-Hahn difference initial value problem of the form

$$\begin{cases} {}_{t_k}D_{\frac{k+1}{k+2},\frac{1}{k+3}}x(t) = \frac{1}{(t^2+40)}\big(\frac{x^2(t)+2|x(t)|}{|x(t)|+1}\big)e^{-t} + \frac{3}{4}, \quad t \in J, t \neq t_k = k, \\ \Delta x(t_k) = \frac{|x(t_k)|}{(4+k)(|x(t_k)|+4)}, \quad k = 1,2,\dots,9, \\ x(0) = \frac{2}{3}. \end{cases} \tag{4.4}$$

Here $J = [0,10]$, $q_k = (k+1)/(k+2)$, $\omega_k = 1/(k+3)$, $k = 0,1,\dots,9$, $m = 9$, $T = 10$, $f(t,x) = (1/(t^2+40))((x^2+2|x|)/(|x|+1))e^{-t} + (3/4)$, and $\varphi_k(x) = (|x|)/((4+k)(|x|+4))$. Observe that $\theta_k = \omega_k/(1-q_k) + t_k = (k^2+4k+2)/(k+3) \in J_k$, $k = 0,1,\dots,9$. Since $|f(t,x) - f(t,y)| \le (1/20)|x-y|$ and $|\varphi_k(x) - \varphi_k(y)| \le (1/20)|x-y|$, then ($H_1$) and ($H_2$) are satisfied with $L_1 = 1/20$ and $L_2 = 1/20$, respectively. We can show that

$$L_1 T + mL_2 = \frac{1}{2} + \frac{9}{20} = \frac{19}{20} < 1.$$

Therefore, by Theorem 4.2, we deduce that the problem (4.4) has a unique solution on $[0,10]$.

4.2 Second-order impulsive q_k,ω_k-Hahn difference equations

In this subsection, we consider the second-order initial value problem of the impulsive q_k,ω_k-Hahn difference equation

$$\begin{cases} {}_{t_k}D^2_{q_k,\omega_k}x(t) = f(t,x(t)), \quad t \in J, t \neq t_k, \\ \Delta x(t_k) = \varphi_k(x(t_k)), \quad k = 1,2,\dots,m, \\ {}_{t_k}D_{q_k,\omega_k}x(t_k^+) - {}_{t_{k-1}}D_{q_{k-1},\omega_{k-1}}x(t_k) = \varphi_k^*(x(t_k)), \quad k = 1,2,\dots,m, \\ x(0) = \alpha, \qquad {}_{t_0}D_{q_0,\omega_0}x(0) = \beta, \end{cases} \tag{4.5}$$

where $\alpha,\beta \in \mathbb{R}$, $0 = t_0 < t_1 < t_2 < \cdots < t_k < \cdots < t_m < t_{m+1} = T$, $f \in C(J \times \mathbb{R},\mathbb{R})$, $\varphi_k,\varphi_k^* \in C(\mathbb{R},\mathbb{R})$, $\Delta x(t_k) = x(t_k^+) - x(t_k)$, $k = 1,2,\dots,m$, and the numbers $0 < q_k < 1$, $\omega_k > 0$ such that $\theta_k \in J_k$ for $k = 0,1,2,\dots,m$.

Lemma 4.4 *A function $x \in PC(J,\mathbb{R})$ is the solution of* (4.5) *if and only if x satisfies the integral equation*

$$\begin{aligned} x(t) &= \alpha + \beta t + \sum_{t_0<t_k<t}\bigg(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s} f\big(u,x(u)\big)\,{}_{t_{k-1}}d_{q_{k-1},\omega_{k-1}}u\,{}_{t_{k-1}}d_{q_{k-1},\omega_{k-1}}s + \varphi_k\big(x(t_k)\big)\bigg) \\ &\quad + t\bigg[\sum_{t_0<t_k<t}\bigg(\int_{t_{k-1}}^{t_k} f\big(s,x(s)\big)\,{}_{t_{k-1}}d_{q_{k-1},\omega_{k-1}}s + \varphi_k^*\big(x(t_k)\big)\bigg)\bigg] \\ &\quad - \sum_{t_0<t_k<t} t_k\bigg(\int_{t_{k-1}}^{t_k} f\big(s,x(s)\big)\,{}_{t_{k-1}}d_{q_{k-1},\omega_{k-1}}s + \varphi_k^*\big(x(t_k)\big)\bigg) \\ &\quad + \int_{t_k}^{t}\int_{t_k}^{s} f\big(u,x(u)\big)\,{}_{t_k}d_{q_k,\omega_k}u\,{}_{t_k}d_{q_k,\omega_k}s, \end{aligned} \tag{4.6}$$

with $\sum_{t_0<t_0} = 0$.

Proof For $t \in J_0$, taking q_0, ω_0-integral for the first equation of (4.5) and using the second initial condition, we get

$$\begin{aligned} {}_{t_0}D_{q_0,\omega_0}x(t) &= {}_{t_0}D_{q_0,\omega_0}x(0) + \int_{t_0}^{t} f\bigl(s,x(s)\bigr)_{t_0}\, d_{q_0,\omega_0}s \\ &= \beta + \int_{t_0}^{t} f\bigl(s,x(s)\bigr)_{t_0}\, d_{q_0,\omega_0}s, \end{aligned} \tag{4.7}$$

which leads to

$${}_{t_0}D_{q_0,\omega_0}x(t_1) = \beta + \int_{t_0}^{t_1} f\bigl(s,x(s)\bigr)_{t_0}\, d_{q_0,\omega_0}s.$$

For $t \in J_0$, the q_0, ω_0-integration for (4.7) and the first initial condition of (4.5) imply

$$x(t) = \alpha + \beta t + \int_{t_0}^{t} \int_{t_0}^{s} f\bigl(u,x(u)\bigr)_{t_0}\, d_{q_0,\omega_0}u_{t_0}\, d_{q_0,\omega_0}s.$$

In particular, for $t = t_1$, we have

$$x(t_1) = \alpha + \beta t_1 + \int_{t_0}^{t_1} \int_{t_0}^{s} f\bigl(u,x(u)\bigr)_{t_0}\, d_{q_0,\omega_0}u_{t_0}\, d_{q_0,\omega_0}s.$$

Let us consider the interval $J_1 = (t_1, t_2]$. By the q_1, ω_1-integration for (4.5) with respect to $t \in J_1$, we have

$${}_{t_1}D_{q_1,\omega_1}x(t) = {}_{t_1}D_{q_1,\omega_1}x\bigl(t_1^+\bigr) + \int_{t_1}^{t} f\bigl(s,x(s)\bigr)_{t_1}\, d_{q_1,\omega_1}s.$$

From the second impulsive condition of (4.5), that is, ${}_{t_1}D_{q_1,\omega_1}x(t_1^+) = {}_{t_0}D_{q_0,\omega_0}x(t_1) + \varphi_1^*(x(t_1))$, we obtain

$${}_{t_1}D_{q_1,\omega_1}x(t) = \beta + \int_{t_0}^{t_1} f\bigl(s,x(s)\bigr)_{t_0}\, d_{q_0,\omega_0}s + \int_{t_1}^{t} f\bigl(s,x(s)\bigr)_{t_1}\, d_{q_1,\omega_1}s + \varphi_1^*\bigl(x(t_1)\bigr). \tag{4.8}$$

For $t \in J_1$, taking the q_1, ω_1-integration for (4.8) and using Example 3.12(i), we get

$$\begin{aligned} x(t) &= x\bigl(t_1^+\bigr) + \left[\beta + \int_{t_0}^{t_1} f\bigl(s,x(s)\bigr)_{t_0}\, d_{q_0,\omega_0}s + \varphi_1^*\bigl(x(t_1)\bigr)\right](t - t_1) \\ &\quad + \int_{t_1}^{t} \int_{t_1}^{s} f\bigl(u,x(u)\bigr)_{t_1}\, d_{q_1,\omega_1}u_{t_1}\, d_{q_1,\omega_1}s. \end{aligned}$$

Applying the first impulsive condition of (4.5), that is, $x(t_1^+) = x(t_1) + \varphi_1(x(t_1))$, we obtain

$$\begin{aligned} x(t) &= \alpha + \beta t_1 + \int_{t_0}^{t_1} \int_{t_0}^{s} f\bigl(u,x(u)\bigr)_{t_0}\, d_{q_0,\omega_0}u_{t_0}\, d_{q_0,\omega_0}s + \varphi_1\bigl(x(t_1)\bigr) \\ &\quad + \left[\beta + \int_{t_0}^{t_1} f\bigl(s,x(s)\bigr)_{t_0}\, d_{q_0,\omega_0}s + \varphi_1^*\bigl(x(t_1)\bigr)\right](t - t_1) \\ &\quad + \int_{t_1}^{t} \int_{t_1}^{s} f\bigl(u,x(u)\bigr)_{t_1}\, d_{q_1,\omega_1}u_{t_1}\, d_{q_1,\omega_1}s \end{aligned}$$

$$
\begin{aligned}
&= \alpha + \beta t + \int_{t_0}^{t_1} \int_{t_0}^{s} f\big(u, x(u)\big)_{t_0}\, d_{q_0,\omega_0} u_{t_0}\, d_{q_0,\omega_0} s + \varphi_1\big(x(t_1)\big) \\
&\quad + \left[\int_{t_0}^{t_1} f\big(s, x(s)\big)_{t_0}\, d_{q_0,\omega_0} s + \varphi_1^*\big(x(t_1)\big)\right](t - t_1) \\
&\quad + \int_{t_1}^{t} \int_{t_1}^{s} f\big(u, x(u)\big)_{t_1}\, d_{q_1,\omega_1} u_{t_1}\, d_{q_1,\omega_1} s.
\end{aligned}
$$

Repeating the above method, for $t \in J$, we obtain (4.6) as desired.

Conversely, it can easily be shown by direct computation that the integral equation (4.6) satisfies the impulsive initial value problem (4.5). This completes the proof. □

From Example 3.12(iii) with $b = t_{k+1}$, we set the notation

$$
\Omega(k) = \frac{(t_{k+1} - t_k)^2 - \omega_k (t_{k+1} - t_k)}{1 + q_k}.
$$

Also, we use the notations

$$
\begin{aligned}
\Psi(U) &= U_1 \left(\sum_{k=1}^{m+1} \Omega(k-1) + T(t_m - t_0) + \sum_{k=1}^{m} t_k (t_k - t_{k-1}) \right) \\
&\quad + mU_2 + U_3 \left(mT + \sum_{k=1}^{m} t_k \right),
\end{aligned}
\tag{4.9}
$$

where $U \in \{L, N\}$.

Theorem 4.5 *Assume that the conditions* (H_1) *and* (H_2) *of Theorem* 4.2 *are satisfied. Further, we suppose that*:

(H_3) *The continuous functions* $\varphi_k^* : \mathbb{R} \to \mathbb{R}$, $k = 1, 2, \ldots, m$, *satisfies*

$$
\left|\varphi_k^*(x) - \varphi_k^*(y)\right| \le L_3 |x - y|, \quad L_3 > 0, \forall x, y \in \mathbb{R}.
$$

If

$$
\Psi(L) < 1, \tag{4.10}
$$

where $\Psi(L)$ *is defined by* (4.9), *then the impulsive* q_k, ω_k*-Hahn difference initial value problem* (4.5) *has a unique solution on* J.

Proof In view of Lemma 4.4, we define an operator $\mathcal{Q} : PC(J, \mathbb{R}) \to PC(J, \mathbb{R})$ by

$$
\begin{aligned}
\mathcal{Q}x(t) &= \alpha + \beta t + \sum_{t_0 < t_k < t} \left(\int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{s} f\big(u, x(u)\big)_{t_{k-1}}\, d_{q_{k-1},\omega_{k-1}} u_{t_{k-1}}\, d_{q_{k-1},\omega_{k-1}} s + \varphi_k\big(x(t_k)\big) \right) \\
&\quad + t \left[\sum_{t_0 < t_k < t} \left(\int_{t_{k-1}}^{t_k} f\big(s, x(s)\big)_{t_{k-1}}\, d_{q_{k-1},\omega_{k-1}} s + \varphi_k^*\big(x(t_k)\big) \right) \right] \\
&\quad - \sum_{t_0 < t_k < t} t_k \left(\int_{t_{k-1}}^{t_k} f\big(s, x(s)\big)_{t_{k-1}}\, d_{q_{k-1},\omega_{k-1}} s + \varphi_k^*\big(x(t_k)\big) \right) \\
&\quad + \int_{t_k}^{t} \int_{t_k}^{s} f\big(u, x(u)\big)_{t_k}\, d_{q_k,\omega_k} u_{t_k}\, d_{q_k,\omega_k} s,
\end{aligned}
$$

with $\sum_{t_0<t_0} = 0$. By transforming the impulsive initial value problem (4.5) into a fixed point problem $x = \mathcal{Q}x$, we will show that the operator $\mathcal{Q}$ has a fixed point which is a unique solution of problem (4.5) via the Banach contraction mapping principle.

Setting $\sup_{t\in J}|f(t,0)| = N_1$, $\max\{|\varphi_k(0)| : k = 1,2,\dots,m\} = N_2$, and $\max\{|\varphi_k^*(0)| : k = 1,2,\dots,m\} = N_3$, we will prove that $\mathcal{Q}B_R \subset B_R$, where $B_R = \{x \in PC(J,\mathbb{R}) : \|x\| \le R\}$ and the positive constant R satisfies

$$R \ge \frac{|\alpha| + |\beta|T + \Psi(N)}{1 - \Psi(L)}. \tag{4.11}$$

For $x \in B_R$, taking into account Example 3.12(iii), we get

$$\begin{aligned}
\bigl|\mathcal{Q}x(t)\bigr| &\le |\alpha| + |\beta|T \\
&\quad + \sum_{t_0<t_k<T}\biggl(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s}\bigl(\bigl|f\bigl(u,x(u)\bigr) - f(u,0)\bigr| \\
&\quad + \bigl|f(u,0)\bigr|\bigr)_{t_{k-1}}\,d_{q_{k-1},\omega_{k-1}}u_{t_{k-1}}\,d_{q_{k-1},\omega_{k-1}}s + \bigl(\bigl|\varphi_k\bigl(x(t_k)\bigr) - \varphi_k(0)\bigr| + \bigl|\varphi_k(0)\bigr|\bigr)\biggr) \\
&\quad + T\biggl[\sum_{t_0<t_k<T}\biggl(\int_{t_{k-1}}^{t_k}\bigl(\bigl|f\bigl(s,x(s)\bigr) - f(s,0)\bigr| + \bigl|f(s,0)\bigr|\bigr)_{t_{k-1}}\,d_{q_{k-1},\omega_{k-1}}s \\
&\quad + \bigl(\bigl|\varphi_k^*\bigl(x(t_k)\bigr) - \varphi_k^*(0)\bigr| + \bigl|\varphi_k^*(0)\bigr|\bigr)\biggr)\biggr] \\
&\quad + \sum_{t_0<t_k<T} t_k\biggl(\int_{t_{k-1}}^{t_k}\bigl(\bigl|f\bigl(s,x(s)\bigr) - f(s,0)\bigr| + \bigl|f(s,0)\bigr|\bigr)_{t_{k-1}}\,d_{q_{k-1},\omega_{k-1}}s \\
&\quad + \bigl(\bigl|\varphi_k^*\bigl(x(t_k)\bigr) - \varphi_k^*(0)\bigr| + \bigl|\varphi_k^*(0)\bigr|\bigr)\biggr) \\
&\quad + \int_{t_m}^{T}\int_{t_m}^{s}\bigl(\bigl|f\bigl(u,x(u)\bigr) - f(u,0)\bigr| + \bigl|f(u,0)\bigr|\bigr)_{t_m}\,d_{q_m,\omega_m}u_{t_m}\,d_{q_m,\omega_m}s \\
&\le |\alpha| + |\beta|T + \sum_{k=1}^{m}\bigl((L_1R + N_1)\Omega(k-1) + L_2R + N_2\bigr) \\
&\quad + T\Biggl[\sum_{k=1}^{m}\bigl((t_k - t_{k-1})(L_1R + N_1) + L_3R + N_3\bigr)\Biggr] \\
&\quad + \sum_{k=1}^{m} t_k\bigl((t_k - t_{k-1})(L_1R + N_1) + L_3R + N_3\bigr) + (L_1R + N_1)\Omega(m) \\
&= |\alpha| + |\beta|T + R\Psi(L) + \Psi(N) \le R.
\end{aligned}$$

Then we have $\|\mathcal{Q}x\| \le R$, which implies $\mathcal{Q}B_R \subset B_R$.

Finally, for $x, y \in PC(J,\mathbb{R})$ and for each $t \in J$, we get

$$\begin{aligned}
\bigl|\mathcal{Q}x(t) - \mathcal{Q}y(t)\bigr| &\le \sum_{t_0<t_k<T}\biggl(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s}\bigl|f\bigl(u,x(u)\bigr) - f\bigl(u,y(u)\bigr)\bigr|_{t_{k-1}}\,d_{q_{k-1},\omega_{k-1}}u_{t_{k-1}}\,d_{q_{k-1},\omega_{k-1}}s \\
&\quad + \bigl|\varphi_k\bigl(x(t_k)\bigr) - \varphi_k\bigl(y(t_k)\bigr)\bigr|\biggr)
\end{aligned}$$

$$
\begin{aligned}
&+ T\Bigg[\sum_{t_0<t_k<T}\bigg(\int_{t_{k-1}}^{t_k}\big|f\big(s,x(s)\big)-f\big(s,y(s)\big)\big|_{t_{k-1}}\,d_{q_{k-1},\omega_{k-1}}s \\
&+ \big|\varphi_k^*\big(x(t_k)\big)-\varphi_k^*\big(y(t_k)\big)\big|\bigg)\Bigg] \\
&+ \sum_{t_0<t_k<T} t_k\bigg(\int_{t_{k-1}}^{t_k}\big|f\big(s,x(s)\big)-f\big(s,y(s)\big)\big|_{t_{k-1}}\,d_{q_{k-1},\omega_{k-1}}s \\
&+ \big|\varphi_k^*\big(x(t_k)\big)-\varphi_k^*\big(y(t_k)\big)\big|\bigg) \\
&+ \int_{t_m}^{T}\int_{t_m}^{s}\big|f\big(u,x(u)\big)-f\big(u,y(u)\big)\big|_{t_m}\,d_{q_m,\omega_m}u_{t_m}\,d_{q_m,\omega_m}s \\
\le{}& \sum_{k=1}^{m}\big[L_1\Omega(k-1)\|x-y\|+L_2\|x-y\|\big] \\
&+ T\Bigg[\sum_{k=1}^{m}\big[(t_k-t_{k-1})L_1\|x-y\|+L_3\|x-y\|\big]\Bigg] \\
&+ \sum_{k=1}^{m} t_k\big[(t_k-t_{k-1})L_1\|x-y\|+L_3\|x-y\|\big]+L_1\Omega(m)\|x-y\| \\
={}& \Psi(L)\|x-y\|.
\end{aligned}
$$

It follows that $\|\mathcal{Q}x-\mathcal{Q}y\| \le \Psi(L)\|x-y\|$. As $\Psi(L)<1$, we deduce from the Banach contraction mapping principle that $\mathcal{Q}$ is a contraction. Therefore, we see that the operator $\mathcal{Q}$ has a fixed point which is a unique solution of the impulsive q_k,ω_k-Hahn difference initial value problem (4.5) on J. The proof is completed. □

Example 4.6 Consider the second-order impulsive q_k,ω_k-Hahn difference initial value problem of the form

$$
\begin{cases}
{}_{t_k}D^2_{\frac{k+3}{4k+6},\frac{k+1}{2k+5}}x(t)=\frac{1}{t^2+10}\big(\frac{x^2(t)+2|x(t)|}{1+|x(t)|}\big)\frac{e^{-\cos^2 t}}{88}+\frac{1}{2}, & t\in J, t\neq t_k=k,\\
\Delta x(t_k)=\frac{|x(t_k)|}{5(k+5)(1+|x(t_k)|)}+\frac{2}{3}, & k=1,2,\ldots,9,\\
{}_{t_k}D_{\frac{k+3}{4k+6},\frac{k+1}{2k+5}}x(t_k^+)-{}_{t_{k-1}}D_{\frac{k+2}{4k+2},\frac{k}{2k+3}}x(t_k)=\frac{|\sin x(t_k)|}{10(\sqrt{k}+40)}+\frac{3}{4}, & k=1,2,\ldots,9,\\
x(0)=\frac{2}{3}, \qquad {}_{t_0}D_{\frac{1}{2},\frac{1}{5}}x(0)=\frac{5}{7}.
\end{cases}
\tag{4.12}
$$

Here $J=[0,10]$, $q_k=(k+3)/(4k+6)$, $\omega_k=(k+1)/(2k+5)$, $k=0,1,\ldots,9$, $m=9$, $T=10$, $\alpha=2/3$, $\beta=5/7$, $f(t,x)=(1/(t^2+10))((x^2+2|x|)/(1+|x|))(e^{-\cos^2 t}/88)+(1/2)$, $\varphi_k(x)=(|x|/(5(k+5)(1+|x|)))+(2/3)$, and $\varphi_k^*(x)=(|\sin x|)/(10(\sqrt{k}+40))+(3/4)$. Observe that $\theta_k=\omega_k/(1-q_k)+t_k=(6k^2+19k+6)/(6k+15)\in J_k$, $k=0,1,\ldots,9$. Also, we can find that $\sum_{k=1}^{10}\Omega(k-1)=4.720324567$.

Since $|f(t,x)-f(t,y)|\le(1/440)|x-y|$, $|\varphi_k(x)-\varphi_k(y)|\le(1/30)|x-y|$, and $|\varphi_k^*(x)-\varphi_k^*(y)|\le(1/410)|x-y|$, (H_1), (H_2), and (H_3) are satisfied with $L_1=1/440$, $L_2=1/30$, and $L_2=1/410$, respectively. From the above information, we find that

$$\Psi(L)=0.9468144850<1.$$

Therefore, by Theorem 4.5, we deduce that the problem (4.12) has a unique solution on $[0,10]$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally in this article. They read and approved the final manuscript.

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