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Two-dimensional product-type system of difference equations solvable in closed form

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Abstract

A solvable two-dimensional product-type system of difference equations of interest is presented. Closed form formulas for its general solution are given.

MSC: Primary 39A10; 39A20

Keywords: system of difference equations; product-type system; solvable in closed form

1 Introduction

Concrete nonlinear difference equations and systems have become of some interest recently. Experts have proposed various classes of the equations and systems hoping that their studies will lead to some new general results or will bring about some new methods in the theory (see, *e.g.*, [1–22]). Many of the papers study or are motivated by the study of symmetric systems (see, *e.g.*, [4–7, 9, 10, 14, 17–22]). It turned out that some of the equations and systems can be solved, which motivated some experts to work on the topic (see, *e.g.*, [1, 8, 11, 13–17, 19–22]; for some old results see, *e.g.*, [23–25]). One of the motivations for the renewed interest in the area has been Stević's method/idea for transforming some nonlinear equations into solvable linear ones (see, for example, [11, 13, 19, 20] and numerous related references therein). It also turned out that many classes of nonlinear difference equations and systems can be transformed to solvable ones by using some tricks and suitable changes of variables (see, *e.g.*, [8, 13, 16, 19] and the related references therein).

Numerous recent equations and systems are closely related to product-type ones, which are solvable for the case of positive initial values (see, *e.g.*, the equation in [12], which is a kind of perturbation of some product-type and the system in [18]; see also the related references therein). If the initial values are not positive, then there appear several problems. Thus, it is of some interest to describe the product-type systems with complex initial values which are solvable. A detailed study of the problem has been started recently by Stević *et al.* in [14, 15, 17, 21, 22] (some subclasses of the class of difference equations studied in [16] are also product-type ones). During the investigation we realized that the solvability of some product-type systems is preserved if some coefficients/multipliers are added. The first system of this type was studied in [14]. Based on this idea, quite recently in [22] it has been shown that the solvability of the system studied in [17] is preserved if two coefficients/multipliers are added. On the other hand, it can be seen that there are only several classes of product-type systems of difference equations which can be *practically* solved in closed form, due to the well-known fact that roots of the polynomials of degree $d \geq 5$ cannot be solved by radicals. Hence, it is of interest to find all the classes of practically solvable product-type systems of difference equations and present formulas for their solutions in terms of the initial values and parameters.

Here we present a new class of product-type systems of difference equations which are solvable under some natural assumptions. Namely, we investigate the solvability of the system

$$z_{n+1} = \alpha z_n^a w_{n-1}^b, \qquad w_{n+1} = \beta w_{n-1}^c z_{n-1}^d, \quad n \in \mathbb{N}_0, \tag{1}$$

where $a, b, c, d \in \mathbb{Z}$, $\alpha, \beta \in \mathbb{C}$ and $z_{-1}, z_0, w_{-1}, w_0 \in \mathbb{C}$. It is interesting that none of the subclasses of the class in (1) has been previously treated in our papers on product-type systems, so that all the formulas presented here should be new. The formulas are obtained by further developing the methods in our previous papers, especially the ones in [14] and [22].

A solution to system (1) need not be defined if its initial values belong to the set

$$\mathcal{U} = \{(z_{-1}, z_0, w_{-1}, w_0) \in \mathbb{C}^4 : z_{-1} = 0 \text{ or } z_0 = 0 \text{ or } w_{-1} = 0 \text{ or } w_0 = 0\}.$$

Thus, from now on we will assume that $z_{-1}, z_0, w_{-1}, w_0 \in \mathbb{C} \setminus \{0\}$. Since the cases $\alpha = 0$ and $\beta = 0$ are trivial or produce solutions which are not well defined we will also assume that $\alpha\beta \neq 0$.

Let us also note that we will use the convention $\sum_{i=k}^{l} a_i = 0$, when $l < k$, throughout the paper.

2 Main results

The main results in this paper are proved in this section.

Theorem 1 *Assume that $b, c, d \in \mathbb{Z}$, $a = 0$, $\alpha, \beta \in \mathbb{C} \setminus \{0\}$, and $z_{-1}, z_0, w_{-1}, w_0 \in \mathbb{C} \setminus \{0\}$. Then system* (1) *is solvable in closed form.*

Proof Since $a = 0$ system (1) is

$$z_{n+1} = \alpha w_{n-1}^b, \qquad w_{n+1} = \beta w_{n-1}^c z_{n-1}^d, \quad n \in \mathbb{N}_0. \tag{2}$$

Using the first equation in (2) in the second one, we obtain

$$w_{n+1} = \beta \alpha^d w_{n-1}^c w_{n-3}^{bd}, \quad n \geq 2, \tag{3}$$

from which it follows that

$$w_{2n+1} = \beta \alpha^d w_{2n-1}^c w_{2n-3}^{bd}, \quad n \in \mathbb{N}, \tag{4}$$

and

$$w_{2n+2} = \beta \alpha^d w_{2n}^c w_{2n-2}^{bd}, \quad n \in \mathbb{N}. \tag{5}$$

Case $bd = 0$. In this case equations (4) and (5) become

$$w_{2n+1} = \beta\alpha^d w_{2n-1}^c, \quad n \in \mathbb{N}, \tag{6}$$

and

$$w_{2n+2} = \beta\alpha^d w_{2n}^c, \quad n \in \mathbb{N}, \tag{7}$$

from which it follows that

$$\begin{aligned} w_{2n+1} &= \left(\beta\alpha^d\right)^{\sum_{j=0}^{n-1} c^j} w_1^{c^n} = \left(\beta\alpha^d\right)^{\sum_{j=0}^{n-1} c^j} \left(\beta w_{-1}^c z_{-1}^d\right)^{c^n} \\ &= \beta^{\sum_{j=0}^{n} c^j} \alpha^{d\sum_{j=0}^{n-1} c^j} w_{-1}^{c^{n+1}} z_{-1}^{dc^n}, \quad n \in \mathbb{N}, \end{aligned} \tag{8}$$

$$\begin{aligned} w_{2n} &= \left(\beta\alpha^d\right)^{\sum_{j=0}^{n-2} c^j} w_2^{c^{n-1}} = \left(\beta\alpha^d\right)^{\sum_{j=0}^{n-2} c^j} \left(\beta w_0^c z_0^d\right)^{c^{n-1}} \\ &= \beta^{\sum_{j=0}^{n-1} c^j} \alpha^{d\sum_{j=0}^{n-2} c^j} w_0^{c^n} z_0^{dc^{n-1}}, \quad n \geq 2. \end{aligned} \tag{9}$$

Hence

$$w_{2n+1} = \beta^{\frac{1-c^{n+1}}{1-c}} \alpha^{d\frac{1-c^n}{1-c}} w_{-1}^{c^{n+1}} z_{-1}^{dc^n}, \quad n \in \mathbb{N}, \tag{10}$$

$$w_{2n} = \beta^{\frac{1-c^n}{1-c}} \alpha^{d\frac{1-c^{n-1}}{1-c}} w_0^{c^n} z_0^{dc^{n-1}}, \quad n \geq 2, \tag{11}$$

when $c \neq 1$, and

$$w_{2n+1} = \beta^{n+1}\alpha^{dn} w_{-1} z_{-1}^d, \quad n \in \mathbb{N}, \tag{12}$$

$$w_{2n} = \beta^n \alpha^{d(n-1)} w_0 z_0^d, \quad n \geq 2, \tag{13}$$

when $c = 1$.

By using (8) and (9) in the first equation in (2) with $n \to 2n$ and $n \to 2n-1$, respectively, we get

$$z_{2n+1} = \alpha w_{2n-1}^b = \alpha\beta^{b\sum_{j=0}^{n-1} c^j} w_{-1}^{bc^n}, \quad n \geq 2, \tag{14}$$

$$z_{2n} = \alpha w_{2n-2}^b = \alpha\beta^{b\sum_{j=0}^{n-2} c^j} w_0^{bc^{n-1}}, \quad n \geq 3. \tag{15}$$

Hence, from (14) and (15) we have

$$z_{2n+1} = \alpha\beta^{b\frac{1-c^n}{1-c}} w_{-1}^{bc^n}, \quad n \geq 2, \tag{16}$$

$$z_{2n} = \alpha\beta^{b\frac{1-c^{n-1}}{1-c}} w_0^{bc^{n-1}}, \quad n \geq 3, \tag{17}$$

when $c \neq 1$, and

$$z_{2n+1} = \alpha\beta^{bn} w_{-1}^b, \quad n \geq 2, \tag{18}$$

$$z_{2n} = \alpha\beta^{b(n-1)} w_0^b, \quad n \geq 3, \tag{19}$$

when $c = 1$.

Case $bd \neq 0$. Let $\gamma := \beta\alpha^d$

$$a_1 := c, \qquad b_1 = bd, \qquad x_1 := 1. \tag{20}$$

Then (4) and (5) can be written as

$$w_{2n+1} = \gamma^{x_1} w_{2n-1}^{a_1} w_{2n-3}^{b_1}, \quad n \in \mathbb{N}, \tag{21}$$

and

$$w_{2n+2} = \gamma^{x_1} w_{2n}^{a_1} w_{2n-2}^{b_1}, \quad n \in \mathbb{N}. \tag{22}$$

By using (21) with $n \to n-1$ into (21), we get

$$\begin{aligned} w_{2n+1} &= \gamma^{x_1} \left(\gamma w_{2n-3}^{a_1} w_{2n-5}^{b_1}\right)^{a_1} w_{2n-3}^{b_1} \\ &= \gamma^{x_1+a_1} w_{2n-3}^{a_1 a_1 + b_1} w_{2n-5}^{b_1 a_1} \\ &= \gamma^{x_2} w_{2(n-2)+1}^{a_2} w_{2(n-3)+1}^{b_2}, \end{aligned} \tag{23}$$

for $n \geq 2$, where

$$a_2 := a_1 a_1 + b_1, \qquad b_2 := b_1 a_1, \qquad x_2 := x_1 + a_1. \tag{24}$$

Assume that

$$w_{2n+1} = \gamma^{x_k} w_{2(n-k)+1}^{a_k} w_{2(n-k-1)+1}^{b_k}, \tag{25}$$

for some $k \geq 2$ and every $n \geq k$, where

$$a_k := a_1 a_{k-1} + b_{k-1}, \qquad b_k := b_1 a_{k-1}, \qquad x_k := x_{k-1} + a_{k-1}. \tag{26}$$

Using (21) with $n \to n-k$ into (25) we get

$$\begin{aligned} w_{2n+1} &= \gamma^{x_k} \left(\gamma w_{2(n-k-1)+1}^{a_1} w_{2(n-k-2)+1}^{b_1}\right)^{a_k} w_{2(n-k-1)+1}^{b_k} \\ &= \gamma^{x_k + a_k} w_{2(n-k-1)+1}^{a_1 a_k + b_k} w_{2(n-k-2)+1}^{b_1 a_k} \\ &= \gamma^{x_{k+1}} w_{2(n-k-1)+1}^{a_{k+1}} w_{2(n-k-2)+1}^{b_{k+1}}, \end{aligned} \tag{27}$$

for every $n \geq k+1$, where

$$a_{k+1} := a_1 a_k + b_k, \qquad b_{k+1} := b_1 a_k, \qquad x_{k+1} := x_k + a_k. \tag{28}$$

Equalities (23), (24), (27), (28), along with the induction show that (25) and (26) hold for all natural numbers k and n such that $2 \leq k \leq n$. Moreover, because of (21), equality (25) holds for $1 \leq k \leq n$.

For $n = k$, (25) becomes

$$w_{2n+1} = \gamma^{x_n} w_1^{a_n} w_{-1}^{b_n}, \quad n \in \mathbb{N}. \tag{29}$$

Using the equalities $w_1 = \beta w_{-1}^c z_{-1}^d$, $a_{n+1} = ca_n + b_n$, and $x_{n+1} = x_n + a_n$ in (29) it follows that

$$\begin{aligned} w_{2n+1} &= \left(\beta\alpha^d\right)^{x_n} \left(\beta w_{-1}^c z_{-1}^d\right)^{a_n} w_{-1}^{b_n} \\ &= \alpha^{dx_n} \beta^{x_{n+1}} w_{-1}^{ca_n+b_n} z_{-1}^{da_n} \\ &= \alpha^{dx_n} \beta^{x_{n+1}} w_{-1}^{a_{n+1}} z_{-1}^{da_n}, \quad n \in \mathbb{N}. \end{aligned} \tag{30}$$

Using (30) in the first equation in (2), we get

$$z_{2n+1} = \alpha^{1+bdx_{n-1}} \beta^{bx_n} w_{-1}^{ba_n} z_{-1}^{bda_{n-1}}, \quad n \geq 2. \tag{31}$$

By using the same procedure it is proved that

$$w_{2n+2} = \gamma^{x_k} w_{2(n-k+1)}^{a_k} w_{2(n-k)}^{b_k}, \tag{32}$$

for all natural numbers k and n such that $1 \leq k \leq n$, where $(a_k)_{k\in\mathbb{N}}$, $(b_k)_{k\in\mathbb{N}}$, $(x_k)_{k\in\mathbb{N}}$ satisfy (20) and (26).

For $n = k$, (32) becomes

$$w_{2n+2} = \gamma^{x_n} w_2^{a_n} w_0^{b_n}, \quad n \in \mathbb{N}. \tag{33}$$

Since $w_2 = \beta w_0^c z_0^d$, $x_{n+1} = x_n + a_n$, and $a_{n+1} = ca_n + b_n$, from (33) we have

$$\begin{aligned} w_{2n+2} &= \left(\beta\alpha^d\right)^{x_n} \left(\beta w_0^c z_0^d\right)^{a_n} w_0^{b_n} \\ &= \alpha^{dx_n} \beta^{x_{n+1}} w_0^{a_{n+1}} z_0^{da_n}, \quad n \in \mathbb{N}. \end{aligned} \tag{34}$$

Using (34) in the first equation in (2), we get

$$z_{2n+2} = \alpha^{1+bdx_{n-1}} \beta^{bx_n} w_0^{ba_n} z_0^{bda_{n-1}}, \quad n \geq 2. \tag{35}$$

From the first two equations in (26) we have

$$a_k = a_1 a_{k-1} + b_1 a_{k-2}, \quad k \geq 3. \tag{36}$$

From (36) and since $b_k = b_1 a_{k-1}$, we see that $(b_k)_{k\in\mathbb{N}}$ is also a solution of (36).

From (26) with $k = 1$ one obtains

$$a_1 = a_1 a_0 + b_0, \qquad b_1 = b_1 a_0, \qquad x_1 = x_0 + a_0. \tag{37}$$

From this and since $b_1 = bd \neq 0$, from the second equation in (37) we get $a_0 = 1$, which along with the fact $x_1 = 1$ and the other two relations in (37) implies $b_0 = x_0 = 0$.

This and (26) with $k = 0$ imply

$$1 = a_0 = a_1 a_{-1} + b_{-1}, \qquad 0 = b_0 = b_1 a_{-1}, \qquad 0 = x_0 = x_{-1} + a_{-1}, \tag{38}$$

which along with $b_1 \neq 0$ and the second equation in (38) implies $a_{-1} = 0$. This along with the other two relations in (38) implies that we must have $b_{-1} = 1$ and $x_{-1} = 0$.

Hence $(a_k)_{k\geq -1}$ and $(b_k)_{k\geq -1}$ are solutions to (36) satisfying the (shifted) initial conditions

$$a_{-1} = 0, \qquad a_0 = 1; \qquad b_{-1} = 1, \qquad b_0 = 0, \tag{39}$$

while $(x_k)_{k\geq -1}$ satisfies the third equation in (26) and

$$x_{-1} = x_0 = 0, \qquad x_1 = 1. \tag{40}$$

From the third equation in (26) along with $x_1 = 1$ and $a_0 = 1$, we have

$$x_k = 1 + \sum_{j=1}^{k-1} a_j = \sum_{j=0}^{k-1} a_j. \tag{41}$$

The characteristic equation associated to (36) is $\lambda^2 - c\lambda - bd = 0$, from which it follows that

$$\lambda_{1,2} = \frac{c \pm \sqrt{c^2 + 4bd}}{2},$$

are the corresponding characteristic roots.

If $c^2 + 4bd \neq 0$, then

$$a_n = c_1 \lambda_1^n + c_2 \lambda_2^n,$$

which along with $a_{-1} = 0$ and $a_0 = 1$ yields

$$a_n = \frac{\lambda_1^{n+1} - \lambda_2^{n+1}}{\lambda_1 - \lambda_2}. \tag{42}$$

From this and since $b_n = b_1 a_{n-1}$, we have

$$b_n = bd \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2}. \tag{43}$$

If $c + bd \neq 1$, which is equivalent to $\lambda_1 \neq 1 \neq \lambda_2$, from (41) and (42), it follows that

$$x_n = \sum_{j=0}^{n-1} \frac{\lambda_1^{j+1} - \lambda_2^{j+1}}{\lambda_1 - \lambda_2} = \frac{(\lambda_2 - 1)\lambda_1^{n+1} - (\lambda_1 - 1)\lambda_2^{n+1} + \lambda_1 - \lambda_2}{(\lambda_1 - 1)(\lambda_2 - 1)(\lambda_1 - \lambda_2)}. \tag{44}$$

If $c + bd = 1$, that is, if one of the characteristic roots is one, say λ_2, then $\lambda_1 = -bd$, so that

$$x_n = \sum_{j=0}^{n-1} \frac{\lambda_1^{j+1} - 1}{\lambda_1 - 1} = \frac{1}{(\lambda_1 - 1)} \left(\lambda_1 \frac{\lambda_1^n - 1}{\lambda_1 - 1} - n \right) = \frac{(-bd)^{n+1} + (n+1)bd + n}{(1 + bd)^2}. \tag{45}$$

If $c^2 + 4bd = 0$, then

$$a_n = (\hat{c}_1 + \hat{c}_2 n)\left(\frac{c}{2}\right)^n.$$

This along with $a_{-1} = 0$ and $a_0 = 1$ yields

$$a_n = (n+1)\left(\frac{c}{2}\right)^n. \tag{46}$$

Using the relation $b_n = b_1 a_{n-1}$ along with the fact $bd = -c^2/4$, we get

$$b_n = bdn\left(\frac{c}{2}\right)^{n-1} = -n\left(\frac{c}{2}\right)^{n+1}. \tag{47}$$

From (41) and (46), we have

$$x_n = \sum_{j=0}^{n-1}(j+1)\left(\frac{c}{2}\right)^j = \frac{1-(n+1)(\frac{c}{2})^n + n(\frac{c}{2})^{n+1}}{(1-\frac{c}{2})^2}, \tag{48}$$

if $c \neq 2$. If $c = 2$, we obtain

$$x_n = \sum_{j=0}^{n-1}(j+1) = \frac{n(n+1)}{2}, \tag{49}$$

completing the proof of the result. □

Corollary 1 *Consider system* (1) *with* $b, c, d \in \mathbb{Z}$, $a = 0$, *and* $\alpha, \beta \in \mathbb{C} \setminus \{0\}$. *Assume that* $z_{-1}, z_0, w_{-1}, w_0 \in \mathbb{C} \setminus \{0\}$. *Then the following statements are true.*

(a) *If* $bd = 0$ *and* $c \neq 1$, *then the general solution to system* (1) *is given by* (10), (11), (16), *and* (17).

(b) *If* $bd = 0$ *and* $c = 1$, *then the general solution to system* (1) *is given by* (12), (13), (18), *and* (19).

(c) *If* $bd \neq 0$, $c^2 + 4bd \neq 0$, *and* $c + bd \neq 1$, *then the general solution to system* (1) *is given by* (30), (31), (34), *and* (35), *where the sequence* $(a_n)_{n\geq -1}$ *is given by formula* (42), *while* $(x_n)_{n\geq -1}$ *is given by* (44).

(d) *If* $bd \neq 0$, $c^2 + 4bd \neq 0$, *and* $c + bd = 1$, *then the general solution to system* (1) *is given by* (30), (31), (34), *and* (35), *where the sequence* $(a_n)_{n\geq -1}$ *is given by formula* (42), *while* $(x_n)_{n\geq -1}$ *is given by* (45).

(e) *If* $bd \neq 0$, $c^2 + 4bd = 0$, *and* $c \neq 2$, *then the general solution to system* (1) *is given by* (30), (31), (34), *and* (35), *where the sequence* $(a_n)_{n\geq -1}$ *is given by formula* (46), *while* $(x_n)_{n\geq -1}$ *is given by* (48).

(f) *If* $bd \neq 0$, $c^2 + 4bd = 0$, *and* $c = 2$, *then the general solution to system* (1) *is given by* (30), (31), (34), *and* (35), *where the sequence* $(a_n)_{n\geq -1}$ *is given by formula* (46) *with* $c = 2$, *while* $(x_n)_{n\geq -1}$ *is given by* (49).

Theorem 2 *Assume that* $a, c, d \in \mathbb{Z}$, $b = 0$, $\alpha, \beta \in \mathbb{C} \setminus \{0\}$, *and* $z_{-1}, z_0, w_{-1}, w_0 \in \mathbb{C} \setminus \{0\}$. *Then system* (1) *is solvable in closed form.*

Proof Since $b = 0$, we have

$$z_{n+1} = \alpha z_n^a, \qquad w_{n+1} = \beta w_{n-1}^c z_{n-1}^d, \quad n \in \mathbb{N}_0. \tag{50}$$

From the first equation in (50) we get

$$z_n = \alpha^{\sum_{j=0}^{n-1} a^j} z_0^{a^n}, \quad n \in \mathbb{N}. \tag{51}$$

Hence, if $a \neq 1$, we have

$$z_n = \alpha^{\frac{1-a^n}{1-a}} z_0^{a^n}, \quad n \in \mathbb{N}, \tag{52}$$

while if $a = 1$,

$$z_n = \alpha^n z_0, \quad n \in \mathbb{N}. \tag{53}$$

Using (51) in the second equation in (50), it follows that

$$w_{n+1} = \beta \alpha^{d \sum_{j=0}^{n-2} a^j} z_0^{da^{n-1}} w_{n-1}^c, \quad n \geq 2. \tag{54}$$

Using (54) twice, we get

$$\begin{aligned} w_{2n} &= \beta \alpha^{d \sum_{j=0}^{2n-3} a^j} z_0^{da^{2n-2}} w_{2n-2}^c \\ &= \beta \alpha^{d \sum_{j=0}^{2n-3} a^j} z_0^{da^{2n-2}} \left(\beta \alpha^{d \sum_{j=0}^{2n-5} a^j} z_0^{da^{2n-4}} w_{2n-4}^c\right)^c \\ &= \beta^{1+c} \alpha^{d \sum_{j=0}^{2n-3} a^j + dc \sum_{j=0}^{2n-5} a^j} z_0^{da^{2n-2} + dca^{2n-4}} w_{2n-4}^{c^2}, \end{aligned} \tag{55}$$

for every $n \geq 3$, and

$$\begin{aligned} w_{2n+1} &= \beta \alpha^{d \sum_{j=0}^{2n-2} a^j} z_0^{da^{2n-1}} w_{2n-1}^c \\ &= \beta \alpha^{d \sum_{j=0}^{2n-2} a^j} z_0^{da^{2n-1}} \left(\beta \alpha^{d \sum_{j=0}^{2n-4} a^j} z_0^{da^{2n-3}} w_{2n-3}^c\right)^c \\ &= \beta^{1+c} \alpha^{d \sum_{j=0}^{2n-2} a^j + dc \sum_{j=0}^{2n-4} a^j} z_0^{da^{2n-1} + dca^{2n-3}} w_{2n-3}^{c^2}, \quad n \geq 2. \end{aligned} \tag{56}$$

Assume that, for a natural number k, it has been proved that

$$w_{2n} = \beta^{\sum_{j=0}^{k-1} c^j} \alpha^{d \sum_{i=0}^{k-1} c^i \sum_{j=0}^{2n-2i-3} a^j} z_0^{d \sum_{j=0}^{k-1} c^j a^{2n-2j-2}} w_{2n-2k}^{c^k}, \tag{57}$$

for $n \geq k+1$, and

$$w_{2n+1} = \beta^{\sum_{j=0}^{k-1} c^j} \alpha^{d \sum_{i=0}^{k-1} c^i \sum_{j=0}^{2n-2i-2} a^j} z_0^{d \sum_{j=0}^{k-1} c^j a^{2n-2j-1}} w_{2n-2k+1}^{c^k}, \tag{58}$$

for every $n \geq k$.

Using (54) with $n \to 2n-2k-1$ and $n \to 2n-2k$, in (57) and (58), we obtain

$$\begin{aligned} w_{2n} &= \beta^{\sum_{j=0}^{k-1} c^j} \alpha^{d\sum_{i=0}^{k-1} c^i \sum_{j=0}^{2n-2i-3} a^j} z_0^{d\sum_{j=0}^{k-1} c^j a^{2n-2j-2}} \\ &\quad \times \bigl(\beta\alpha^{d\sum_{j=0}^{2n-2k-3} a^j} z_0^{da^{2n-2k-2}} w_{2n-2k-2}^c\bigr)^{c^k} \\ &= \beta^{\sum_{j=0}^{k} c^j} \alpha^{d\sum_{i=0}^{k} c^i \sum_{j=0}^{2n-2i-3} a^j} z_0^{d\sum_{j=0}^{k} c^j a^{2n-2j-2}} w_{2n-2k-2}^{c^{k+1}}, \end{aligned} \tag{59}$$

for $n \geq k+2$, and

$$\begin{aligned} w_{2n+1} &= \beta^{\sum_{j=0}^{k-1} c^j} \alpha^{d\sum_{i=0}^{k-1} c^i \sum_{j=0}^{2n-2i-2} a^j} z_0^{d\sum_{j=0}^{k-1} c^j a^{2n-2j-1}} \\ &\quad \times \bigl(\beta\alpha^{d\sum_{j=0}^{2n-2k-2} a^j} z_0^{da^{2n-2k-1}} w_{2n-2k-1}^c\bigr)^{c^k} \\ &= \beta^{\sum_{j=0}^{k} c^j} \alpha^{d\sum_{i=0}^{k} c^i \sum_{j=0}^{2n-2i-2} a^j} z_0^{d\sum_{j=0}^{k} c^j a^{2n-2j-1}} w_{2n-2k-1}^{c^{k+1}}, \end{aligned} \tag{60}$$

for every $n \geq k+1$.

From (55), (56), (59), (60), and the induction it follows that (57) holds for all natural numbers k and n such that $1 \leq k \leq n-1$, while (58) holds for all k and n such that $1 \leq k \leq n$.

By taking $k = n-1$ in (57), we get

$$w_{2n} = \beta^{\sum_{j=0}^{n-2} c^j} \alpha^{d\sum_{i=0}^{n-2} c^i \sum_{j=0}^{2n-2i-3} a^j} z_0^{d\sum_{j=0}^{n-2} c^j a^{2n-2j-2}} w_2^{c^{n-1}}, \quad n \geq 2. \tag{61}$$

By using the relation $w_2 = \beta w_0^c z_0^d$ in (61) we get

$$\begin{aligned} w_{2n} &= \beta^{\sum_{j=0}^{n-2} c^j} \alpha^{d\sum_{i=0}^{n-2} c^i \sum_{j=0}^{2n-2i-3} a^j} z_0^{d\sum_{j=0}^{n-2} c^j a^{2n-2j-2}} \bigl(\beta w_0^c z_0^d\bigr)^{c^{n-1}} \\ &= \beta^{\sum_{j=0}^{n-1} c^j} \alpha^{d\sum_{i=0}^{n-2} c^i \sum_{j=0}^{2n-2i-3} a^j} z_0^{d\sum_{j=0}^{n-1} c^j a^{2n-2j-2}} w_0^{c^n}, \quad n \in \mathbb{N}. \end{aligned} \tag{62}$$

By taking $k = n$ in (58), and using the relation $w_1 = \beta w_{-1}^c z_{-1}^d$, we get

$$\begin{aligned} w_{2n+1} &= \beta^{\sum_{j=0}^{n-1} c^j} \alpha^{d\sum_{i=0}^{n-1} c^i \sum_{j=0}^{2n-2i-2} a^j} z_0^{d\sum_{j=0}^{n-1} c^j a^{2n-2j-1}} w_1^{c^n} \\ &= \beta^{\sum_{j=0}^{n-1} c^j} \alpha^{d\sum_{i=0}^{n-1} c^i \sum_{j=0}^{2n-2i-2} a^j} z_0^{d\sum_{j=0}^{n-1} c^j a^{2n-2j-1}} \bigl(\beta w_{-1}^c z_{-1}^d\bigr)^{c^n} \\ &= \beta^{\sum_{j=0}^{n} c^j} \alpha^{d\sum_{i=0}^{n-1} c^i \sum_{j=0}^{2n-2i-2} a^j} z_0^{d\sum_{j=0}^{n-1} c^j a^{2n-2j-1}} w_{-1}^{c^{n+1}} z_{-1}^{dc^n}, \end{aligned} \tag{63}$$

for $n \in \mathbb{N}$. It is also easy to check that (63) holds also for $n = 0$ when $c \neq 0$.

Subcase $a \neq 1 \neq c$, $c \neq a^2$. In this case we have

$$\begin{aligned} w_{2n} &= \beta^{\frac{1-c^n}{1-c}} \alpha^{d\sum_{i=0}^{n-2} c^i \frac{1-a^{2n-2i-2}}{1-a}} z_0^{d\frac{a^{2n}-c^n}{a^2-c}} w_0^{c^n}, \\ &= \beta^{\frac{1-c^n}{1-c}} \alpha^{\frac{d}{1-a}(\frac{1-c^{n-1}}{1-c} - a^2 \frac{a^{2n-2}-c^{n-1}}{a^2-c})} z_0^{d\frac{a^{2n}-c^n}{a^2-c}} w_0^{c^n}, \\ &= \beta^{\frac{1-c^n}{1-c}} \alpha^{\frac{d(a^2-c+(1-a^2)c^n+(c-1)a^{2n})}{(1-a)(1-c)(a^2-c)}} z_0^{d\frac{a^{2n}-c^n}{a^2-c}} w_0^{c^n}, \end{aligned} \tag{64}$$

for $n \in \mathbb{N}$, and

$$\begin{aligned} w_{2n+1} &= \beta^{\frac{1-c^{n+1}}{1-c}} \alpha^{d\sum_{i=0}^{n-1} c^i \frac{1-a^{2n-2i-1}}{1-a}} z_0^{ad\frac{a^{2n}-c^n}{a^2-c}} w_{-1}^{c^{n+1}} z_{-1}^{dc^n} \\ &= \beta^{\frac{1-c^{n+1}}{1-c}} \alpha^{\frac{d}{1-a}(\frac{1-c^n}{1-c}-a\frac{a^{2n}-c^n}{a^2-c})} z_0^{ad\frac{a^{2n}-c^n}{a^2-c}} w_{-1}^{c^{n+1}} z_{-1}^{dc^n} \\ &= \beta^{\frac{1-c^{n+1}}{1-c}} \alpha^{\frac{d(a^2-c+(a+c)(1-a)c^n-(1-c)a^{2n+1})}{(1-a)(1-c)(a^2-c)}} z_0^{ad\frac{a^{2n}-c^n}{a^2-c}} w_{-1}^{c^{n+1}} z_{-1}^{dc^n}, \end{aligned} \tag{65}$$

for every $n \in \mathbb{N}$.

Subcase $a \neq 1 \neq c$, $c = a^2$. In this case we have

$$\begin{aligned} w_{2n} &= \beta^{\sum_{j=0}^{n-1} a^{2j}} \alpha^{d\sum_{i=0}^{n-2} a^{2i} \sum_{j=0}^{2n-2i-3} a^j} z_0^{d\sum_{j=0}^{n-1} a^{2j} a^{2n-2j-2}} w_0^{a^{2n}}, \\ &= \beta^{\frac{1-a^{2n}}{1-a^2}} \alpha^{d\sum_{i=0}^{n-2} a^{2i} \frac{1-a^{2n-2i-2}}{1-a}} z_0^{dna^{2n-2}} w_0^{a^{2n}}, \\ &= \beta^{\frac{1-a^{2n}}{1-a^2}} \alpha^{\frac{d}{1-a}(\frac{1-a^{2n-2}}{1-a^2}-(n-1)a^{2n-2})} z_0^{dna^{2n-2}} w_0^{a^{2n}}, \\ &= \beta^{\frac{1-a^{2n}}{1-a^2}} \alpha^{\frac{d(1-na^{2n-2}+(n-1)a^{2n})}{(a-1)^2(a+1)}} z_0^{dna^{2n-2}} w_0^{a^{2n}}, \quad n \geq 2, \end{aligned} \tag{66}$$

$$\begin{aligned} w_{2n+1} &= \beta^{\sum_{j=0}^{n} a^{2j}} \alpha^{d\sum_{i=0}^{n-1} a^{2i} \sum_{j=0}^{2n-2i-2} a^j} z_0^{d\sum_{j=0}^{n-1} a^{2j} a^{2n-2j-1}} w_{-1}^{a^{2n+2}} z_{-1}^{da^{2n}} \\ &= \beta^{\frac{1-a^{2n+2}}{1-a^2}} \alpha^{d\sum_{i=0}^{n-1} a^{2i} \frac{1-a^{2n-2i-1}}{1-a}} z_0^{dna^{2n-1}} w_{-1}^{a^{2n+2}} z_{-1}^{da^{2n}} \\ &= \beta^{\frac{1-a^{2n+2}}{1-a^2}} \alpha^{\frac{d}{1-a}(\frac{1-a^{2n}}{1-a^2}-na^{2n-1})} z_0^{dna^{2n-1}} w_{-1}^{a^{2n+2}} z_{-1}^{da^{2n}} \\ &= \beta^{\frac{1-a^{2n+2}}{1-a^2}} \alpha^{\frac{d(1-na^{2n-1}-a^{2n}+na^{2n+1})}{(a+1)(a-1)^2}} z_0^{dna^{2n-1}} w_{-1}^{a^{2n+2}} z_{-1}^{da^{2n}}, \end{aligned} \tag{67}$$

for every $n \in \mathbb{N}$.

Subcase $a^2 \neq 1 = c$. In this case we have

$$\begin{aligned} w_{2n} &= \beta^{\sum_{j=0}^{n-1} 1} \alpha^{d\sum_{i=0}^{n-2} \sum_{j=0}^{2n-2i-3} a^j} z_0^{d\sum_{j=0}^{n-1} a^{2n-2j-2}} w_0, \\ &= \beta^n \alpha^{d\sum_{i=0}^{n-2} \frac{1-a^{2n-2i-2}}{1-a}} z_0^{d\frac{a^{2n}-1}{a^2-1}} w_0, \\ &= \beta^n \alpha^{\frac{d}{1-a}(n-1-a^2\frac{a^{2n-2}-1}{a^2-1})} z_0^{d\frac{a^{2n}-1}{a^2-1}} w_0, \\ &= \beta^n \alpha^{\frac{d(a^{2n}-na^2+n-1)}{(a-1)^2(a+1)}} z_0^{d\frac{a^{2n}-1}{a^2-1}} w_0, \quad n \in \mathbb{N}, \end{aligned} \tag{68}$$

$$\begin{aligned} w_{2n+1} &= \beta^{\sum_{j=0}^{n} 1} \alpha^{d\sum_{i=0}^{n-1} \sum_{j=0}^{2n-2i-2} a^j} z_0^{d\sum_{j=0}^{n-1} a^{2n-2j-1}} w_{-1} z_{-1}^d \\ &= \beta^{n+1} \alpha^{d\sum_{i=0}^{n-1} \frac{1-a^{2n-2i-1}}{1-a}} z_0^{ad\frac{a^{2n}-1}{a^2-1}} w_{-1} z_{-1}^d \\ &= \beta^{n+1} \alpha^{\frac{d}{1-a}(n-a\frac{a^{2n}-1}{a^2-1})} z_0^{ad\frac{a^{2n}-1}{a^2-1}} w_{-1} z_{-1}^d \\ &= \beta^{n+1} \alpha^{\frac{d(a^{2n+1}+n(1-a^2)-a)}{(a-1)^2(a+1)}} z_0^{ad\frac{a^{2n}-1}{a^2-1}} w_{-1} z_{-1}^d, \end{aligned} \tag{69}$$

for every $n \in \mathbb{N}$.

Subcase $a = -1$, $c = 1$. In this case we have

$$\begin{aligned} w_{2n} &= \beta^{\sum_{j=0}^{n-1} 1} \alpha^{d \sum_{i=0}^{n-2} \sum_{j=0}^{2n-2i-3} (-1)^j} z_0^{d \sum_{j=0}^{n-1} (-1)^{2n-2j-2}} w_0, \\ &= \beta^n z_0^{dn} w_0, \quad n \in \mathbb{N}, \end{aligned} \tag{70}$$

$$\begin{aligned} w_{2n+1} &= \beta^{\sum_{j=0}^{n} 1} \alpha^{d \sum_{i=0}^{n-1} \sum_{j=0}^{2n-2i-2} (-1)^j} z_0^{d \sum_{j=0}^{n-1} (-1)^{2n-2j-1}} w_{-1} z_{-1}^d \\ &= \beta^{n+1} \alpha^{d \sum_{i=0}^{n-1} \frac{1-(-1)^{2n-2i-1}}{2}} z_0^{d \sum_{j=0}^{n-1} (-1)} w_{-1} z_{-1}^d \\ &= \beta^{n+1} \alpha^{dn} z_0^{-dn} w_{-1} z_{-1}^d, \end{aligned} \tag{71}$$

for every $n \in \mathbb{N}_0$.

Subcase $a = 1$, $c \neq 1$. In this case we have

$$\begin{aligned} w_{2n} &= \beta^{\sum_{j=0}^{n-1} c^j} \alpha^{d \sum_{i=0}^{n-2} c^i \sum_{j=0}^{2n-2i-3} 1} z_0^{d \sum_{j=0}^{n-1} c^j} w_0^{c^n} \\ &= \beta^{\frac{1-c^n}{1-c}} \alpha^{d \sum_{i=0}^{n-2} (2n-2i-2) c^i} z_0^{d \frac{1-c^n}{1-c}} w_0^{c^n} \\ &= \beta^{\frac{1-c^n}{1-c}} \alpha^{d((2n-2)\frac{1-c^{n-1}}{1-c} - 2c \frac{1-(n-1)c^{n-2}+(n-2)c^{n-1}}{(1-c)^2})} z_0^{d \frac{1-c^n}{1-c}} w_0^{c^n} \\ &= \beta^{\frac{1-c^n}{1-c}} \alpha^{\frac{2d(n-1-nc+c^n)}{(1-c)^2}} z_0^{d \frac{1-c^n}{1-c}} w_0^{c^n}, \quad n \in \mathbb{N}, \end{aligned} \tag{72}$$

$$\begin{aligned} w_{2n+1} &= \beta^{\sum_{j=0}^{n} c^j} \alpha^{d \sum_{i=0}^{n-1} c^i \sum_{j=0}^{2n-2i-2} 1} z_0^{d \sum_{j=0}^{n-1} c^j} w_{-1}^{c^{n+1}} z_{-1}^{dc^n} \\ &= \beta^{\frac{1-c^{n+1}}{1-c}} \alpha^{d \sum_{i=0}^{n-1} (2n-2i-1) c^i} z_0^{d \frac{1-c^n}{1-c}} w_{-1}^{c^{n+1}} z_{-1}^{dc^n} \\ &= \beta^{\frac{1-c^{n+1}}{1-c}} \alpha^{d((2n-1)\frac{1-c^n}{1-c} - 2c \sum_{i=1}^{n-1} i c^{i-1})} z_0^{d \frac{1-c^n}{1-c}} w_{-1}^{c^{n+1}} z_{-1}^{dc^n} \\ &= \beta^{\frac{1-c^{n+1}}{1-c}} \alpha^{d((2n-1)\frac{1-c^n}{1-c} - 2c \frac{1-nc^{n-1}+(n-1)c^n}{(1-c)^2})} z_0^{d \frac{1-c^n}{1-c}} w_{-1}^{c^{n+1}} z_{-1}^{dc^n} \\ &= \beta^{\frac{1-c^{n+1}}{1-c}} \alpha^{\frac{d(2n-1-(2n+1)c+c^n+c^{n+1})}{(1-c)^2}} z_0^{d \frac{1-c^n}{1-c}} w_{-1}^{c^{n+1}} z_{-1}^{dc^n}, \end{aligned} \tag{73}$$

for every $n \in \mathbb{N}$.

Subcase $a = c = 1$. In this case we have

$$\begin{aligned} w_{2n} &= \beta^{\sum_{j=0}^{n-1} 1} \alpha^{d \sum_{i=0}^{n-2} \sum_{j=0}^{2n-2i-3} 1} z_0^{d \sum_{j=0}^{n-1} 1} w_0 \\ &= \beta^n \alpha^{d \sum_{i=0}^{n-2} (2n-2i-2)} z_0^{dn} w_0 \\ &= \beta^n \alpha^{d(n-1)n} z_0^{dn} w_0, \quad n \in \mathbb{N}, \end{aligned} \tag{74}$$

$$\begin{aligned} w_{2n+1} &= \beta^{\sum_{j=0}^{n} 1} \alpha^{d \sum_{i=0}^{n-1} \sum_{j=0}^{2n-2i-2} 1} z_0^{d \sum_{j=0}^{n-1} 1} w_{-1} z_{-1}^d \\ &= \beta^{n+1} \alpha^{d \sum_{i=0}^{n-1} (2n-2i-1)} z_0^{dn} w_{-1} z_{-1}^d \\ &= \beta^{n+1} \alpha^{dn^2} z_0^{dn} w_{-1} z_{-1}^d, \end{aligned} \tag{75}$$

for every $n \in \mathbb{N}$. □

Corollary 2 *Consider system* (1) *with $a, c, d \in \mathbb{Z}$, $b = 0$, and $\alpha, \beta \in \mathbb{C} \setminus \{0\}$. Assume that $z_{-1}, z_0, w_{-1}, w_0 \in \mathbb{C} \setminus \{0\}$. Then the following statements are true.*

(a) *If* $a \neq 1 \neq c$ *and* $c \neq a^2$, *then the general solution to system* (1) *is given by* (52), (64), *and* (65).
(b) *If* $a \neq 1 \neq c$ *and* $c = a^2 \neq 0$, *then the general solution to system* (1) *is given by* (52), (66), *and* (67).
(c) *If* $a^2 \neq 1 = c$, *then the general solution to system* (1) *is given by* (52), (68), *and* (69).
(d) *If* $a = -1$ *and* $c = 1$, *then the general solution to system* (1) *is given by* (52), (70), *and* (71).
(e) *If* $a = 1$ *and* $c \neq 1$, *then the general solution to system* (1) *is given by* (53), (72), *and* (73).
(f) *If* $a = c = 1$, *then the general solution to system* (1) *is given by* (53), (74), *and* (75).

Theorem 3 *Assume that* $a, b, c \in \mathbb{Z}$, $d = 0$, $\alpha, \beta \in \mathbb{C} \setminus \{0\}$, *and* $z_{-1}, z_0, w_{-1}, w_0 \in \mathbb{C} \setminus \{0\}$. *Then system* (1) *is solvable in closed form.*

Proof In this case system (1) becomes

$$z_{n+1} = \alpha z_n^a w_{n-1}^b, \qquad w_{n+1} = \beta w_{n-1}^c, \quad n \in \mathbb{N}_0. \tag{76}$$

From the second equation in (76) it easily follows that

$$w_{2n} = \beta^{\sum_{j=0}^{n-1} c^j} w_0^{c^n}, \quad n \in \mathbb{N} \quad \text{and} \quad w_{2n+1} = \beta^{\sum_{j=0}^{n} c^j} w_{-1}^{c^{n+1}}, \quad n \in \mathbb{N}_0, \tag{77}$$

which, for the case $c \neq 1$, implies that

$$w_{2n} = \beta^{\frac{1-c^n}{1-c}} w_0^{c^n}, \quad n \in \mathbb{N}, \tag{78}$$

and

$$w_{2n+1} = \beta^{\frac{1-c^{n+1}}{1-c}} w_{-1}^{c^{n+1}}, \quad n \in \mathbb{N}_0, \tag{79}$$

while, for the case $c = 1$, we have

$$w_{2n} = \beta^n w_0, \quad n \in \mathbb{N}, \tag{80}$$

and

$$w_{2n+1} = \beta^{n+1} w_{-1}, \quad n \in \mathbb{N}_0. \tag{81}$$

Employing (77) in the first equation in (76) we obtain

$$z_{2n} = \alpha \beta^{b \sum_{j=0}^{n-2} c^j} w_0^{bc^{n-1}} z_{2n-1}^a, \quad n \geq 2, \tag{82}$$

$$z_{2n+1} = \alpha \beta^{b \sum_{j=0}^{n-1} c^j} w_{-1}^{bc^n} z_{2n}^a, \quad n \in \mathbb{N}. \tag{83}$$

Combining (82) and (83) it follows that

$$\begin{aligned} z_{2n} &= \alpha \beta^{b \sum_{j=0}^{n-2} c^j} w_0^{bc^{n-1}} \left(\alpha \beta^{b \sum_{j=0}^{n-2} c^j} w_{-1}^{bc^{n-1}} z_{2n-2}^a\right)^a \\ &= \alpha^{1+a} \beta^{b(1+a) \sum_{j=0}^{n-2} c^j} \left(w_0^b w_{-1}^{ab}\right)^{c^{n-1}} z_{2n-2}^{a^2}, \end{aligned} \tag{84}$$

for $n \geq 2$, and

$$
\begin{aligned}
z_{2n+1} &= \alpha\beta^{b\sum_{j=0}^{n-1}c^j} w_{-1}^{bc^n}\left(\alpha\beta^{b\sum_{j=0}^{n-2}c^j} w_0^{bc^{n-1}} z_{2n-1}^a\right)^a \\
&= \alpha^{1+a}\beta^{b(\sum_{j=0}^{n-1}c^j + a\sum_{j=0}^{n-2}c^j)}\left(w_0^{ab}w_{-1}^{bc}\right)^{c^{n-1}} z_{2n-1}^{a^2}, \quad n \in \mathbb{N}.
\end{aligned}
\tag{85}
$$

Assume that, for some natural number k we have proved that

$$
z_{2n} = \alpha^{(1+a)\sum_{j=0}^{k-1}a^{2j}}\beta^{b(1+a)\sum_{i=0}^{k-1}a^{2i}\sum_{j=0}^{n-i-2}c^j}\left(w_0^b w_{-1}^{ab}\right)^{\sum_{j=0}^{k-1}a^{2j}c^{n-j-1}} z_{2n-2k}^{a^{2k}},
\tag{86}
$$

for $n \geq k+1$ and

$$
z_{2n+1} = \alpha^{(1+a)\sum_{j=0}^{k-1}a^{2j}}\beta^{b\sum_{i=0}^{k-1}a^{2i}(\sum_{j=0}^{n-i-1}c^j + a\sum_{j=0}^{n-i-2}c^j)}\left(w_0^{ab}w_{-1}^{bc}\right)^{\sum_{j=0}^{k-1}a^{2j}c^{n-j-1}} z_{2n-2k+1}^{a^{2k}},
\tag{87}
$$

for every $n \geq k$.

By using (84) with $n \to n-k$ into (86), and (85) with $n \to n-k$ into (87), it follows that

$$
\begin{aligned}
z_{2n} &= \alpha^{(1+a)\sum_{j=0}^{k-1}a^{2j}}\beta^{b(1+a)\sum_{i=0}^{k-1}a^{2i}\sum_{j=0}^{n-i-2}c^j}\left(w_0^b w_{-1}^{ab}\right)^{\sum_{j=0}^{k-1}a^{2j}c^{n-j-1}} \\
&\quad\times\left(\alpha^{1+a}\beta^{b(1+a)\sum_{j=0}^{n-k-2}c^j}\left(w_0^b w_{-1}^{ab}\right)^{c^{n-k-1}} z_{2n-2k-2}^{a^2}\right)^{a^{2k}} \\
&= \alpha^{(1+a)\sum_{j=0}^{k}a^{2j}}\beta^{b(1+a)\sum_{i=0}^{k}a^{2i}\sum_{j=0}^{n-i-2}c^j}\left(w_0^b w_{-1}^{ab}\right)^{\sum_{j=0}^{k}a^{2j}c^{n-j-1}} z_{2n-2k-2}^{a^{2k+2}},
\end{aligned}
\tag{88}
$$

for $n \geq k+2$ and

$$
\begin{aligned}
z_{2n+1} &= \alpha^{(1+a)\sum_{j=0}^{k-1}a^{2j}}\beta^{b\sum_{i=0}^{k-1}a^{2i}(\sum_{j=0}^{n-i-1}c^j + a\sum_{j=0}^{n-i-2}c^j)}\left(w_0^{ab}w_{-1}^{bc}\right)^{\sum_{j=0}^{k-1}a^{2j}c^{n-j-1}} \\
&\quad\times\left(\alpha^{1+a}\beta^{b(\sum_{j=0}^{n-k-1}c^j + a\sum_{j=0}^{n-k-2}c^j)}\left(w_0^{ab}w_{-1}^{bc}\right)^{c^{n-k-1}} z_{2n-2k-1}^{a^2}\right)^{a^{2k}} \\
&= \alpha^{(1+a)\sum_{j=0}^{k}a^{2j}}\beta^{b\sum_{i=0}^{k}a^{2i}(\sum_{j=0}^{n-i-1}c^j + a\sum_{j=0}^{n-i-2}c^j)}\left(w_0^{ab}w_{-1}^{bc}\right)^{\sum_{j=0}^{k}a^{2j}c^{n-j-1}} z_{2n-2k-1}^{a^{2k+2}},
\end{aligned}
\tag{89}
$$

for every $n \geq k+1$.

From the equalities in (84), (85), (88), (89), and by induction we see that (86) holds for all natural numbers k and n such that $1 \leq k \leq n-1$, while (87) holds for $1 \leq k \leq n$.

If we choose $k = n-1$ in (86) and $k = n$ in (87) we get

$$
\begin{aligned}
z_{2n} &= \alpha^{(1+a)\sum_{j=0}^{n-2}a^{2j}}\beta^{b(1+a)\sum_{i=0}^{n-2}a^{2i}\sum_{j=0}^{n-i-2}c^j}\left(w_0^b w_{-1}^{ab}\right)^{\sum_{j=0}^{n-2}a^{2j}c^{n-j-1}} z_2^{a^{2n-2}} \\
&= \alpha^{(1+a)\sum_{j=0}^{n-2}a^{2j}}\beta^{b(1+a)\sum_{i=0}^{n-2}a^{2i}\sum_{j=0}^{n-i-2}c^j}\left(w_0^b w_{-1}^{ab}\right)^{\sum_{j=0}^{n-2}a^{2j}c^{n-j-1}}\left(\alpha^{1+a}z_0^{a^2}w_{-1}^{ab}w_0^b\right)^{a^{2n-2}} \\
&= \alpha^{(1+a)\sum_{j=0}^{n-1}a^{2j}}\beta^{b(1+a)\sum_{i=0}^{n-2}a^{2i}\sum_{j=0}^{n-i-2}c^j}\left(w_0^b w_{-1}^{ab}\right)^{\sum_{j=0}^{n-1}a^{2j}c^{n-j-1}} z_0^{a^{2n}},
\end{aligned}
\tag{90}
$$

for every $n \in \mathbb{N}$, and

$$
\begin{aligned}
z_{2n+1} &= \alpha^{(1+a)\sum_{j=0}^{n-1}a^{2j}}\beta^{b\sum_{i=0}^{n-1}a^{2i}(\sum_{j=0}^{n-i-1}c^j + a\sum_{j=0}^{n-i-2}c^j)}\left(w_0^{ab}w_{-1}^{bc}\right)^{\sum_{j=0}^{n-1}a^{2j}c^{n-j-1}} z_1^{a^{2n}} \\
&= \alpha^{(1+a)\sum_{j=0}^{n-1}a^{2j}}\beta^{b\sum_{i=0}^{n-1}a^{2i}(\sum_{j=0}^{n-i-1}c^j + a\sum_{j=0}^{n-i-2}c^j)}\left(w_0^{ab}w_{-1}^{bc}\right)^{\sum_{j=0}^{n-1}a^{2j}c^{n-j-1}}\left(\alpha z_0^a w_{-1}^b\right)^{a^{2n}} \\
&= \alpha^{\sum_{j=0}^{2n}a^j}\beta^{b\sum_{i=0}^{n-1}a^{2i}(\sum_{j=0}^{n-i-1}c^j + a\sum_{j=0}^{n-i-2}c^j)} w_0^{ab\sum_{j=0}^{n-1}a^{2j}c^{n-j-1}} w_{-1}^{b\sum_{j=0}^{n}a^{2j}c^{n-j}} z_0^{a^{2n+1}},
\end{aligned}
\tag{91}
$$

for every $n \in \mathbb{N}$.

Subcase $c \neq a^2 \neq 1 \neq c$. In this case we have

$$\begin{aligned} z_{2n} &= \alpha^{\frac{1-a^{2n}}{1-a}} \beta^{b(1+a)\sum_{i=0}^{n-2} a^{2i} \frac{1-c^{n-i-1}}{1-c}} \left(w_0^b w_{-1}^{ab}\right)^{\frac{a^{2n}-c^n}{a^2-c}} z_0^{a^{2n}} \\ &= \alpha^{\frac{1-a^{2n}}{1-a}} \beta^{\frac{b(1+a)}{1-c}(\frac{1-a^{2n-2}}{1-a^2} - c\frac{a^{2n-2}-c^{n-1}}{a^2-c})} \left(w_0^b w_{-1}^{ab}\right)^{\frac{a^{2n}-c^n}{a^2-c}} z_0^{a^{2n}} \\ &= \alpha^{\frac{1-a^{2n}}{1-a}} \beta^{\frac{b(1+a)(a^2-c+c^n-a^{2n}+ca^{2n}-a^2c^n)}{(1-c)(1-a^2)(a^2-c)}} \left(w_0^b w_{-1}^{ab}\right)^{\frac{a^{2n}-c^n}{a^2-c}} z_0^{a^{2n}}, \end{aligned} \tag{92}$$

$$\begin{aligned} z_{2n+1} &= \alpha^{\sum_{j=0}^{2n} a^j} \beta^{b\sum_{i=0}^{n-1} a^{2i}(\sum_{j=0}^{n-i-1} c^j + a\sum_{j=0}^{n-i-2} c^j)} w_0^{ab\sum_{j=0}^{n-1} a^{2j}c^{n-j-1}} w_{-1}^{b\sum_{j=0}^{n} a^{2j}c^{n-j}} z_0^{a^{2n+1}} \\ &= \alpha^{\frac{1-a^{2n+1}}{1-a}} \beta^{b\sum_{i=0}^{n-1} a^{2i}(\frac{1-c^{n-i}}{1-c} + a\frac{1-c^{n-i-1}}{1-c})} w_0^{ab\frac{a^{2n}-c^n}{a^2-c}} w_{-1}^{b\frac{a^{2n+2}-c^{n+1}}{a^2-c}} z_0^{a^{2n+1}} \\ &= \alpha^{\frac{1-a^{2n+1}}{1-a}} \beta^{\frac{b}{1-c}(\frac{1-a^{2n}}{1-a} - (c+a)\frac{a^{2n}-c^n}{a^2-c})} w_0^{ab\frac{a^{2n}-c^n}{a^2-c}} w_{-1}^{b\frac{a^{2n+2}-c^{n+1}}{a^2-c}} z_0^{a^{2n+1}} \\ &= \alpha^{\frac{1-a^{2n+1}}{1-a}} \beta^{\frac{b(a^2-c-a(1-c)a^{2n}+(c+a)(1-a)c^n)}{(1-c)(1-a)(a^2-c)}} w_0^{ab\frac{a^{2n}-c^n}{a^2-c}} w_{-1}^{b\frac{a^{2n+2}-c^{n+1}}{a^2-c}} z_0^{a^{2n+1}}, \end{aligned} \tag{93}$$

for every $n \in \mathbb{N}$.

Subcase $a^2 \neq 1 \neq c$, $c = a^2$. In this case we have

$$\begin{aligned} z_{2n} &= \alpha^{(1+a)\sum_{j=0}^{n-1} a^{2j}} \beta^{b(1+a)\sum_{i=0}^{n-2} a^{2i}\sum_{j=0}^{n-i-2} a^{2j}} \left(w_0^b w_{-1}^{ab}\right)^{\sum_{j=0}^{n-1} a^{2j}a^{2n-2j-2}} z_0^{a^{2n}} \\ &= \alpha^{\frac{1-a^{2n}}{1-a}} \beta^{b(1+a)\sum_{i=0}^{n-2} a^{2i}\frac{1-a^{2n-2i-2}}{1-a^2}} \left(w_0^b w_{-1}^{ab}\right)^{na^{2n-2}} z_0^{a^{2n}} \\ &= \alpha^{\frac{1-a^{2n}}{1-a}} \beta^{\frac{b}{1-a}(\frac{1-a^{2n-2}}{1-a^2} - (n-1)a^{2n-2})} \left(w_0^b w_{-1}^{ab}\right)^{na^{2n-2}} z_0^{a^{2n}} \\ &= \alpha^{\frac{1-a^{2n}}{1-a}} \beta^{\frac{b(1-na^{2n-2}+(n-1)a^{2n})}{(a+1)(a-1)^2}} \left(w_0^b w_{-1}^{ab}\right)^{na^{2n-2}} z_0^{a^{2n}}, \end{aligned} \tag{94}$$

$$\begin{aligned} z_{2n+1} &= \alpha^{\sum_{j=0}^{2n} a^j} \beta^{b\sum_{i=0}^{n-1} a^{2i}(\sum_{j=0}^{n-i-1} a^{2j} + a\sum_{j=0}^{n-i-2} a^{2j})} w_0^{ab\sum_{j=0}^{n-1} a^{2j}a^{2n-2j-2}} w_{-1}^{b\sum_{j=0}^{n} a^{2j}a^{2n-2j}} z_0^{a^{2n+1}} \\ &= \alpha^{\frac{1-a^{2n+1}}{1-a}} \beta^{b\sum_{i=0}^{n-1} a^{2i}(\frac{1-a^{2n-2i}}{1-a^2} + a\frac{1-a^{2n-2i-2}}{1-a^2})} w_0^{bna^{2n-1}} w_{-1}^{b(n+1)a^{2n}} z_0^{a^{2n+1}} \\ &= \alpha^{\frac{1-a^{2n+1}}{1-a}} \beta^{\frac{b}{1-a}(\frac{1-a^{2n}}{1-a^2} - na^{2n-1})} w_0^{bna^{2n-1}} w_{-1}^{b(n+1)a^{2n}} z_0^{a^{2n+1}} \\ &= \alpha^{\frac{1-a^{2n+1}}{1-a}} \beta^{\frac{b(1-a^{2n}-na^{2n-1}+na^{2n+1})}{(a+1)(a-1)^2}} w_0^{bna^{2n-1}} w_{-1}^{b(n+1)a^{2n}} z_0^{a^{2n+1}}, \end{aligned} \tag{95}$$

for every $n \in \mathbb{N}$.

Subcase $a^2 \neq 1 = c$. In this case we have

$$\begin{aligned} z_{2n} &= \alpha^{(1+a)\sum_{j=0}^{n-1} a^{2j}} \beta^{b(1+a)\sum_{i=0}^{n-2} a^{2i}\sum_{j=0}^{n-i-2} 1} \left(w_0^b w_{-1}^{ab}\right)^{\sum_{j=0}^{n-1} a^{2j}} z_0^{a^{2n}} \\ &= \alpha^{\frac{1-a^{2n}}{1-a}} \beta^{b(1+a)\sum_{i=0}^{n-2} a^{2i}(n-i-1)} \left(w_0^b w_{-1}^{ab}\right)^{\frac{1-a^{2n}}{1-a}} z_0^{a^{2n}} \\ &= \alpha^{\frac{1-a^{2n}}{1-a}} \beta^{b(1+a)((n-1)\frac{1-a^{2n-2}}{1-a^2} - a^2\frac{1-(n-1)a^{2n-4}+(n-2)a^{2n-2}}{(1-a^2)^2})} \left(w_0^b w_{-1}^{ab}\right)^{\frac{1-a^{2n}}{1-a}} z_0^{a^{2n}} \\ &= \alpha^{\frac{1-a^{2n}}{1-a}} \beta^{\frac{b(n-1-na^2+a^{2n})}{(a+1)(a-1)^2}} \left(w_0^b w_{-1}^{ab}\right)^{\frac{1-a^{2n}}{1-a}} z_0^{a^{2n}}, \end{aligned} \tag{96}$$

$$\begin{aligned} z_{2n+1} &= \alpha^{\sum_{j=0}^{2n} a^j} \beta^{b\sum_{i=0}^{n-1} a^{2i}(\sum_{j=0}^{n-i-1} 1 + a\sum_{j=0}^{n-i-2} 1)} w_0^{ab\sum_{j=0}^{n-1} a^{2j}} w_{-1}^{b\sum_{j=0}^{n} a^{2j}} z_0^{a^{2n+1}} \\ &= \alpha^{\frac{1-a^{2n+1}}{1-a}} \beta^{b\sum_{i=0}^{n-1} a^{2i}((1+a)n-a-(1+a)i)} w_0^{ab\frac{1-a^{2n}}{1-a}} w_{-1}^{b\frac{1-a^{2n+2}}{1-a}} z_0^{a^{2n+1}} \end{aligned}$$

$$= \alpha^{\frac{1-a^{2n+1}}{1-a}} \beta^{b(((1+a)n-a)\frac{1-a^{2n}}{1-a^2} - a^2\frac{1-na^{2n-2}+(n-1)a^{2n}}{(a+1)(1-a)^2})} w_0^{ab\frac{1-a^{2n}}{1-a}} w_{-1}^{b\frac{1-a^{2n+2}}{1-a}} z_0^{a^{2n+1}}$$
$$= \alpha^{\frac{1-a^{2n+1}}{1-a}} \beta^{\frac{b(n-a-na^2+a^{2n+1})}{(a+1)(1-a)^2}} w_0^{ab\frac{1-a^{2n}}{1-a}} w_{-1}^{b\frac{1-a^{2n+2}}{1-a}} z_0^{a^{2n+1}}, \quad (97)$$

for every $n \geq -1$.

Subcase $a = -1$, $c = 1$. In this case we have

$$z_{2n} = \left(w_0^b w_{-1}^{-b}\right)^{\sum_{j=0}^{n-1}(-1)^{2j}} z_0^{(-1)^{2n}}$$
$$= \left(w_0^b w_{-1}^{-b}\right)^n z_0, \quad (98)$$
$$z_{2n+1} = \alpha^{\sum_{j=0}^{2n}(-1)^j} \beta^{b\sum_{i=0}^{n-1}(-1)^{2i}(\sum_{j=0}^{n-i-1}1 - \sum_{j=0}^{n-i-2}1)} w_0^{-b\sum_{j=0}^{n-1}(-1)^{2j}} w_{-1}^{b\sum_{j=0}^{n}(-1)^{2j}} z_0^{(-1)^{2n+1}}$$
$$= \alpha\beta^{bn} w_0^{-bn} w_{-1}^{b(n+1)} z_0^{-1}, \quad (99)$$

for every $n \in \mathbb{N}$.

Subcase $a = 1 \neq c$. In this case we have

$$z_{2n} = \alpha^{2\sum_{j=0}^{n-1}1} \beta^{2b\sum_{i=0}^{n-2}\sum_{j=0}^{n-i-2}c^j} \left(w_0^b w_{-1}^b\right)^{\sum_{j=0}^{n-1}c^{n-j-1}} z_0$$
$$= \alpha^{2n} \beta^{2b\sum_{i=0}^{n-2}\frac{1-c^{n-i-1}}{1-c}} \left(w_0^b w_{-1}^b\right)^{\frac{1-c^n}{1-c}} z_0$$
$$= \alpha^{2n} \beta^{\frac{2b}{1-c}(n-1-c\frac{1-c^{n-1}}{1-c})} \left(w_0^b w_{-1}^b\right)^{\frac{1-c^n}{1-c}} z_0$$
$$= \alpha^{2n} \beta^{\frac{2b(n-1-nc+c^n)}{(1-c)^2}} \left(w_0^b w_{-1}^b\right)^{\frac{1-c^n}{1-c}} z_0, \quad (100)$$
$$z_{2n+1} = \alpha^{\sum_{j=0}^{2n}1} \beta^{b\sum_{i=0}^{n-1}(\sum_{j=0}^{n-i-1}c^j + \sum_{j=0}^{n-i-2}c^j)} w_0^{b\sum_{j=0}^{n-1}c^{n-j-1}} w_{-1}^{b\sum_{j=0}^{n}c^{n-j}} z_0$$
$$= \alpha^{2n+1} \beta^{b\sum_{i=0}^{n-1}(\frac{1-c^{n-i}}{1-c} + \frac{1-c^{n-i-1}}{1-c})} w_0^{b\frac{1-c^n}{1-c}} w_{-1}^{b\frac{1-c^{n+1}}{1-c}} z_0$$
$$= \alpha^{2n+1} \beta^{\frac{b}{1-c}(2n - c\frac{1-c^n}{1-c} - \frac{1-c^n}{1-c})} w_0^{b\frac{1-c^n}{1-c}} w_{-1}^{b\frac{1-c^{n+1}}{1-c}} z_0$$
$$= \alpha^{2n+1} \beta^{\frac{b(2n-1-(2n+1)c+c^n+c^{n+1})}{(1-c)^2}} w_0^{b\frac{1-c^n}{1-c}} w_{-1}^{b\frac{1-c^{n+1}}{1-c}} z_0, \quad (101)$$

for $n \in \mathbb{N}$.

Subcase $a = c = 1$. In this case we have

$$z_{2n} = \alpha^{2\sum_{j=0}^{n-1}1} \beta^{2b\sum_{i=0}^{n-2}\sum_{j=0}^{n-i-2}1} \left(w_0^b w_{-1}^b\right)^{\sum_{j=0}^{n-1}1} z_0$$
$$= \alpha^{2n} \beta^{2b\sum_{i=0}^{n-2}(n-i-1)} \left(w_0^b w_{-1}^b\right)^n z_0$$
$$= \alpha^{2n} \beta^{b(n-1)n} \left(w_0^b w_{-1}^b\right)^n z_0, \quad (102)$$
$$z_{2n+1} = \alpha^{\sum_{j=0}^{2n}1} \beta^{b\sum_{i=0}^{n-1}(\sum_{j=0}^{n-i-1}1 + \sum_{j=0}^{n-i-2}1)} w_0^{b\sum_{j=0}^{n-1}1} w_{-1}^{b\sum_{j=0}^{n}1} z_0$$
$$= \alpha^{2n+1} \beta^{b\sum_{i=0}^{n-1}(2n-2i-1)} w_0^{bn} w_{-1}^{b(n+1)} z_0$$
$$= \alpha^{2n+1} \beta^{bn^2} w_0^{bn} w_{-1}^{b(n+1)} z_0, \quad (103)$$

for every $n \in \mathbb{N}$, completing the proof. □

Corollary 3 *Consider system* (1) *with* $a, b, c \in \mathbb{Z}$, $d = 0$, *and* $\alpha, \beta \in \mathbb{C} \setminus \{0\}$. *Assume that* $z_{-1}, z_0, w_{-1}, w_0 \in \mathbb{C} \setminus \{0\}$. *Then the following statements are true.*

(a) *If* $c \neq a^2 \neq 1 \neq c$, *then the general solution to system* (1) *is given by* (78), (79), (92), *and* (93).

(b) *If* $c = a^2 \neq 1 \neq c$, *then the general solution to system* (1) *is given by* (78), (79), (94), *and* (95).

(c) *If* $a^2 \neq 1 = c$, *then the general solution to system* (1) *is given by* (80), (81), (96), *and* (97).

(d) *If* $a = -1$ *and* $c = 1$, *then the general solution to system* (1) *is given by* (80), (81), (98), *and* (99).

(e) *If* $a = 1$ *and* $c \neq 1$, *then the general solution to system* (1) *is given by* (78), (79), (100), *and* (101).

(f) *If* $a = c = 1$, *then the general solution to system* (1) *is given by* (80), (81), (102), *and* (103).

Theorem 4 *Assume that* $a, b, c, d \in \mathbb{Z}$, $bd \neq 0$, $\alpha, \beta \in \mathbb{C} \setminus \{0\}$, *and* $z_{-1}, z_0, w_{-1}, w_0 \in \mathbb{C} \setminus \{0\}$. *Then system* (1) *is solvable in closed form.*

Proof First note that the conditions $\alpha, \beta \in \mathbb{C} \setminus \{0\}$ and $z_{-1}, z_0, w_{-1}, w_0 \in \mathbb{C} \setminus \{0\}$ along with the equations in (1) imply $z_n w_n \neq 0$ for $n \geq -1$. Hence, for every such a solution the first equation in (1) yields

$$w_{n-1}^b = \frac{z_{n+1}}{\alpha z_n^a}, \quad n \in \mathbb{N}_0, \tag{104}$$

while from the second one it follows that

$$w_{n+1}^b = \beta^b w_{n-1}^{bc} z_{n-1}^{bd}, \quad n \in \mathbb{N}_0. \tag{105}$$

From (104) and (105) one obtains

$$z_{n+3} = \alpha^{1-c} \beta^b z_{n+2}^a z_{n+1}^c z_n^{-ac} z_{n-1}^{bd}, \quad n \in \mathbb{N}_0, \tag{106}$$

which is a fourth order product-type difference equation.

Note also that

$$z_1 = \alpha z_0^a w_{-1}^b, \qquad z_2 = \alpha \left(\alpha z_0^a w_{-1}^b\right)^a w_0^b = \alpha^{1+a} z_0^{a^2} w_{-1}^{ab} w_0^b. \tag{107}$$

Let $\delta = \alpha^{1-c} \beta^b$,

$$a_1 = a, \qquad b_1 = c, \qquad c_1 = -ac, \qquad d_1 = bd, \qquad y_1 = 1. \tag{108}$$

Then equation (106) can be written as

$$z_{n+3} = \delta^{y_1} z_{n+2}^{a_1} z_{n+1}^{b_1} z_n^{c_1} z_{n-1}^{d_1}, \quad n \in \mathbb{N}_0. \tag{109}$$

Using (109) with $n \to n-1$ into (109) we get

$$\begin{aligned} z_{n+3} &= \delta^{y_1}\left(\delta z_{n+1}^{a_1} z_n^{b_1} z_{n-1}^{c_1} z_{n-2}^{d_1}\right)^{a_1} z_{n+1}^{b_1} z_n^{c_1} z_{n-1}^{d_1}, \\ &= \delta^{y_1+a_1} z_{n+1}^{a_1a_1+b_1} z_n^{b_1a_1+c_1} z_{n-1}^{c_1a_1+d_1} z_{n-2}^{d_1a_1} \\ &= \delta^{y_2} z_{n+1}^{a_2} z_n^{b_2} z_{n-1}^{c_2} z_{n-2}^{d_2}, \end{aligned} \tag{110}$$

for $n \in \mathbb{N}$, where

$$\begin{aligned} &a_2 := a_1a_1 + b_1, \quad b_2 := b_1a_1 + c_1, \qquad c_2 := c_1a_1 + d_1, \\ &d_2 := d_1a_1, \qquad y_2 := y_1 + a_1. \end{aligned} \tag{111}$$

Assume that, for a k such that $2 \le k \le n+1$, we have proved that

$$z_{n+3} = \delta^{y_k} z_{n+3-k}^{a_k} z_{n+2-k}^{b_k} z_{n+1-k}^{c_k} z_{n-k}^{d_k}, \tag{112}$$

for $n \ge k-1$, and that

$$\begin{aligned} &a_k = a_1a_{k-1} + b_{k-1}, \qquad b_k = b_1a_{k-1} + c_{k-1}, \\ &c_k = c_1a_{k-1} + d_{k-1}, \qquad d_k = d_1a_{k-1}, \end{aligned} \tag{113}$$

$$y_k := y_{k-1} + a_{k-1}. \tag{114}$$

Using (109) with $n \to n-k$ into (110) one obtains

$$\begin{aligned} z_{n+3} &= \delta^{y_k}\left(\delta z_{n+2-k}^{a_1} z_{n+1-k}^{b_1} z_{n-k}^{c_1} z_{n-k-1}^{d_1}\right)^{a_k} z_{n+2-k}^{b_k} z_{n+1-k}^{c_k} z_{n-k}^{d_k} \\ &= \delta^{y_k+a_k} z_{n+2-k}^{a_1a_k+b_k} z_{n+1-k}^{b_1a_k+c_k} z_{n-k}^{c_1a_k+d_k} z_{n-k-1}^{d_1a_k} \\ &= \delta^{y_{k+1}} z_{n+2-k}^{a_{k+1}} z_{n+1-k}^{b_{k+1}} z_{n-k}^{c_{k+1}} z_{n-k-1}^{d_{k+1}}, \end{aligned} \tag{115}$$

for $n \ge k$, where

$$\begin{aligned} &a_{k+1} := a_1a_k + b_k, \qquad b_{k+1} := b_1a_k + c_k, \\ &c_{k+1} := c_1a_k + d_k, \qquad d_{k+1} := d_1a_k, \end{aligned} \tag{116}$$

$$y_{k+1} := y_k + a_k. \tag{117}$$

This along with (110), (111), and the method of induction shows that (112), (113), and (114), hold for every k and n such that $2 \le k \le n+1$. In fact (112) holds for $1 \le k \le n+1$ (see (109)).

Hence, choosing $k = n+1$ in (112), and using (107) we have

$$\begin{aligned} z_{n+3} &= \delta^{y_{n+1}} z_2^{a_{n+1}} z_1^{b_{n+1}} z_0^{c_{n+1}} z_{-1}^{d_{n+1}} \\ &= \left(\alpha^{1-c}\beta^b\right)^{y_{n+1}} \left(\alpha^{1+a} z_0^{a^2} w_{-1}^{ab} w_0^b\right)^{a_{n+1}} \left(\alpha z_0^a w_{-1}^b\right)^{b_{n+1}} z_0^{c_{n+1}} z_{-1}^{d_{n+1}} \\ &= \alpha^{(1-c)y_{n+1}+(1+a)a_{n+1}+b_{n+1}} \beta^{by_{n+1}} z_0^{a^2a_{n+1}+ab_{n+1}+c_{n+1}} \\ &\quad \times w_{-1}^{aba_{n+1}+bb_{n+1}} w_0^{ba_{n+1}} z_{-1}^{d_{n+1}}, \quad n \in \mathbb{N}_0. \end{aligned} \tag{118}$$

From (113) we easily see that $(a_k)_{k\geq 4}$ satisfies the difference equation

$$a_k = a_1 a_{k-1} + b_1 a_{k-2} + c_1 a_{k-3} + d_1 a_{k-4}. \tag{119}$$

Since $b_k = a_{k+1} - a_1 a_k$, $c_k = b_{k+1} - b_1 a_k$, $d_k = d_1 a_{k-1}$, and from the linearity of equation (119) we see that $(b_k)_{k\in\mathbb{N}}$, $(c_k)_{k\in\mathbb{N}}$, and $(d_k)_{k\in\mathbb{N}}$ are also solutions to the equation.

System (113) with $k = 1$ yields

$$\begin{aligned} &a_1 = a_1 a_0 + b_0, \qquad b_1 = b_1 a_0 + c_0, \qquad c_1 = c_1 a_0 + d_0, \\ &d_1 = d_1 a_0, \qquad y_1 = y_0 + a_0. \end{aligned} \tag{120}$$

The condition $d_1 = bd \neq 0$ along with the fourth equation in (120) implies $a_0 = 1$. Using this and $y_1 = 1$ in the other equalities in (120) we get $b_0 = c_0 = d_0 = y_0 = 0$. Repeating the procedure for $k = 0, -1, -2$, is easily obtained

$$\begin{aligned} &a_{-3} = 0, \qquad a_{-2} = 0, \qquad a_{-1} = 0, \qquad a_0 = 1; \\ &b_{-3} = 0, \qquad b_{-2} = 0, \qquad b_{-1} = 1, \qquad b_0 = 0; \\ &c_{-3} = 0, \qquad c_{-2} = 1, \qquad c_{-1} = 0, \qquad c_0 = 0; \\ &d_{-3} = 1, \qquad d_{-2} = 0, \qquad d_{-1} = 0, \qquad d_0 = 0. \end{aligned} \tag{121}$$

Hence, $(a_k)_{k\geq -3}$, $(b_k)_{k\geq -3}$, $(c_k)_{k\geq -3}$, and $(d_k)_{k\geq -3}$ are solutions to (119) satisfying initial conditions (121), while $(y_k)_{k\geq -3}$ satisfies the following conditions:

$$y_{-3} = y_{-2} = y_{-1} = y_0 = 0, \qquad y_1 = 1, \tag{122}$$

and (114), from which it follows that

$$y_k = \sum_{j=1}^{k-1} a_j. \tag{123}$$

Since equation (119) is solvable, it follows that closed form formulas for $(a_k)_{k\geq -3}$, $(b_k)_{k\geq -3}$, $(c_k)_{k\geq -3}$, and $(d_k)_{k\geq -3}$, can be found. From (123), the form of the solution a_k, and by using some known summation formulas it follows that the formula for $(y_k)_{k\geq -3}$ can also be found. From these facts and (118) we see that equation (106) is solvable too.

From the second equation in (1), we have that for every well-defined solution

$$z_{n-1}^d = \frac{w_{n+1}}{\beta w_{n-1}^c}, \quad n \in \mathbb{N}_0, \tag{124}$$

while from the first one it follows that

$$z_{n+1}^d = \alpha^d z_n^{ad} w_{n-1}^{bd}, \quad n \in \mathbb{N}_0. \tag{125}$$

From (124) into (125) one obtains

$$w_{n+3} = \alpha^d \beta^{1-a} w_{n+2}^a w_{n+1}^c w_n^{-ac} w_{n-1}^{bd}, \quad n \in \mathbb{N}_0, \tag{126}$$

which differs from (106) only by the constant multiplier.

We have

$$w_1 = \beta w_{-1}^c z_{-1}^d \quad \text{and} \quad w_2 = \beta w_0^c z_0^d. \tag{127}$$

As above one obtains, for all natural numbers k and n such that $1 \le k \le n+1$,

$$w_{n+3} = \eta^{\hat{y}_k} w_{n+3-k}^{a_k} w_{n+2-k}^{b_k} w_{n+1-k}^{c_k} w_{n-k}^{d_k}, \quad n \ge k-1, \tag{128}$$

where $\eta = \alpha^d \beta^{1-a}$, $(a_k)_{k\in\mathbb{N}}$, $(b_k)_{k\in\mathbb{N}}$, $(c_k)_{k\in\mathbb{N}}$, and $(d_k)_{k\in\mathbb{N}}$ satisfy (113) with initial conditions (108), while $(\hat{y}_k)_{k\in\mathbb{N}}$ satisfies (114) and (122), so that (123) holds where y_k is replaced by $\hat{y}_k$.

From (128) with $k = n+1$ and by using (127) we get

$$\begin{aligned} w_{n+3} &= \eta^{\hat{y}_{n+1}} w_2^{a_{n+1}} w_1^{b_{n+1}} w_0^{c_{n+1}} w_{-1}^{d_{n+1}} \\ &= \left(\alpha^d \beta^{1-a}\right)^{\hat{y}_{n+1}} \left(\beta w_0^c z_0^d\right)^{a_{n+1}} \left(\beta w_{-1}^c z_{-1}^d\right)^{b_{n+1}} w_0^{c_{n+1}} w_{-1}^{d_{n+1}} \\ &= \alpha^{d\hat{y}_{n+1}} \beta^{(1-a)\hat{y}_{n+1}+a_{n+1}+b_{n+1}} w_0^{ca_{n+1}+c_{n+1}} z_0^{da_{n+1}} w_{-1}^{cb_{n+1}+d_{n+1}} z_{-1}^{db_{n+1}}, \end{aligned} \tag{129}$$

for $n \in \mathbb{N}_0$.

As above the solvability of (119) shows that formulas for $(a_k)_{k\ge-3}$, $(b_k)_{k\ge-3}$, $(c_k)_{k\ge-3}$, and $(d_k)_{k\ge-3}$ can be found, and consequently a formula for $(\hat{y}_k)_{k\ge-3}$. This fact along with (129) implies that equation (126) is solvable too. Hence, system (1) is also solvable in this case, as desired. □

Corollary 4 *Consider system* (1) *with* $a, b, c, d \in \mathbb{Z}$, $bd \ne 0$, $\alpha, \beta \in \mathbb{C} \setminus \{0\}$. *Assume that* $z_{-1}, z_0, w_{-1}, w_0 \in \mathbb{C} \setminus \{0\}$. *Then the general solution to system* (1) *is given by* (118) *and* (129), *where the sequences* $(a_k)_{k\in\mathbb{N}}$, $(b_k)_{k\in\mathbb{N}}$, $(c_k)_{k\in\mathbb{N}}$, *and* $(d_k)_{k\in\mathbb{N}}$ *satisfy the difference equation* (119) *with initial conditions in* (121), *while* $(y_k)_{k\in\mathbb{N}}$ *and* $(\hat{y}_k)_{k\in\mathbb{N}}$ *are given by* (123) *and satisfy conditions* (122).

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the manuscript.

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