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# Infinitely many solutions for fractional Laplacian problems with local growth conditions

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**Abstract**
In this paper, we deal with the fractional Laplacian equations

$$(\mathrm{P}) \quad \begin{cases} (-\Delta)^s u = f(x,u), & x \in \Omega, \\ u(x) = 0, & x \in \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $0 < s < 1 < p < +\infty$, $N \in \mathbb{N}, N > 2s$, $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary. Under local growth conditions of $f(x,t)$, infinitely many solutions for problem (P) are obtained via variational methods.

**MSC:** 35R11; 35A15; 35B38; 35P30

**Keywords:** fractional Laplacian equation; local growth condition; Clark's theorem; variational methods

## 1 Introduction and main results

In this article we are concerned with the multiplicity of solutions for the following fractional Laplacian equations:

$$(\mathrm{P}) \quad \begin{cases} (-\Delta)^s u = f(x,u), & x \in \Omega, \\ u(x) = 0, & x \in \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $0 < s < 1 < p < +\infty$, $N > 2s$, $\Omega \subset \mathbb{R}^N$ is an open bounded domain with smooth boundary, $f(x,t)$ is a Carathéodory function defined on $\Omega \times (-\delta,\delta)$ for some $\delta > 0$, and $(-\Delta)^s$ is known as the fractional Laplacian operator, which (up to normalization factors) may be defined as

$$-(-\Delta)^s u = \frac{1}{2} \int_{\mathbb{R}^N} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{N+2s}} \, dy.$$

The topic of fractional Laplacian operators $(-\Delta)^s$ and more generally non-local operators is a classical one in harmonic analysis and partial differential equations. These operators arise in a quite natural way in many different contexts, such as the thin obstacle problem, optimization, finance, materials science, continuum mechanics, etc. Recently, great attention has been focused on the study of them, both for the pure mathematical research and in view of concrete applications. A series of important research results have been got via various methods. Here we only collect some results got through variational methods and critical point theory. First, for an elementary introduction to the fractional Laplacian operator $(-\Delta)^s$ and more generally non-local operators, please check the cited articles [1, 2] and the references therein. In particular, in [2], the authors define the fractional Sobolev spaces $W^{s,p}$ via the Gagliardo approach and give some of their basic properties and prove some continuous and compact embedding results. Second, it is well known that similar to the classical elliptic problem

$$(\mathrm{P}') \quad \begin{cases} -\Delta u = f(x,u), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases}$$

problem (P) also enjoys a variational nature and its solutions can also be constructed as critical points of the associated Euler-Lagrange functional. A natural question is whether or not the classical topological and variational methods may be adapted to problem (P) and to its generalization in order to extend the classical results known for the classical elliptic problem (P′) to a non-local setting. A great attention has been focused on this topic. For example, in [3], based on some analysis of the fractional spaces involved, the authors prove that mountain pass theorem still works for a general integrodifferential operator of fractional type. And one mountain pass solution is got for some non-local elliptic operators. In [4], the authors get one critical point for some non-local elliptic operators with real parameter by mountain pass theorem and linking theorem, respectively. The fact that saddle point theorem still works for some non-local elliptic operators has been proved in [5]. [6] proves that symmetric mountain pass theorem still works for a general integrodifferential operator of fractional type. In [7], Morse theory is applied to study the existence of weak solution for problem (P). The ground state solution is got by the Nehari manifold method for non-local elliptic operators involving concave-convex nonlinearities in [8]. In [9], the existence of multiple nontrivial weak solutions for some parametric non-local equations with the nonlinear term having a sublinear growth at infinity is got via Variational methods. The existence or multiplicity of solutions for fractional elliptic problems have also been investigated in [10–14] and the references therein. The issues of regularity and non-existence of solutions are studied in [15–19]. The corresponding equations in $\mathbb{R}^N$ have also been widely studied, for example [20–26] and the references therein.

In all the works mentioned above, in order to apply Variational methods and Critical point theory, the nonlinearity $f$ is assumed on the whole $\Omega \times \mathbb{R}$ or $\mathbb{R} \times \mathbb{R}^N$ and has to satisfy various global growth conditions. The purpose of this article is to prove that a new version of Clark's theorem (see [27]) is still valid for fractional Laplacian problem (P). Furthermore, in our article, the nonlinearity $f$ just needs to satisfy some local normal growth conditions. More precisely we assume $f$ satisfies the following sublinear conditions near the origin and not any condition at all near infinity.

($f_1$) $f$ is a Carathéodory function defined on $\Omega \times (-\delta, \delta)$ for some $\delta > 0$ which can be chosen small;

($f_2$) there exists a positive constant $q_1 \in (\frac{4}{2_s^*}, 2)$ such that

$$\lim_{|t|\to 0} \frac{F(x,t)}{|t|^{q_1}} = 0, \quad \text{uniformly for a.e. } x \in \Omega;$$

($f_3$) there exists a positive constant $q_2 \in (\frac{4}{2_s^*}, 2)$ such that

$$\lim_{|t|\to 0} \frac{F(x,t)}{|t|^{q_2}} = +\infty, \quad \text{uniformly for a.e. } x \in \Omega;$$

($f_4$) $f(x,t)$ is odd in $t$, for a.e. $x \in \Omega$, $t \in (-\delta, \delta)$.

Our main result is stated as follows.

**Theorem 1.1** *Let* ($f_1$)-($f_4$) *hold, then problem* (P) *enjoys a sequence of nontrivial solutions* $\{u_m\}$ *with* $|u_m|_\infty \to 0$ *as* $m \to \infty$.

**Remark 1.1** In this article, the nonlinearity $f$ just satisfies some sublinear growth condition near the origin, While without any assumptions near infinity. In order to prove our results via variational approach, inspired by the methods of [27, 28], first, we need to modify and extend $f$ to an appropriate $\tilde{f}$ and to show for the associated modified functional the existence of solutions. Second, in order to obtain solutions for the original problem (P), some $L^\infty$-estimates for the solutions of the modified problem are absolutely necessary. However, as far as we have known there is few result about the $L^p$-estimate for fractional Laplacian problem as the class Laplacian problem. Similar bounds were obtained before only in some special cases, for a semilinear fractional Laplacian equation with reaction term independent of $u$, or for the eigenvalue problem of some fractional elliptic operators. Recently, in [29], the authors provided a method to give a priori $L^\infty$ bounds for the weak solutions of problems similar to (P). Inspired by this method, we are able to get a suitable estimate of $L^\infty$ norm of the weak solutions (for more details please check Lemma 3.2 of our article). Finally, by the Sobolev embedding theorem and Lemma 2.2, we can get infinity many solutions for the original problem (P).

**Remark 1.2** The key step of our article is to get a suitable estimate of $L^\infty$ norm of the weak solutions. In condition ($f_2$), the assumption $q_1 > \frac{4}{2_s^*}$ will be applied to give a $L^\infty$-estimate for the weak solutions.

Throughout the article, the letter $C$ will denote various positive constants whose values may change from line to line but are not essential to the analysis of the problem. We denote the usual norm of $L^q(\Omega)$ by $|\cdot|_q$ for $1 \le q \le \infty$. Moreover, let $0 < s < 1$ be real numbers, and the fractional critical exponent be defined as $2_s^* = \frac{2N}{N-2s}$.

The paper is organized as follows. In Section 2, we introduce some preliminary notions and notations and set the functional framework of the our problem. In Section 3, we will prove our main result Theorem 1.1.

## 2 Preliminary

In this preliminary section, for the reader's convenience, we collect some basic results that will be used in the forthcoming section.

First, we introduce a variational setting for problem (P). The Gagliardo seminorm is defined for a measurable function $u : \mathbb{R}^N \mapsto \mathbb{R}$ by

$$[u]_{s,2} = \left( \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} \, dx \, dy \right)^{\frac{1}{2}}.$$

The fractional Sobolev space is

$$W^{s,2}(\mathbb{R}^N) := \{u \in L^2(\mathbb{R}^N) : u \text{ is measurable}, [u]_{s,2} < \infty\},$$

endowed with the norm

$$\|u\|_{s,2} = \left(\int_{\mathbb{R}^N} |u|^2\,dx + [u]_{s,2}^2\right)^{\frac{1}{2}}.$$

In this paper, we will work in the closed linear subspace

$$X(\Omega) = \{u \in W^{s,2}(\mathbb{R}^N) : u(x) = 0, \text{a.e. } x \in \mathbb{R}^N \setminus \Omega\},$$

which can be equivalently renormed by setting $\|\cdot\| = [\cdot]_{s,2}$ (see Theorem 7.1 of [2]). It is readily seen that $(X(\Omega), \|\cdot\|)$ is a Hilbert space with the inner product

$$\langle u, v\rangle := \int_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}}\,dx\,dy.$$

A weak solution of problem (P) is a function $u \in X(\Omega)$ such that

$$\langle u, v\rangle = \int_\Omega f(x,u)v\,dx, \quad \text{for all } v \in X(\Omega). \tag{2.1}$$

**Definition 2.1** Let $E$ be a Banach space, we say that a functional $\Phi \in C^1(E,\mathbb{R})$ satisfies Palais-Smale condition at the level $c \in \mathbb{R}$ ($(\mathrm{PS})_c$ in short) if any sequence $\{u_n\} \subset E$ satisfying $\Phi(u_n) \to c$, $\Phi'(u_n) \to 0$ as $n \to \infty$, has a convergent subsequence. $\Phi$ satisfies (PS) condition if $\Phi$ satisfies $(\mathrm{PS})_c$ condition at any $c \in \mathbb{R}$.

The following Sobolev type embedding theorem holds.

**Lemma 2.1** ([2]) *The embedding $X(\Omega) \hookrightarrow L^q(\Omega)$ is continuous for all $q \in [1, 2_s^*]$, and compact for $q \in [1, 2_s^*)$.*

We also need the following new version of Clark's theorem; see Theorem 1.1 in [27].

**Lemma 2.2** *Assume that $X$ is a Banach space, $\Phi \in C^1(X,\mathbb{R})$ satisfying (PS) condition is bounded from below and even, $\Phi(0) = 0$. If for any $k \in \mathbb{N}$, there exist a $k$ dimensional subspace $X^k$ and $\rho_k > 0$ such that $\sup_{X^k \cap S_{\rho_k}} \Phi < 0$, where $S_{\rho_k} = \{u \in X : \|u\| = \rho_k\}$, then at least one of the following results holds.*

(1) *There exists a sequence of critical points $\{u_k\}$ satisfying $\Phi(u_k) < 0$ for all $k$ and $\|u_k\| \to 0$ as $k \to \infty$.*
(2) *There exists $r > 0$ such that for any $a \in (0, r)$ there exists a critical point $u$ such that $\|u\| = a$ and $\Phi(u) = 0$.*

## 3 Proof of main result

In this section we will prove our main result, Theorem 1.1. Since $(f_1)$-$(f_4)$ describe the behaviors of $f$ just in $\Omega \times (-\delta, \delta)$, the functional $\int_\Omega F(x,u)\,dx$ is not well defined in $X(\Omega)$.

To overcome this difficulty, we need to modify and extend $f$ to an appropriate $\tilde{f}$ in the spirit of the arguments developed by [28]. First of all, it follows from ($f_2$) and ($f_3$) that for small $|t|$

$$\bigl|F(x,t)\bigr| > |t|^{q_2}, \qquad \bigl|F(x,t)\bigr| < |t|^{q_1}, \quad \text{for a.e. } x \in \Omega. \tag{3.1}$$

Let $\rho \in C^1(\mathbb{R},[0,1])$ be an even cut-off function verifying $t\rho'(t) \le 0$ and

$$\rho(t) = \begin{cases} 1, & \text{if } |t| \le \tau, \\ 0, & \text{if } |t| \ge 2\tau, \end{cases} \tag{3.2}$$

where $\tau \in (0, \frac{\delta}{2})$ is chosen such that (3.1) and (3.2) hold for $|t| \le 2\tau$. Set

$$\tilde{F}(x,t) = \rho(t)F(x,t) + \bigl(1-\rho(t)\bigr)|t|^{q_2}, \qquad \tilde{f}(x,t) = \frac{\partial}{\partial t}\tilde{F}(x,t).$$

It is easy to see that $\tilde{F}$ is even in $t$ and $\tilde{f}$ is a Carathéodory function defined on $\Omega \times \mathbb{R}$. We then consider the following problem:

$$(\tilde{\mathrm{P}}) \qquad \begin{cases} (-\Delta)^s u = \tilde{f}(x,u), & x \in \Omega, \\ u(x) = 0, & x \in \mathbb{R}^N \backslash \Omega, \end{cases}$$

and its associated functional

$$\tilde{\Phi}(u) = \frac{1}{2}\|u\|^2 - \int_\Omega \tilde{F}(x,u)\,dx, \quad u \in X(\Omega).$$

By the definition of $\tilde{F}$, one can see that $\tilde{\Phi} \in C^1(X(\Omega),\mathbb{R})$. It is also easy to see that $\tilde{f}(x,t) = f(x,t)$ for $(x,t) \in \Omega \times [-\tau,\tau]$ and a critical point $u$ of $\tilde{\Phi}$ is a solution of the original problem (P) if and only if $|u|_\infty \le \tau$.

In order to get our main result by Lemma 2.2. First of all, we check that $\tilde{\Phi}$ is coercive, i.e. $\tilde{\Phi}(u) \to \infty$, as $\|u\| \to \infty$, and $\tilde{\Phi}$ satisfies the (PS) condition.

**Lemma 3.1** *The functional $\tilde{\Phi}$ is bounded from below and satisfies* (PS) *condition.*

*Proof* By (3.1) and the definition of $\tilde{F}$, we have

$$\tilde{F}(x,t) \le C\bigl(|t|^{q_1} + |t|^{q_2}\bigr), \quad \text{for } (x,t) \in \Omega \times \mathbb{R}, \tag{3.3}$$

where $C$ is a positive constant. Then Lemma 2.1 implies that

$$\begin{aligned} \tilde{\Phi}(u) &= \frac{1}{2}\|u\|^2 - \int_\Omega \tilde{F}(x,u)\,dx \\ &\ge \frac{1}{2}\|u\|^2 - C\int_\Omega \bigl(|u|^{q_1} + |u|^{q_2}\bigr)\,dx \\ &\ge \frac{1}{2}\|u\|^2 - C\bigl(\|u\|^{q_1} + \|u\|^{q_2}\bigr). \end{aligned}$$

Since $\frac{4}{2_s^*} < q_1 < q_2 < 2$, it follows that

$$\tilde{\Phi}(u) \to +\infty, \quad \text{as } \|u\| \to \infty. \tag{3.4}$$

Therefore, $\tilde{\Phi}$ is coercive and bounded from blow.

Next, we will prove that $\tilde{\Phi}$ satisfies (PS) condition. For any $c \in \mathbb{R}$, let $\{u_n\} \subset X(\Omega)$ be a $(\mathrm{PS})_c$ sequence, then

$$\tilde{\Phi}(u_n) \to c, \qquad \tilde{\Phi}'(u_n) \to 0, \quad \text{as } n \to \infty. \tag{3.5}$$

By (3.4), it follows that $\{u_n\}$ is bounded in $X(\Omega)$. By Lemma 2.1, we can assume that, up to a subsequence, for some $u \in X(\Omega)$,

$$\begin{aligned} &u_n \rightharpoonup u, \quad \text{as } n \to \infty, \text{in } X(\Omega), \\ &u_n \to u, \quad \text{as } n \to \infty, \text{in } L^q(\Omega), q \in \left(1, 2_s^*\right), \\ &u_n(x) \to u(x), \quad \text{as } n \to \infty, \text{a.e. } x \in \Omega. \end{aligned}$$

It follows from (3.5) and $u_n \rightharpoonup u$ in $X(\Omega)$ that

$$\left\langle \tilde{\Phi}'(u_n), u_n - u \right\rangle \to 0, \quad \text{as } n \to \infty.$$

As $q_1, q_2 \in (1, 2_s^*)$, by the Hölder inequality and (3.3), we have

$$\int_\Omega \tilde{f}(x, u_n)(u_n - u)\, dx \to 0, \quad \text{as } n \to \infty.$$

Then

$$\langle u_n, u_n - u \rangle = \left\langle \tilde{\Phi}'(u_n), u_n - u \right\rangle + \int_\Omega \tilde{f}(x, u_n)(u_n - u)\, dx \to 0, \quad \text{as } n \to \infty.$$

Since $\langle u, u_n - u \rangle \to 0$, as $n \to \infty$,

$$\|u_n - u\| \to 0, \quad \text{as } n \to \infty,$$

and the functional $\tilde{\Phi}$ satisfies the $(\mathrm{PS})_c$ condition for any $c \in \mathbb{R}$.

Second, it is easy to see that $\tilde{\Phi}$ is even and $\tilde{\Phi}(0) = 0$. Thus, in order to use Lemma 2.2 it suffices to find a subspace $X^k$ and $\rho_k > 0$ for any $k \in \mathbb{N}$ such that

$$\sup_{u \in X^k \bigcap S_{\rho_k}} \tilde{\Phi}(u) < 0.$$

For any $k \in \mathbb{N}$, we can choose $k$ independent functions $\varphi_i \in X(\Omega)$ for $i = 1, \ldots, k$, and define $X^k := \overline{\operatorname{span}\{\varphi_1, \varphi_2, \ldots, \varphi_k\}}$. By $(f_2)$, $(f_3)$, and the definition of $\tilde{F}$, we have $\tilde{F}(x,t) \geq C|t|^{q_2}, t \in \mathbb{R}$ for some $C > 0$. Then by the fact that all norms on $X^k$ are equivalent

$$\tilde{\Phi}(u) = \frac{1}{2}\|u\|^2 - \int_\Omega \tilde{F}(x, u)\, dx \leq \frac{1}{2}\|u\|^2 - C \int_\Omega |u|^{q_2}\, dx < 0,$$

for all $u \in X^k$ with $\|u\| = \rho_k$ for small enough $\rho_k > 0$. That is to say, by choosing $\rho_k > 0$ small enough we can get

$$\sup_{u \in X^k \cap S_{\rho_k}} \tilde{\Phi}(u) < 0.$$

Then all conditions of Lemma 2.2 are verified, we can get a sequence of critical points $\{u_m\} \subseteq X(\Omega)$ for $\tilde{\Phi}$ with $\tilde{\Phi}(u_m) \le 0$ and $\|u_m\| \to 0$, as $m \to \infty$. □

Finally, in order to get the weak solutions of the original problem (P), we will prove that the above sequence of critical points $\{u_m\}$ for $\tilde{\Phi}$ enjoys the following property.

**Lemma 3.2** *Under the assumptions of Theorem* 1.1, *the above sequence of critical points* $\{u_m\}$ *for* $\tilde{\Phi}$ *satisfies*

$$|u_m|_\infty \to 0, \quad \text{as } m \to \infty.$$

*Proof* We modify the proof of Theorem 3.1 of [29]. For the convenience of the reader, here we give a detailed proof. Compare with that result in [29], our result is a little more precise for our case. First of all, by the definition of $\tilde{f}$ and ($f_2$), it is easy to see that there exists $C > 0$ such that

$$\left|\tilde{f}(x,t)\right| \le C\left(|t|^{q_1-1} + |t|\right), \quad (x,t) \in \Omega \times \mathbb{R}, \text{and } 1 + \frac{q_1}{2} > 1 + \frac{2}{2_s^*}.$$

Then all the conditions of Theorem 3.1 of [29] hold. For a weak solution $u \in X(\Omega)$ of the above problem ($\tilde{\mathrm{P}}$) with $u^+ \not\equiv 0$, we choose $\rho \ge \max\{1, \frac{1}{|u|_2}\}$, set $v = \frac{u}{\rho |u|_2}$, then $v \in X(\Omega)$, $|v|_2 = \frac{1}{\rho}$, and $v$ is a weak solution of the auxiliary problem

$$(\mathrm{A}\tilde{\mathrm{P}}) \quad \begin{cases} (-\Delta)^s v = \frac{1}{\rho|u|_2}\tilde{f}(x, \rho|u|_2 v), & x \in \Omega, \\ v(x) = 0, & x \in \mathbb{R}^N \backslash \Omega. \end{cases}$$

For any $\tau > 0$, we set $v_n = (v - \tau + \frac{\tau}{2^n})^+$ for all $n \in \mathbb{N}$. It is easy to see that $v_n \in X(\Omega)$, $v_0 = v^+$, and for all $n \in \mathbb{N}$ we have $0 \le v_{n+1}(x) \le v_n(x)$ and $v_n(x) \to (v(x) - \tau)^+$, a.e. $x \in \Omega$ as $n \to \infty$. Moreover, the following inclusion holds (up to a Lebesgue null set):

$$\{x \in \Omega : v_{n+1} > 0\} \subseteq \left\{x \in \Omega : \ 0 < v < \left(2^{n+1} - 1\right)v_n\right\} \cap \left\{x \in \Omega : v_n > \frac{\tau}{2^{n+1}}\right\}. \tag{3.6}$$

For every $n \in \mathbb{N}$, we set $R_n = |v_n|_2^2$, then $R_0 = |v^+|_2^2 \le \frac{1}{\rho^2}$, and $R_n \in [0,1]$ is nonincreasing in $n$. We will prove that $R_n \to 0$ as $n \to \infty$. By Hölder's inequality, the fractional Sobolev inequality (see Theorem 6.5 of [2]), (3.6), and the Chebyshev inequality we have, for all $n \in \mathbb{N}$,

$$\begin{aligned} R_{n+1} &\le \left|\{x \in \Omega : v_{n+1} > 0\}\right|^{1-\frac{2}{2_s^*}} |v_{n+1}|_{2_s^*}^2 \le C\left|\left\{x \in \Omega : v_n > \frac{\tau}{2^{n+1}}\right\}\right|^{1-\frac{2}{2_s^*}} \|v_{n+1}\|^2 \\ &\le C\tau^{2(\frac{2}{2_s^*}-1)} 2^{(2-\frac{4}{2_s^*})(n+1)} R_n^{1-\frac{2}{2_s^*}} \|v_{n+1}\|^2. \end{aligned} \tag{3.7}$$

By testing $(\mathrm{A}\tilde{\mathrm{P}})$ with $v_{n+1}$, and applying (3.6), we can see that

$$\begin{aligned}\|v_{n+1}\|^2 &\leq \int_\Omega \frac{1}{\rho|u|_2}\tilde{f}\big(x,\rho|u|_2 v\big)v_{n+1}\,dx \\ &\leq C\int_{\{v_{n+1}>0\}}\big((\rho|u|_2)^{q_1-2}|v|^{q_1-1}+|v|\big)v_{n+1}\,dx \\ &\leq C\int_{\{v_{n+1}>0\}}\big(\big(2^{n+1}-1\big)^{q_1-1}|v_n|^{q_1}+\big(2^{n+1}-1\big)|v_n|^2\big)\,dx \\ &\leq C2^{(n+1)}R_n^{\frac{q_1}{2}}. \end{aligned} \tag{3.8}$$

Combining (3.7) with (3.8), we have

$$\begin{aligned} R_{n+1} &\leq C\tau^{2(\frac{2}{2_s^*}-1)}2^{(3-\frac{4}{2_s^*})(n+1)}R_n^{1+\frac{q_1}{2}-\frac{2}{2_s^*}} \\ &= C\tau^{2(\frac{2}{2_s^*}-1)}2^{(3-\frac{4}{2_s^*})}H^nR_n^{1+\beta} \\ &\leq H^n\big(C_0(\tau)R_n\big)^{1+\beta}, \end{aligned} \tag{3.9}$$

where $H=2^{3-\frac{4}{2_s^*}}$, $\beta=\frac{q_1}{2}-\frac{2}{2_s^*}$, $C_0(\tau)>1$ is big enough. Similar to [29], provided $\rho=\max\{(\frac{C_0^{1+\beta}(\tau)}{\nu})^{\frac{1}{2\beta}},\frac{1}{|u|_2}\}$ is big enough, by induction we can also prove that for all $n\in\mathbb{N}$

$$R_n\leq\frac{\nu^n}{\rho^2},\quad \text{where } \nu=\frac{1}{H^{\frac{1}{\beta}}}\in(0,1). \tag{3.10}$$

In fact, we already know that $R_0\leq\frac{1}{\rho^2}$. Assuming that (3.10) holds for some $n\in\mathbb{N}$, by (3.9) we have

$$R_{n+1}\leq H^n\big(C_0(\tau)R_n\big)^{1+\beta}\leq H^nC_0^{1+\beta}(\tau)\left(\frac{\nu^n}{\rho^2}\right)^{1+\beta}\leq\frac{C_0^{1+\beta}(\tau)}{\rho^{2\beta}}\frac{\nu^n}{\rho^2}\leq\frac{\nu^{n+1}}{\rho^2}.$$

By (3.10), we have $R_n\to0, n\to\infty$. This implies that $v_n(x)\to0$, a.e. $x\in\Omega$, as $n\to\infty$. So $v(x)\leq\tau$, a.e. $x\in\Omega$. An analogous argument can be applied to $-v$. Therefore, we have $|v|_\infty\leq\tau$, hence $u\in L^\infty(\Omega)$ and by the fractional Sobolev embedding result

$$|u|_\infty\leq\tau\rho|u|_2=\tau,\quad \text{for } |u|_2 \text{ small enough such that } \rho=\frac{1}{|u|_2}.$$

In fact, since the solutions $\{u_m\}$ that we have got above satisfy $\|u_m\|\to0$, as $m\to\infty$, there exists $M\in\mathbb{N}$ such that for any $m>M(\tau)$, $\rho=\frac{1}{|u_m|_2}$, i.e. $|u_m|_\infty\leq\tau$ for any $m>M(\tau)$. That is to say $|u_m|_\infty\to0$, as $m\to\infty$. The proof is complete. □

Therefore, from the above discussion we can see that the original problem (P) also enjoys a sequence of nontrivial solutions $\{u_m\}$ satisfying $|u_m|_\infty\to0$, as $m\to\infty$. Thus the proof of Theorem 1.1 is complete. □

**Competing interests**

The authors declare that they have no competing interests.

**Authors' contributions**

All authors contributed equally to the manuscript and typed, read, and approved the final manuscript.

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