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New approximate solutions to fractional nonlinear systems of partial differential equations using the FNDM

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Abstract

In this article, we focus our study on finding approximate analytical solutions to systems of nonlinear PDEs using the fractional natural decomposition method (FNDM). We apply the FNDM to obtain approximate numerical solutions for two different types of nonlinear time-fractional systems of partial differential equations. The theoretical analysis of the FNDM is investigated for these systems of equations and is calculated in the explicit form of a power series with easily computable terms. The analysis shows that our analytical solutions converge very rapidly to the exact solutions and the effectiveness of the FNDM is numerically confirmed.

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Keywords: fractional natural decomposition method; system of fractional differential equations; Caputo fractional derivative

1 Introduction

During the last three decades, several numerical methods have been developed in the field of fractional calculus [1–14]. The use of fractional differentiation for modeling physical problems has been wide spread in recent years [15, 16]. Fractional calculus has applications in physics, fluid flow, chemical physics, control theory of dynamical systems, electrical networks, modeling of earth quake, measurement of viscoelastic material properties, etc. The natural decomposition method (NDM) was first introduced by Rawashdeh and Maitama in 2014 [17–19], to solve linear and nonlinear ODEs and PDEs that appear in many mathematical physics and engineering applications. The NDM is based on the natural transform method (NTM) [20–23] and the Adomian decomposition method (ADM) [24, 25] and it provides solutions in an infinite series form, and the obtained series may converge to a closed form solution if the exact solution exists. For concrete problems where the exact solution does not exist, the truncated series may be used for numerical purposes. For nonlinear models, the NDM has shown dependable results and gives analytical approximation that converges very rapidly. Recently, Rawashdeh and Al-Jammal [26] used the FNDM to find analytical solutions for nonlinear fractional ODES. Many numerical methods were used in the past to solve fractional systems of nonlinear partial differential equations, such as the fractional Sumudu transform [27, 28], the fractional matrix method [3], the fractional Adomian decomposition method (FADM) [4, 10], the fractional reduced differential

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transform method (FRDTM) [8, 9], the fractional Laplace decomposition method (FLDM) [29], the fractional homotopy analysis method (FHAM) [30, 31], and the fractional homotopy perturbation method (FHPM) [29, 32].

In this paper, we find approximate solutions to the following fractional systems of FNLPDEs.

First, the nonlinear time-fractional coupled Burgers' system of equations:

$$\begin{aligned} &D_t^{\alpha} w = w_{xx} + 2ww_x - (wv)_x \quad (0 < \alpha \leq 1), \\ &D_t^{\beta} v = v_{xx} + 2vv_x - (wv)_x \quad (0 < \beta \leq 1), \end{aligned} \tag{1.1}$$

subject to the initial conditions

$$w(x,0) = \sin(x); \qquad v(x,0) = \sin(x). \tag{1.2}$$

Second, the time-fractional nonlinear system in dimension three:

$$\begin{aligned} &D_t^{\alpha} h + v_x w_y - v_y w_x = -h \quad (0 < \alpha \leq 1), \\ &D_t^{\beta} v + h_y w_x + h_x w_y = v \quad (0 < \beta \leq 1), \\ &D_t^{\gamma} w + h_x v_y + h_y v_x = w \quad (0 < \gamma \leq 1), \end{aligned} \tag{1.3}$$

subject to the initial conditions

$$h(x,y,0) = e^{x+y}; \qquad v(x,y,0) = e^{x-y}; \qquad w(x,y,0) = e^{y-x}. \tag{1.4}$$

The goal of our study is to use the FNDM [33] to find approximate solutions to two different types of systems of nonlinear partial differential equations for $0 < \alpha, \beta, \gamma < 1$, and exact solutions in the case when $\alpha = \beta = \gamma = 1$.

The rest of this paper is organized as follows: In Section 2, we give some preliminaries and definitions of fractional calculus. In Sections 3 and 4, the natural transform method is introduced. Section 4 is devoted to the application of the method to two applications and present graphs to show the effectiveness of the FNDM for some values of x and t. In Section 5, we present tables for different values of α, β, γ, and t. Section 6 is for a discussion and our conclusion of this paper.

2 Preliminaries of fractional calculus

In this section, we give some of the main definitions and notations related to fractional calculus. These basic definitions are due to Liouville [1, 2, 5, 7].

Definition 2.1 A real function $f(x), x > 0$ is said to be in the space $C_\mu, \mu \in \mathbb{R}$, if there exists a real number q ($> \mu$), such that $f(x) = x^q g(x)$, where $g(x) \in C[0,\infty)$, and it is said to be in the space C_μ^m if $f^{(m)} \in C_\mu, m \in \mathbb{N}$.

Definition 2.2 For an integrable function $f \in C_\mu$, the Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, is defined as

$$\begin{cases} J^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\,dt, & \text{when } \alpha > 0, x > 0, \\ J^0 f(x) = f(x). \end{cases}$$

Caputo and Mainardi [2] presented a modified fractional differentiation operator D^α in their work on the theory of viscoelasticity to overcome the disadvantages of the Riemann-Liouville derivative when one tries to model real world problems.

Definition 2.3 The fractional derivative of $f \in C_{-1}^m$ in the Caputo sense can be defined as

$$\begin{aligned} D^\alpha f(x) &= J^{m-\alpha} D^m f(x) \\ &= \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} f^{(m)}(t)\,dt, \quad m-1 < \alpha \leq m, m \in \mathbb{N}, x > 0. \end{aligned}$$

Definition 2.4 [34] A one-parameter function of the Mittag-Leffler type is defined by the series expansion:

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad \alpha > 0, z \in \mathbb{C}. \tag{2.1}$$

Lemma 2.1 [6] *If* $m-1 < \alpha \leq m, m \in \mathbb{N}$, *and* $f \in C_\mu^m, \mu \geq -1$, *then*

$$\begin{cases} D^\alpha J^\alpha f(x) = f(x), & \text{if } x > 0, \\ J^\alpha D^\alpha f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^+) \frac{x^k}{k!}, & \text{if } m-1 < \alpha < m. \end{cases}$$

It should be mentioned here that the Caputo fractional derivative is used because it allows traditional initial and boundary conditions to be included in the formulation of the problem.

Remark 2.1 Note that Γ represents the Gamma function, which is defined by

$$\Gamma(z) := \int_0^\infty e^{-t} t^{z-1}\,dt, \quad z \in \mathbb{C}. \tag{2.2}$$

Notice that the Gamma function is the continuous extension to the fractional function. Throughout this paper, we will be using the recursive relation $\Gamma(z+1) = z\Gamma(z), z > 0$, to calculate the values of the Gamma function of all real numbers by finding only the values of the Gamma function between 1 and 2.

3 Definitions and properties of the *N* transform

In this section, we present some background about the nature of the natural transform method (NTM). Given a function $f(t)$, $t \in \mathbb{R}$, then the general integral transform is defined by [20–23]:

$$\Im\big[f(t)\big](s) = \int_{-\infty}^{\infty} K(s,t) f(t)\,dt, \tag{3.1}$$

where $K(s,t)$ represent the kernel of the transform, s is a real (complex) number which is independent of t. Note that when $K(s,t)$ is $e^{-st}, tJ_n(st)$, and $t^{s-1}(st)$, then equation (3.1) gives, respectively, the Laplace transform, the Hankel transform, and the Mellin transform. Now, for $f(t), t \in (-\infty, \infty)$, consider the integral transforms defined by

$$\Im\big[f(t)\big](u) = \int_{-\infty}^{\infty} K(t) f(ut)\,dt \tag{3.2}$$

and

$$\Im\big[f(t)\big](s,u) = \int_{-\infty}^{\infty} K(s,t)f(ut)\,dt. \tag{3.3}$$

It is worth mentioning that when $K(t) = e^{-t}$, equation (3.2) gives the integral Sumudu transform, where the parameter s is replaced by u. Moreover, for any value of n the generalized Laplace and Sumudu transform are, respectively, defined by [20–23]

$$\ell\big[f(t)\big] = F(s) = s^n \int_0^{\infty} e^{-s^{n+1}t} f\big(s^n t\big)\,dt \tag{3.4}$$

and

$$\mathbb{S}\big[f(t)\big] = G(u) = u^n \int_0^{\infty} e^{-u^n t} f\big(tu^{n+1}\big)\,dt. \tag{3.5}$$

Note that when $n = 0$, equation (3.4) and equation (3.5) are the Laplace and Sumudu transform, respectively. The natural transform of the function $f(t)$ for $t \in \mathbb{R}$ is defined by [21–23]

$$\mathbb{N}\big[f(t)\big] = R(s,u) = \int_{-\infty}^{\infty} e^{-st} f(ut)\,dt; \quad s,u \in (-\infty,\infty), \tag{3.6}$$

where $\mathbb{N}[f(t)]$ is the natural transformation of the time function $f(t)$ and the variables s and u are the natural transform variables. Moreover, if the function $f(t)H(t)$ is defined on the positive real axis, where $H(\cdot)$ is the Heaviside function, $t \in (0,\infty)$, and we suppose that

$$A = \big\{f(t) : \exists M, \tau_1, \tau_2 > 0, \text{with } \big|f(t)\big| < Me^{\frac{|t|}{\tau_j}}, \text{for } t \in (-1)^j \times [0,\infty), j \in \mathbb{Z}^+\big\}.$$

Then we define the natural transform (N transform) as

$$\mathbb{N}\big[f(t)H(t)\big] = \mathbb{N}^+\big[f(t)\big] = R^+(s,u) = \int_0^{\infty} e^{-st} f(ut)\,dt; \quad s,u \in (0,\infty). \tag{3.7}$$

Note that if $u = 1$ equation (3.7) can be reduced to the Laplace transform and if $s = 1$ equation (3.7) can be reduced to the Sumudu transform.

Important properties Some basic properties of the N transforms are given as follows [20–23]:

1. $\mathbb{N}^+[1] = \frac{1}{s}$.
2. $\mathbb{N}^+[t^\alpha] = \frac{\Gamma(\alpha+1)u^\alpha}{s^{\alpha+1}}$, where $\alpha > -1$.

4 Analysis of the fractional natural decomposition method

In this section, we present the methodology of the FNDM and present some theorems of the FNDM. One of the authors (Rawashdeh) of this paper proved these theorems in

[33]. Also, in [35] the authors used a different approach to prove Theorem 4.1 and Theorem 4.3.

Theorem 4.1 *If $R(s,u)$ is the natural transform of $f(t)$, then the natural transform of the Riemann-Liouville fractional integral for $f(t)$ of order α, denoted by $J^\alpha f(t)$, is given by*

$$\mathbb{N}^+\left[J^\alpha f(t)\right] = \frac{u^\alpha}{s^\alpha} R(s,u). \tag{4.1}$$

Theorem 4.2 *If n is any positive integer, where $n-1 \leq \alpha < n$ and $R(s,u)$ is the natural transform of the function $f(t)$, then the natural transform, $R_\alpha(s,u)$, of the Riemann-Liouville fractional derivative of the function $f(t)$ of order α, denoted by $D^\alpha f(t)$, is given by*

$$\mathbb{N}^+\left[D^\alpha f(t)\right] = R_\alpha(s,u) = \frac{s^\alpha}{u^\alpha} R(s,u) - \sum_{k=0}^{n-1} \frac{s^k}{u^{k+1}} \left(D^{\alpha-k-1} f(t)\right)_{t=0}. \tag{4.2}$$

Theorem 4.3 *If n is any positive integer, where $n-1 \leq \alpha < n$ and $R(s,u)$ is the natural transform of the function $f(t)$, then the natural transform, $R_\alpha^c(s,u)$ of the Caputo fractional derivative of the function $f(t)$ of order α, denoted by ${}^cD^\alpha f(t)$, is given by*

$$\mathbb{N}^+\left[{}^cD^\alpha f(t)\right] = R_\alpha^c(s,u) = \frac{s^\alpha}{u^\alpha} R(s,u) - \sum_{k=0}^{n-1} \frac{s^{\alpha-(k+1)}}{u^{\alpha-k}} \left[D^k f(t)\right]_{t=0}. \tag{4.3}$$

Methodology of the FNDM We illustrate the FNDM algorithm by considering the general fractional nonlinear system PDEs with the initial conditions of the form

$$\begin{aligned} &D_t^\alpha v(x,t) + Rw(x,t) + Fv(x,t) = h_1(x,t), \\ &D_t^\beta w(x,t) + Rv(x,t) + Fw(x,t) = h_2(x,t), \end{aligned} \tag{4.4}$$

subject to the initial conditions

$$\begin{aligned} &v(x,0) = g_1(x), \\ &w(x,0) = g_2(x), \end{aligned} \tag{4.5}$$

where $D_t^\alpha v(x,t)$, $D_t^\beta w(x,t)$ are the Caputo fractional derivatives of the functions $v(x,t)$, $w(x,t)$, respectively, R is the linear differential operator, F represents the general nonlinear differential operator, and $h_1(x,t)$, $h_2(x,t)$ are the source terms.

Apply the N transform and Theorem 4.3 to equation (4.4) to get

$$\begin{aligned} V(x,s,u) = {} & \frac{u^\alpha}{s^\alpha} \sum_{k=0}^{n-1} \frac{s^{\alpha-(k+1)}}{u^{\alpha-k}} \left[D^k v(x,t)\right]_{t=0} \\ & + \frac{u^\alpha}{s^\alpha} \mathbb{N}^+\left[h_1(x,t)\right] - \frac{u^\alpha}{s^\alpha} \mathbb{N}^+\left[Rw(x,t) + Fv(x,t)\right], \end{aligned} \tag{4.6}$$

$$\begin{aligned} W(x,s,u) = {} & \frac{u^\beta}{s^\beta} \sum_{k=0}^{n-1} \frac{s^{\beta-(k+1)}}{u^{\beta-k}} \left[D^k w(x,t)\right]_{t=0} \\ & + \frac{u^\beta}{s^\beta} \mathbb{N}^+\left[h_2(x,t)\right] - \frac{u^\beta}{s^\beta} \mathbb{N}^+\left[Rv(x,t) + Fw(x,t)\right]. \end{aligned} \tag{4.7}$$

Apply the inverse natural transform of equation (4.6) and equation (4.7) to obtain

$$\begin{aligned} v(x,t) &= G(x,t) - \mathbb{N}^{-1}\left[\frac{u^\alpha}{s^\alpha}\mathbb{N}^+\bigl[Rw(x,t) + Fv(x,t)\bigr]\right], \\ w(x,t) &= H(x,t) - \mathbb{N}^{-1}\left[\frac{u^\beta}{s^\beta}\mathbb{N}^+\bigl[Rv(x,t) + Fw(x,t)\bigr]\right]. \end{aligned} \tag{4.8}$$

Note that $G(x,t)$ and $H(x,t)$ are arising from the nonhomogeneous term and the prescribed initial conditions. Now we assume an infinite series solutions form:

$$\begin{aligned} v(x,t) &= \sum_{n=0}^{\infty} v_n(x,t), \qquad Fv(x,t) = \sum_{n=0}^{\infty} A_n, \\ w(x,t) &= \sum_{n=0}^{\infty} w_n(x,t), \qquad Fw(x,t) = \sum_{n=0}^{\infty} B_n. \end{aligned} \tag{4.9}$$

Using equation (4.9) we can rewrite equation (4.8)

$$\begin{aligned} \sum_{n=0}^{\infty} v_n(x,t) &= G(x,t) - \mathbb{N}^{-1}\left[\frac{u^\alpha}{s^\alpha}\mathbb{N}^+\left[R\sum_{n=0}^{\infty} w_n(x,t)\right] + \sum_{n=0}^{\infty} A_n\right], \\ \sum_{n=0}^{\infty} w_n(x,t) &= H(x,t) - \mathbb{N}^{-1}\left[\frac{u^\beta}{s^\beta}\mathbb{N}^+\left[R\sum_{n=0}^{\infty} v_n(x,t)\right] + \sum_{n=0}^{\infty} B_n\right], \end{aligned} \tag{4.10}$$

where the A_n, B_n are the polynomials representing the nonlinear term $Fv(x,t)$, $Fw(x,t)$, respectively. By comparing both sides of equation (4.10) we conclude

$$\begin{aligned} v_0(x,t) &= G(x,t), \\ v_1(x,t) &= -\mathbb{N}^{-1}\left[\frac{u^\alpha}{s^\alpha}\mathbb{N}^+\bigl[Rw_0(x,t)\bigr] + A_0\right], \\ v_2(x,t) &= -\mathbb{N}^{-1}\left[\frac{u^\alpha}{s^\alpha}\mathbb{N}^+\bigl[Rw_1(x,t)\bigr] + A_1\right], \\ w_0(x,t) &= H(x,t), \\ w_1(x,t) &= -\mathbb{N}^{-1}\left[\frac{u^\beta}{s^\beta}\mathbb{N}^+\bigl[Rv_0(x,t)\bigr] + B_0\right], \\ w_2(x,t) &= -\mathbb{N}^{-1}\left[\frac{u^\beta}{s^\beta}\mathbb{N}^+\bigl[Rv_1(x,t)\bigr] + B_1\right]. \end{aligned}$$

We continue in this manner to get the general recursive relation given by

$$\begin{aligned} v_{n+1}(x,t) &= -\mathbb{N}^{-1}\left[\frac{u^\alpha}{s^\alpha}\mathbb{N}^+\bigl[Rw_n(x,t)\bigr] + A_n\right], \quad n \geq 1, \\ w_{n+1}(x,t) &= -\mathbb{N}^{-1}\left[\frac{u^\beta}{s^\beta}\mathbb{N}^+\bigl[Rv_n(x,t)\bigr] + B_n\right], \quad n \geq 1. \end{aligned} \tag{4.11}$$

Finally, the approximate solutions are given by

$$v(x,t)=\sum_{n=0}^{\infty}v_n(x,t),\qquad w(x,t)=\sum_{n=0}^{\infty}w_n(x,t).$$

5 Illustrative examples

In this section, we test the FNDM on two applications and then compare our approximate solutions with the exact solutions.

Example 5.1 Consider the time-fractional nonlinear coupled Burgers' system of equations:

$$\begin{aligned} D_t^{\alpha}w &= w_{xx}+2ww_x-(wv)_x \quad (0<\alpha\leq 1),\\ D_t^{\beta}v &= v_{xx}+2vv_x-(wv)_x \quad (0<\beta\leq 1), \end{aligned} \tag{5.1}$$

subject to the initial conditions

$$w(x,0)=\sin(x);\qquad v(x,0)=\sin(x). \tag{5.2}$$

Apply the N transform and Theorem 4.3 to equation (5.1) to get

$$\begin{aligned} \mathbb{N}^{+}\bigl[D_t^{\alpha}w(x,t)\bigr] &= \mathbb{N}^{+}\left[\frac{\partial^2 w}{\partial x^2}\right]+\mathbb{N}^{+}\left[2w\frac{\partial w}{\partial x}\right]-\mathbb{N}^{+}\bigl[(wv)_x\bigr],\\ \mathbb{N}^{+}\bigl[D_t^{\beta}v(x,t)\bigr] &= \mathbb{N}^{+}\left[\frac{\partial^2 v}{\partial x^2}\right]+\mathbb{N}^{+}\left[2v\frac{\partial v}{\partial x}\right]-\mathbb{N}^{+}\bigl[(wv)_x\bigr]. \end{aligned} \tag{5.3}$$

So equation (5.3) becomes

$$\begin{aligned} &\frac{s^{\alpha}}{u^{\alpha}}\mathbb{N}^{+}\bigl[w(x,t)\bigr]-\sum_{k=0}^{n-1}\frac{s^{\alpha-(k+1)}}{u^{\alpha-k}}\bigl[D^k w\bigr]_{t=0}=\mathbb{N}^{+}\bigl[w_{xx}+2ww_x-(wv)_x\bigr],\\ &\frac{s^{\beta}}{u^{\beta}}\mathbb{N}^{+}\bigl[v(x,t)\bigr]-\sum_{k=0}^{n-1}\frac{s^{\beta-(k+1)}}{u^{\beta-k}}\bigl[D^k v\bigr]_{t=0}=\mathbb{N}^{+}\bigl[v_{xx}+2vv_x-(wv)_x\bigr]. \end{aligned} \tag{5.4}$$

Thus from equation (5.2) and equation (5.4) we conclude

$$\begin{aligned} \mathbb{N}^{+}\bigl[w(x,t)\bigr] &= \frac{1}{s}\sin(x)+\frac{u^{\alpha}}{s^{\alpha}}\mathbb{N}^{+}\bigl[w_{xx}+2ww_x-(wv)_x\bigr],\\ \mathbb{N}^{+}\bigl[v(x,t)\bigr] &= \frac{1}{s}\sin(x)+\frac{u^{\beta}}{s^{\beta}}\mathbb{N}^{+}\bigl[v_{xx}+2vv_x-(wv)_x\bigr]. \end{aligned} \tag{5.5}$$

Apply the inverse N transform of equation (5.5) to obtain

$$\begin{aligned} w(x,t) &= \sin(x)+\mathbb{N}^{-1}\left[\frac{u^{\alpha}}{s^{\alpha}}\mathbb{N}^{+}\bigl[w_{xx}+2ww_x-(wv)_x\bigr]\right],\\ v(x,t) &= \sin(x)+\mathbb{N}^{-1}\left[\frac{u^{\beta}}{s^{\beta}}\mathbb{N}^{+}\bigl[v_{xx}+2vv_x-(wv)_x\bigr]\right]. \end{aligned} \tag{5.6}$$

Assume infinite series solutions for the unknown functions $v(x,t)$, $w(x,t)$ as follows:

$$w(x,t)=\sum_{n=0}^{\infty}w_n(x,t); \qquad v(x,t)=\sum_{n=0}^{\infty}v_n(x,t). \tag{5.7}$$

Note that $ww_x=\sum_{n=0}^{\infty}A_n$, $vv_x=\sum_{n=0}^{\infty}C_n$, and $(wv)_x=\sum_{n=0}^{\infty}B_n$ are the Adomian polynomials that represent the nonlinear terms. Using equation (5.7), we can rewrite equation (5.6) in the form

$$\sum_{n=0}^{\infty}w_n(x,t)=\sin(x)+\mathbb{N}^{-1}\left[\frac{u^\alpha}{s^\alpha}\mathbb{N}^+\left[\sum_{n=0}^{\infty}w_{nxx}+2\sum_{n=0}^{\infty}A_n-\sum_{n=0}^{\infty}B_n\right]\right], \quad n\geq 0, \tag{5.8}$$

$$\sum_{n=0}^{\infty}v_n(x,t)=\sin(x)+\mathbb{N}^{-1}\left[\frac{u^\beta}{s^\beta}\mathbb{N}^+\left[\sum_{n=0}^{\infty}v_{nxx}+2\sum_{n=0}^{\infty}C_n-\sum_{n=0}^{\infty}B_n\right]\right], \tag{5.9}$$

where $n\geq 0$. Then by comparing both sides of equation (5.8) and equation (5.9) above, we can easily generate the recursive relation

$$\begin{aligned} &w_0(x,t)=\sin(x), \qquad v_0(x,t)=\sin(x),\\ &w_1(x,t)=\mathbb{N}^{-1}\left[\frac{u^\alpha}{s^\alpha}\mathbb{N}^+[w_{0xx}+2A_0-B_0]\right],\\ &v_1(x,t)=\mathbb{N}^{-1}\left[\frac{u^\beta}{s^\beta}\mathbb{N}^+[v_{0xx}+2C_0-B_0]\right]. \end{aligned} \tag{5.10}$$

Thus,

$$\begin{aligned} w_1(x,t)&=\mathbb{N}^{-1}\left[\frac{u^\alpha}{s^\alpha}\mathbb{N}^+[w_{0xx}+2w_0w_{0x}-w_0v_{0x}-v_0w_{0x}]\right]\\ &=\mathbb{N}^{-1}\left[\frac{u^\alpha}{s^\alpha}\mathbb{N}^+\bigl[-\sin(x)\bigr]\right]\\ &=-\sin(x)\mathbb{N}^{-1}\left[\frac{u^\alpha}{s^\alpha}\frac{1}{s}\right]=-\sin(x)\frac{t^\alpha}{\Gamma(\alpha+1)} \end{aligned}$$

and

$$\begin{aligned} v_1(x,t)&=\mathbb{N}^{-1}\left[\frac{u^\beta}{s^\beta}\mathbb{N}^+[v_{0xx}+2v_0v_{0x}-w_0v_{0x}-v_0w_{0x}]\right]\\ &=\mathbb{N}^{-1}\left[\frac{u^\beta}{s^\beta}\mathbb{N}^+\bigl[-\sin(x)\bigr]\right]\\ &=-\sin(x)\mathbb{N}^{-1}\left[\frac{u^\beta}{s^\beta}\frac{1}{s}\right]=-\sin(x)\frac{t^\beta}{\Gamma(\beta+1)}. \end{aligned}$$

We continue to get

$$\begin{aligned} w_2(x,t)&=\mathbb{N}^{-1}\left[\frac{u^\alpha}{s^\alpha}\mathbb{N}^+\bigl[w_{1xx}+2w_0w_1+2w_0w_{1x}-(v_1w_0+w_1v_0)_x\bigr]\right]\\ &=\mathbb{N}^{-1}\left[\frac{u^\alpha}{s^\alpha}\mathbb{N}^+\left[\sin(x)\left(\frac{2\cos(x)t^\beta}{\Gamma(\beta+1)}+\frac{(1-2\cos(x))t^\alpha}{\Gamma(\alpha+1)}\right)\right]\right] \end{aligned}$$

$$
\begin{aligned}
&= \sin(x)\mathbb{N}^{-1}\left[\frac{u^\alpha}{s^\alpha}\left(\frac{2\cos(x)u^\beta}{s^{\beta+1}} + \frac{(1-2\cos(x))u^\alpha}{s^{\alpha+1}}\right)\right] \\
&= \sin(x)\left[\left(\frac{2\cos(x)t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{(1-2\cos(x))t^{2\alpha}}{\Gamma(2\alpha+1)}\right)\right].
\end{aligned}
$$

Similarly,

$$
v_2(x,t) = \sin(x)\left[\left(\frac{2\cos(x)t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{(1-2\cos(x))t^{2\beta}}{\Gamma(2\beta+1)}\right)\right].
$$

We continue in this manner to obtain the following approximate solutions:

$$
\begin{aligned}
w(x,t) &= \sum_{n=0}^{\infty} w_n(x,t) \\
&= w_0(x,t) + w_1(x,t) + w_2(x,t) + \cdots \\
&= \sin(x) - \sin(x)\frac{t^\alpha}{\Gamma(\alpha+1)} + \sin(x)\left[\left(\frac{2\cos(x)t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{(1-2\cos(x))t^{2\alpha}}{\Gamma(2\alpha+1)}\right)\right] + \cdots, \\
v(x,t) &= \sum_{n=0}^{\infty} v_n(x,t) \\
&= v_0(x,t) + v_1(x,t) + v_2(x,t) + \cdots \\
&= \sin(x) - \sin(x)\frac{t^\beta}{\Gamma(\beta+1)} + \sin(x)\left(\frac{2\cos(x)t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{(1-2\cos(x))t^{2\beta}}{\Gamma(2\beta+1)}\right) + \cdots.
\end{aligned}
$$

Choosing $\alpha = 1$, $\beta = 1$, and using a Taylor series expansion, the above approximate solution becomes

$$
\begin{aligned}
w(x,t) &= \sum_{n=0}^{\infty} w_n(x,t) \\
&= \sin(x) - \sin(x)t + \sin(x)\left(\frac{2\cos(x)t^2}{2} + \frac{(1-2\cos(x))t^2}{2}\right) + \cdots \\
&= \sin(x)e^{-t}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
v(x,t) &= \sum_{n=0}^{\infty} v_n(x,t) \\
&= \sin(x) - \sin(x)t + \sin(x)\left(\frac{2\cos(x)t^2}{2} + \frac{(1-2\cos(x))t^2}{2}\right) + \cdots \\
&= \sin(x)e^{-t}.
\end{aligned}
$$

These are in fact the exact solutions of equation (5.1) in the case when $\alpha = 1$, $\beta = 1$. Hence, the approximate solution is rapidly convergent to the exact solution. The numerical results of the approximate solution obtained by FNDM and the exact solution are shown in Figures 1-6 for different values of x, t, α, and β.

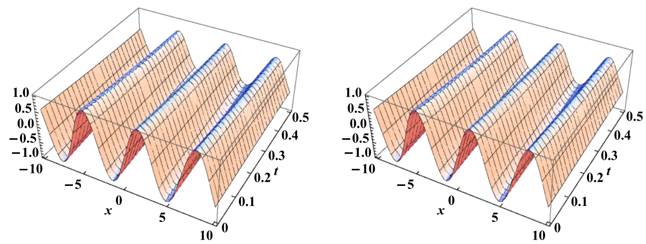

Figure 1 The approximate and exact solutions of $w(x,t)$ when $\alpha = \beta = 1$.

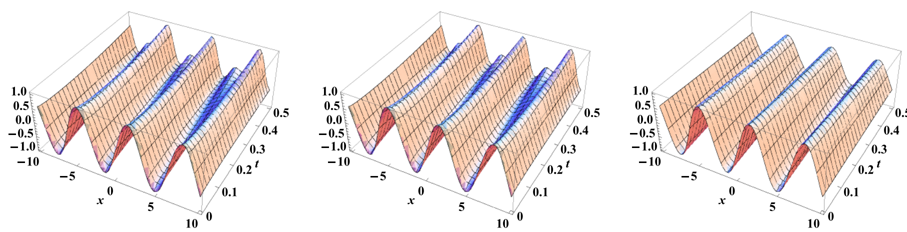

Figure 2 The approximate solutions of $w(x,t)$ when $\alpha = \beta = 0.5$, $\alpha = \beta = 0.75$, and $\alpha = \beta = 0.90$.

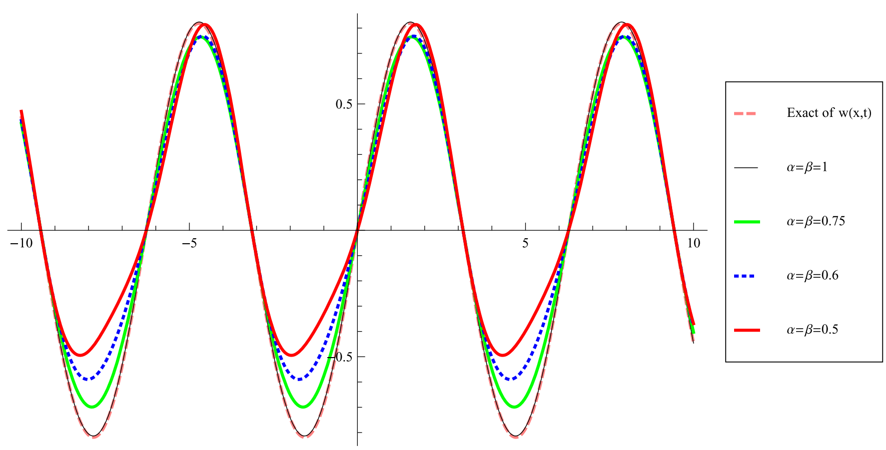

Figure 3 The approximate and exact solutions of $w(x,t)$ for different values of α and β.

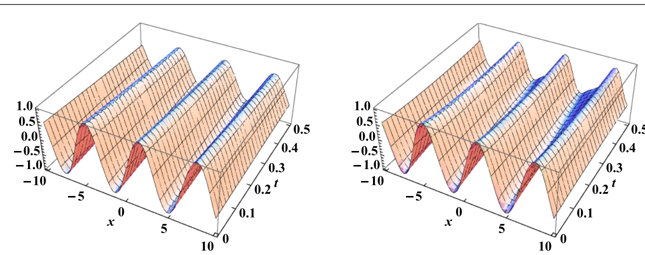

Figure 4 The approximate and exact solutions of $v(x,t)$ when $\alpha = \beta = 1$.

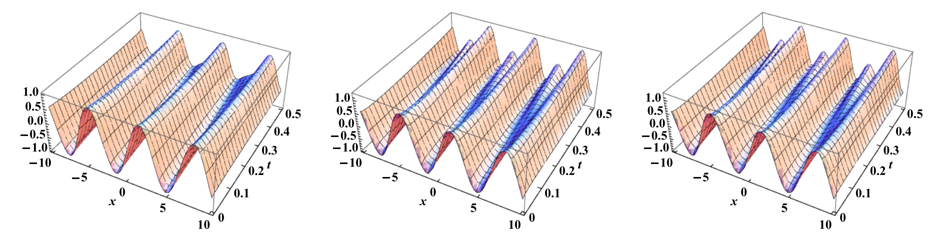

Figure 5 **The approximate solutions of $v(x,t)$ when $\alpha = \beta = 0.5$, $\alpha = \beta = 0.75$, and $\alpha = \beta = 0.90$.**

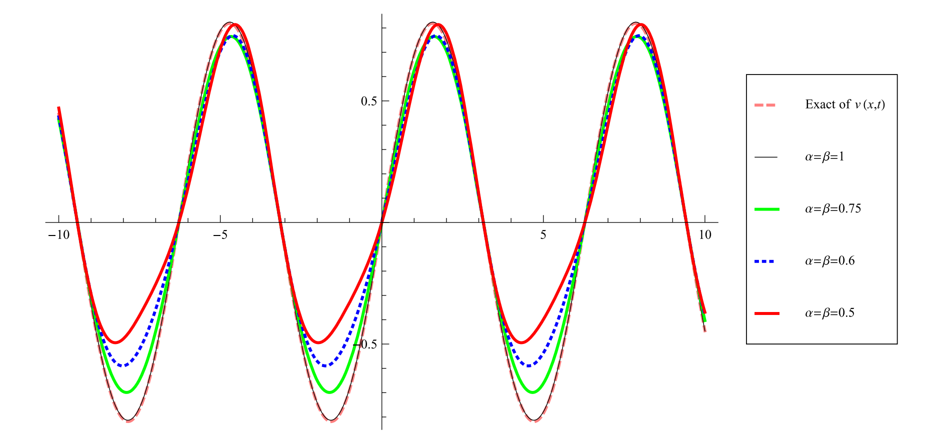

Figure 6 **The approximate and exact solutions of $v(x,t)$ for different values of α and β.**

Example 5.2 Consider the time-fractional nonlinear system of equations in three dimensions:

$$\begin{aligned} &D_t^\alpha h + v_x w_y - v_y w_x = -h \quad (0 < \alpha \le 1), \\ &D_t^\beta v + h_y w_x + h_x w_y = v \quad (0 < \beta \le 1), \\ &D_t^\gamma w + h_x v_y + h_y v_x = w \quad (0 < \gamma \le 1), \end{aligned} \tag{5.11}$$

subject to the initial conditions

$$h(x,y,0) = e^{x+y}; \qquad v(x,y,0) = e^{x-y}; \qquad w(x,y,0) = e^{y-x}. \tag{5.12}$$

Apply the N transform and Theorem 4.3 to equation (5.11) to get

$$\begin{aligned} &\mathbb{N}^+\big[D_t^\alpha h\big] = \mathbb{N}^+[-h + v_y w_x - v_x w_y], \\ &\mathbb{N}^+\big[D_t^\beta v\big] = \mathbb{N}^+[v - h_y w_x - h_x w_y], \\ &\mathbb{N}^+\big[D_t^\gamma w\big] = \mathbb{N}^+[w - h_x v_y - h_y v_x]. \end{aligned} \tag{5.13}$$

So equation (5.13) becomes

$$\frac{s^\alpha}{u^\alpha}\mathbb{N}^+[h] - \sum_{k=0}^{n-1} \frac{s^{\alpha-(k+1)}}{u^{\alpha-k}}\big[D^k h\big]_{t=0} = \mathbb{N}^+[-h + v_y w_x - v_x w_y], \tag{5.14}$$

$$\frac{s^\beta}{u^\beta}\mathbb{N}^+[v] - \sum_{k=0}^{n-1}\frac{s^{\beta-(k+1)}}{u^{\beta-k}}\left[D^k v\right]_{t=0} = \mathbb{N}^+[v - h_y w_x - h_x w_y],$$
$$\frac{s^\gamma}{u^\gamma}\mathbb{N}^+[w] - \sum_{k=0}^{n-1}\frac{s^{\gamma-(k+1)}}{u^{\gamma-k}}\left[D^k w\right]_{t=0} = \mathbb{N}^+[w - h_x v_y - h_y v_x].$$

Thus from equation (5.12) and equation (5.14) we conclude

$$\begin{aligned}
\mathbb{N}^+[h] &= \frac{e^{x+y}}{s} + \frac{u^\alpha}{s^\alpha}\mathbb{N}^+[-h + v_y w_x - v_x w_y],\\
\mathbb{N}^+[v] &= \frac{e^{x-y}}{s} + \frac{u^\beta}{s^\beta}\mathbb{N}^+[v - h_y w_x - h_x w_y],\\
\mathbb{N}^+[w] &= \frac{e^{y-x}}{s} + \frac{u^\gamma}{s^\gamma}\mathbb{N}^+[w - h_x v_y - h_y v_x].
\end{aligned} \tag{5.15}$$

Apply the inverse N transform of equation (5.15) to obtain

$$\begin{aligned}
h(x,y,t) &= e^{x+y} + \mathbb{N}^{-1}\left[\frac{u^\alpha}{s^\alpha}\mathbb{N}^+[-h + v_y w_x - v_x w_y]\right],\\
v(x,y,t) &= e^{x-y} + \mathbb{N}^{-1}\left[\frac{u^\beta}{s^\beta}\mathbb{N}^+[v - h_y w_x - h_x w_y]\right],\\
w(x,y,t) &= e^{y-x} + \mathbb{N}^{-1}\left[\frac{u^\gamma}{s^\gamma}\mathbb{N}^+[w - h_x v_y - h_y v_x]\right].
\end{aligned} \tag{5.16}$$

So assume infinite series solutions for the unknown functions $h(x,y,t)$, $v(x,y,t)$, and $w(x,y,t)$ of the form

$$\begin{aligned}
h(x,y,t) &= \sum_{n=0}^{\infty} h_n(x,y,t),\\
v(x,y,t) &= \sum_{n=0}^{\infty} v_n(x,y,t),\\
w(x,y,t) &= \sum_{n=0}^{\infty} w_n(x,y,t).
\end{aligned} \tag{5.17}$$

Note that

$$\begin{aligned}
v_y w_x &= \sum_{n=0}^{\infty} A_n, \qquad v_x w_y = \sum_{n=0}^{\infty} \bar{A}_n,\\
h_y w_x &= \sum_{n=0}^{\infty} B_n, \qquad h_x w_y = \sum_{n=0}^{\infty} \bar{B}_n,\\
v_y h_x &= \sum_{n=0}^{\infty} C_n, \qquad v_x h_y = \sum_{n=0}^{\infty} \bar{C}_n
\end{aligned} \tag{5.18}$$

represent the Adomian polynomials of the nonlinear terms. Using equation (5.17), we can rewrite equation (5.16) in the form

$$\begin{aligned}
\sum_{n=0}^{\infty} h_n(x,y,t) &= e^{x+y} + \mathbb{N}^{-1}\left[\frac{u^{\alpha}}{s^{\alpha}}\mathbb{N}^{+}\left[\sum_{n=0}^{\infty} h_n + \sum_{n=0}^{\infty} A_n - \sum_{n=0}^{\infty} \bar{A}_n\right]\right], \\
\sum_{n=0}^{\infty} v_n(x,y,t) &= e^{x-y} + \mathbb{N}^{-1}\left[\frac{u^{\beta}}{s^{\beta}}\mathbb{N}^{+}\left[\sum_{n=0}^{\infty} v_n + \sum_{n=0}^{\infty} B_n - \sum_{n=0}^{\infty} \bar{B}_n\right]\right], \\
\sum_{n=0}^{\infty} w_n(x,y,t) &= e^{y-x} + \mathbb{N}^{-1}\left[\frac{u^{\gamma}}{s^{\gamma}}\mathbb{N}^{+}\left[\sum_{n=0}^{\infty} w_n + \sum_{n=0}^{\infty} C_n - \sum_{n=0}^{\infty} \bar{C}_n\right]\right],
\end{aligned} \tag{5.19}$$

where $n \geq 0$. Then by comparing both sides of equation (5.19) above, we can easily generate the recursive relations

$$h_0(x,y,t) = e^{x+y}, \qquad v_0(x,y,t) = e^{x-y}, \qquad w_0(x,y,t) = e^{y-x}.$$

Thus,

$$\begin{aligned}
h_1(x,y,t) &= \mathbb{N}^{-1}\left[\frac{u^{\alpha}}{s^{\alpha}}\mathbb{N}^{+}[-h_0 + v_{0y}w_{0x} - v_{0x}w_{0y}]\right] \\
&= \mathbb{N}^{-1}\left[\frac{u^{\alpha}}{s^{\alpha}}\mathbb{N}^{+}\left[-e^{x+y} + e^{x-y}e^{-x+y} - e^{x-y}e^{-x+y}\right]\right] \\
&= -e^{x+y}\frac{t^{\alpha}}{\Gamma(\alpha+1)}, \\
v_1(x,y,t) &= \mathbb{N}^{-1}\left[\frac{u^{\beta}}{s^{\beta}}\mathbb{N}^{+}[v_0 + h_{0y}w_{0x} - h_{0x}w_{0y}]\right] \\
&= \mathbb{N}^{-1}\left[\frac{u^{\beta}}{s^{\beta}}\mathbb{N}^{+}\left[e^{x-y} + e^{x+y}e^{-x+y} - e^{x+y}e^{-x+y}\right]\right] \\
&= e^{x-y}\frac{t^{\beta}}{\Gamma(\beta+1)}, \\
w_1(x,y,t) &= \mathbb{N}^{-1}\left[\frac{u^{\gamma}}{s^{\gamma}}\mathbb{N}^{+}[w_0 - h_{0x}v_{0y} - h_{0y}v_{0x}]\right] \\
&= \mathbb{N}^{-1}\left[\frac{u^{\gamma}}{s^{\gamma}}\mathbb{N}^{+}\left[e^{-x+y} + e^{x+y}e^{x-y} - e^{x+y}e^{x-y}\right]\right] \\
&= e^{y-x}\frac{t^{\gamma}}{\Gamma(\gamma+1)}.
\end{aligned}$$

We continue in this manner to obtain

$$\begin{aligned}
h_2(x,y,t) &= \mathbb{N}^{-1}\left[\frac{u^{\alpha}}{s^{\alpha}}\mathbb{N}^{+}[-h_1 + v_{0y}w_{1x} + v_{1y}w_{0x} - v_{0x}w_{1y} - v_{1x}w_{0y}]\right] = e^{x+y}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}, \\
v_2(x,y,t) &= \mathbb{N}^{-1}\left[\frac{u^{\beta}}{s^{\beta}}\mathbb{N}^{+}\left[v_1 - (h_{1y}w_{0x} + h_{0y}w_{1x}) - (h_{0x}w_{1y} + h_{1x}w_{0y})\right]\right] = e^{x-y}\frac{t^{2\beta}}{\Gamma(2\beta+1)},
\end{aligned}$$

and

$$w_2(x,y,t) = \mathbb{N}^{-1}\left[\frac{u^\gamma}{s^\gamma}\mathbb{N}^+\left[v_1 - (h_{1x}v_{0y} + h_{0x}v_{1y}) - (v_{0x}h_{1y} + v_{1x}h_{0y})\right]\right] = e^{y-x}\frac{t^{2\gamma}}{\Gamma(2\gamma+1)}.$$

Thus the approximate solutions are given by

$$\begin{aligned}
h(x,y,t) &= \sum_{n=0}^{\infty} h_n(x,y,t) \\
&= h_0(x,y,t) + h_1(x,y,t) + h_2(x,y,t) + \cdots \\
&= e^{x+y} - e^{x+y}\frac{t^\alpha}{\Gamma(\alpha+1)} + e^{x+y}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - e^{x+y}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots \\
&= e^{x+y}\left[1 - \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots\right] \\
&= e^{x+y}\mathrm{E}_\alpha\left(-t^\alpha\right), \\
v(x,y,t) &= \sum_{n=0}^{\infty} v_n(x,y,t) \\
&= v_0(x,y,t) + v_1(x,y,t) + v_2(x,y,t) + \cdots \\
&= e^{x-y} + e^{x-y}\frac{t^\beta}{\Gamma(\beta+1)} + e^{x-y}\frac{t^{2\beta}}{\Gamma(2\beta+1)} + e^{x-y}\frac{t^{3\beta}}{\Gamma(3\beta+1)} + \cdots \\
&= e^{x-y}\left[1 + \frac{t^\beta}{\Gamma(\beta+1)} + \frac{t^{2\beta}}{\Gamma(2\beta+1)} + \frac{t^{3\beta}}{\Gamma(3\beta+1)} + \cdots\right] \\
&= e^{x-y}\mathrm{E}_\beta\left(t^\beta\right), \\
w(x,y,t) &= \sum_{n=0}^{\infty} w_n(x,y,t) \\
&= w_0(x,y,t) + w_1(x,y,t) + w_2(x,y,t) + \cdots \\
&= e^{y-x} + e^{y-x}\frac{t^\gamma}{\Gamma(\gamma+1)} + e^{y-x}\frac{t^{2\gamma}}{\Gamma(2\gamma+1)} + e^{y-x}\frac{t^{3\gamma}}{\Gamma(3\gamma+1)} + \cdots \\
&= e^{y-x}\left[1 - \frac{t^\gamma}{\Gamma(\gamma+1)} + \frac{t^{2\gamma}}{\Gamma(2\gamma+1)} - \frac{t^{3\gamma}}{\Gamma(3\gamma+1)} + \cdots\right] \\
&= e^{y-x}\mathrm{E}_\gamma\left(t^\gamma\right).
\end{aligned}$$

Choosing $\alpha = 1$, $\beta = 1$, $\gamma = 1$, and using a Taylor series expansion, the above approximate solution becomes

$$\begin{aligned}
h(x,y,t) &= e^{x+y}\mathrm{E}_1(-t) = e^{x+y-t}, \\
v(x,y,t) &= e^{x-y}\mathrm{E}_1(t) = e^{x-y+t}, \\
w(x,y,t) &= e^{y-x}\mathrm{E}_1(t) = e^{y-x+t}.
\end{aligned}$$

These are in fact the exact solutions of equation (5.11) in the case when $\alpha = 1$, $\beta = 1$, $\gamma = 1$. Hence, the approximate solution is rapidly convergent to the exact solution.

The numerical results of the approximate solution obtained by FNDM and exact solution are shown in Figures 7-15 when $y = 0.4$ for different values of x, t, α, β, and γ.

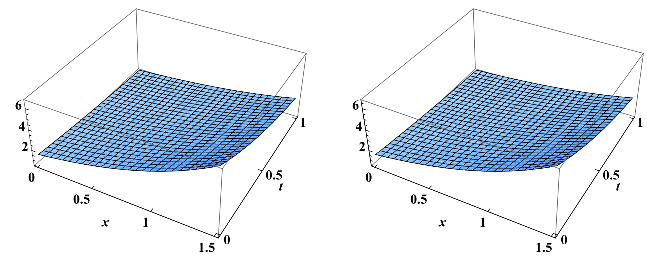

Figure 7 The approximate and exact solutions of $h(x,t)$ when $y = 0.4$ and $\alpha = 1$.

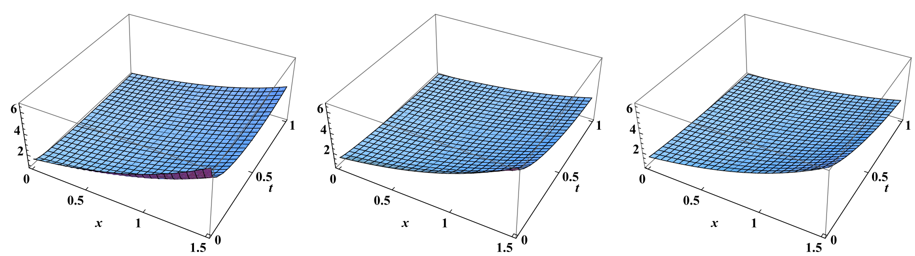

Figure 8 The approximate solutions of $h(x,t)$ when $y = 0.4$ and $\alpha = 0.5$, $\alpha = 0.75$, and $\alpha = 0.90$.

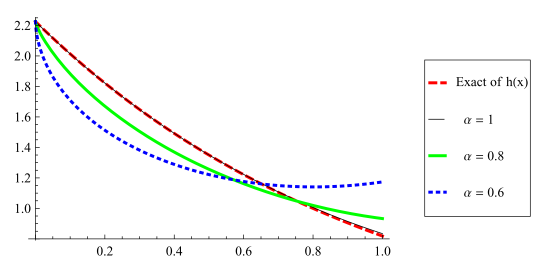

Figure 9 The approximate and exact solutions of $h(x,t)$ for $y = 0.4$ and different values of α.

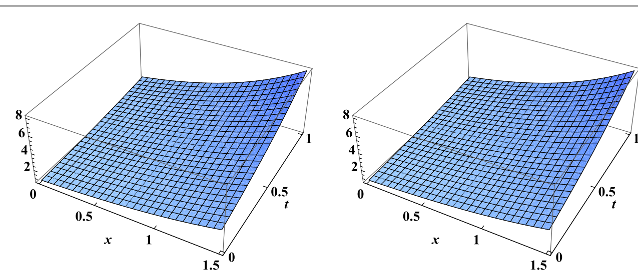

Figure 10 The approximate and exact solutions of $v(x,t)$ when $\alpha = \beta = 1$.

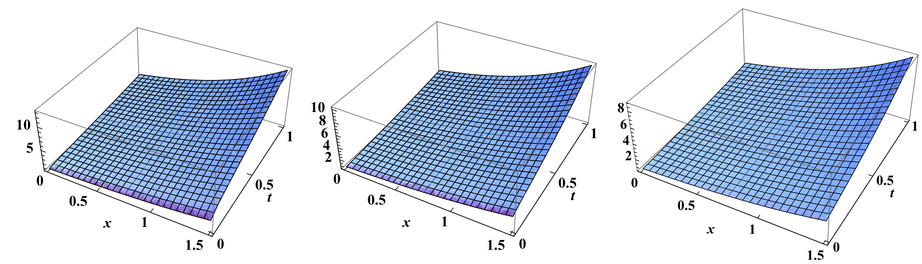

Figure 11 The approximate solutions of $v(x,t)$ when $y = 0.4$ and $\alpha = 0.5$, $\alpha = 0.75$, and $\alpha = 0.90$.

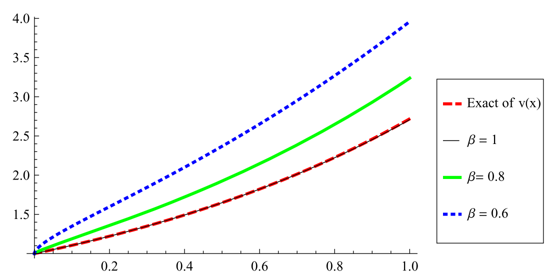

Figure 12 The approximate and exact solutions of $v(x,t)$ for different values of β.

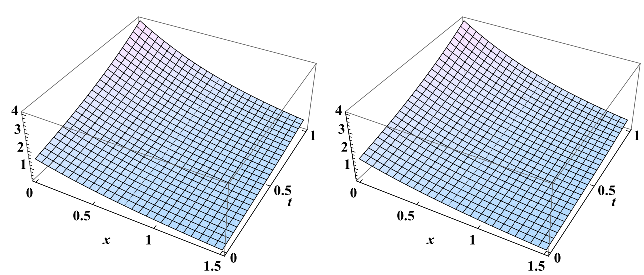

Figure 13 The approximate and exact solutions of $w(x,t)$ when $\alpha = \beta = 1$.

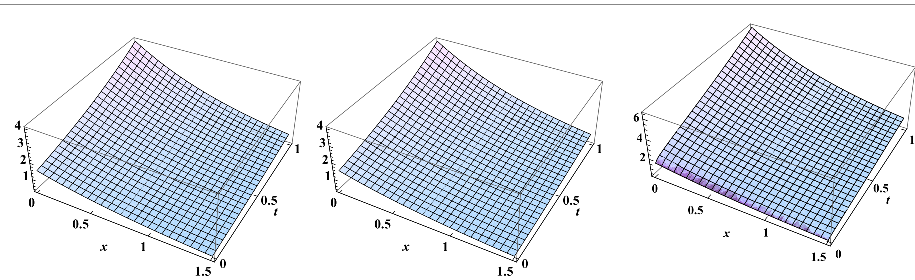

Figure 14 The approximate solutions of $w(x,t)$ when $y = 0.4$ and $\alpha = 0.5$, $\alpha = 0.75$, and $\alpha = 0.90$.

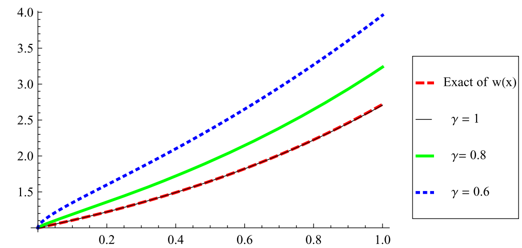

Figure 15 The approximate and exact solutions of $w(x,t)$ for different values of γ.

Table 1 The approximate and exact solution of $v(x,t)$ and $w(x,t)$ for Example 5.1 for different values of α and β

x	t	$\alpha=\beta=0.75$	$\alpha=\beta=0.9$	$\alpha=\beta=1$	
				Numerical	Exact
−10	0.2	0.403596	0.429046	0.446097	0.445407
	0.4	0.349827	0.358413	0.369934	0.364668
−5	0.2	0.711403	0.756262	0.786318	0.785101
	0.4	0.616625	0.63176	0.652069	0.642786
5	0.2	−0.711403	−0.756262	−0.786318	−0.785101
	0.4	−0.616625	−0.63176	−0.652069	−0.642786
10	0.2	−0.403596	−0.429046	−0.446097	−0.445407
	0.4	−0.349827	−0.358413	−0.369934	−0.364668

Table 2 The approximate and exact solutions of $h(x,t)$ for Example 5.2 for different values of α and $y=0.4$

x	t	$\alpha=0.75$	$\alpha=0.9$	$\alpha=1$	
				Numerical	Exact
0.5	0.3	1.6256	1.74178	1.82217	1.82212
	0.6	1.27791	1.31047	1.35131	1.34986
	0.9	1.10414	1.02931	1.0105	1
1	0.3	2.68017	2.8717	3.00424	3.00417
	0.6	2.10692	2.1606	2.22793	2.22554
	0.9	1.82042	1.69705	1.66603	1.64872
1.5	0.3	4.41885	4.73464	4.95316	4.95303
	0.6	3.47372	3.56223	3.67323	3.6693
	0.9	3.00136	2.79796	2.74682	2.71828

6 Numerical tables

In this section, we shall illustrate the accuracy and efficiency of the FNDM. In Table 1, we consider the same values of x and t for $w(x,t)$ and $v(x,t)$, specifically, $x=\{-10,-5,5,10\}$ and $t=\{0.2,0.4\}$. In Tables 2, 3, 4 we consider the same values of x and t for $h(x,t)$, $v(x,t)$, and $w(x,t)$ specifically, $x=\{0.5,1,1.5\}$ and $t=\{0.3,0.6,0.9\}$.

7 Conclusion

In this paper, the FNDM has been successfully applied to obtain numerical solutions to the time-fractional coupled Burgers' system of equations and another nonlinear time-fractional PDE. We successfully found exact solutions to both physical models in the cases when $\alpha=\beta=\gamma=1$. The FNDM introduces a significant improvement in the fields over existing techniques. Our goal in the future is to apply the FNDM to other systems of fractional differential equations that arise in other areas of science.

Table 3 The approximate and exact solutions of $v(x,t)$ for Example 5.2 for different values of β and $y = 0.4$

x	t	$\beta = 0.75$	$\beta = 0.9$	$\beta = 1$	
				Numerical	Exact
0.5	0.3	1.76307	1.58084	1.4918	1.49182
	0.6	2.48848	2.17346	2.01296	2.01375
	0.9	3.40244	2.95156	2.71191	2.71828
1	0.3	2.90682	2.60637	2.45956	2.4596
	0.6	4.10281	3.58343	3.31881	3.32012
	0.9	5.60968	4.86629	4.47118	4.48169
1.5	0.3	4.79253	4.29717	4.05514	4.0552
	0.6	6.76439	5.90808	5.47179	5.47395
	0.9	9.2488	8.02316	7.37174	7.38906

Table 4 The approximate and exact solutions of $w(x,t)$ for Example 5.2 for different values of γ and $y = 0.4$

x	t	$\gamma = 0.75$	$\gamma = 0.9$	$\gamma = 1$	
				Numerical	Exact
0.5	0.3	1.44348	1.29428	1.22138	1.2214
	0.6	2.0374	1.77948	1.64807	1.64872
	0.9	2.78569	2.41653	2.22032	2.22554
1	0.3	0.875516	0.785023	0.740807	0.740818
	0.6	1.23574	1.07931	0.999606	1
	0.9	1.6896	1.4657	1.34669	1.34986
1.5	0.3	0.531027	0.47614	0.449322	0.449329
	0.6	0.749516	0.654634	0.606291	0.606531
	0.9	1.0248	0.888992	0.816812	0.818731

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the manuscript. All authors read and approved the final manuscript.

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