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# The general mixed nonlinear Schrödinger equation: Darboux transformation, rogue wave solutions, and modulation instability

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**Abstract**

In this paper, the Darboux transformation method has been successfully applied to a general mixed nonlinear Schrödinger equation and some rogue wave solutions are proposed. First of all, the determinant representation of an *n*-fold DT is given explicitly. Then starting with a periodic seed solution, we obtain some rogue wave solutions of the general mixed nonlinear Schrödinger equation through iteration of a generalized DT. Second, the three-dimensional images and density profiles of the rogue waves are plotted to show the structures of these rogue wave solutions. Finally, we give evidence for the connection between the occurrence of rogue wave solutions and the modulation instability.

**Keywords:** Darboux transformation; rogue wave solution; general mixed nonlinear Schrödinger equation; Lax pair; modulation instability

## 1 Introduction

In the past decades, there are extensive advancements in the field of nonlinear integrable systems. How to get the exact soliton solutions of these integrable systems has become one of the important research topics in theory and practical applications. Quite a few approaches for finding exact solutions of such nonlinear systems are well established, such as the inverse scattering method [1], the Darboux transformation method [2, 3], the Bäcklund transformation [4–6], the Riemann-Hilbert formulation [7–9], the Hirota bilinear method [10, 11], Lie group analysis [12], the similarity transformation method [13–16], and the homotopy analysis method [17–20], $F$-expansion method [21] and so on [22, 23]. Among these approaches, the DT is well known to be a powerful method for finding exact solutions of integrable systems [2].

In recent years, more researchers have begun to pay attention to rouge waves (RWs), which has been introduced and become an interesting objective in some fields, such as optics [24–27], super fluids [28], Bose-Einstein condensates [29–31], and so on. RW is one terrible reason to generate large marine catastrophes, which is a presence that may always show up with special giant waves in a very short time, and there is no sign before it appears. For example, the giant waves of several tens of meters will suddenly appear in a relatively calm sea. In optics, the wave will appear as a very bright spot, which mathematical physicists call rogue waves [32]. But now, the study of the rogue waves is still only in its infancy. Mechanisms and the probability of its occurrence are not clear. Because observing of rogue waves on the ocean is very difficult and dangerous, unreliable and few records and observations are available, which led to a lack of research on the rogue wave scientific community recognizing the factual basis.

One of the models to describe the rogue waves is the nonlinear Schrödinger (NLS) equation,

$$iq_t + \frac{1}{2}q_{xx} + |q|^2 q = 0, \tag{1}$$

which is an important integrable nonlinear wave equation. Recently, much work was done to study various NLS equations [33–36].

By DT [37], one can get its first-order and high-order rogue wave solutions [38]. Recently, more general rogue waves of the higher-order NLS equation have been calculated [39, 40] and the spatial-temporal structures have also been discovered and analyzed.

It is well known that there is an integrable mixed NLS equation [41],

$$q_t - iq_{xx} + a_1\left(q^* q^2\right)_x + ibq^* q^2 = 0, \tag{2}$$

where $q$ represents a complex field envelop and the asterisk denotes complex conjugation, $a$ and $b$ are two nonnegative constants. The mixed NLS equation is used to model the propagations of the Alfvén waves in plasmas and the ultrashort light pulse in optics.

Motivated by the work of [41], we will consider the following general mixed NLS equation:

$$iq_t + q_{xx} + i\beta\left(|q|^2 q\right)_x + i\alpha q_x + |q|^2 q + rq = 0, \tag{3}$$

$$-ip_t + p_{xx} - i\beta\left(|q|^2 p\right)_x - i\alpha p_x + |q|^2 p + rp = 0, \tag{4}$$

where $\alpha$, $\beta$, and $r$ are real constants. Under the condition $q^* = p$, this equation can be simplified as

$$iq_t + q_{xx} + i\beta\left(|q|^2 q\right)_x + i\alpha q_x + |q|^2 q + rq = 0. \tag{5}$$

This equation can be taken as one version of the Wadati-Konno-Ichikawa (WKI) system [42], and the corresponding Lax pair is given based on the WKI spectral problem

$$\Psi_x = U\Psi = \left(i\beta I\lambda^2 + Q_1\lambda + Q_0\right)\Psi, \tag{6a}$$

$$\Psi_t = V\Psi = \left(-2i\beta^2 I\lambda^4 + V_3\lambda^3 + V_2\lambda^2 + V_1\lambda + V_0\right)\Psi, \tag{6b}$$

with $U$ and $V$ being $2 \times 2$ matrices, and

$$I = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad Q_1 = \begin{pmatrix} \sqrt{2} & (\frac{\alpha\beta+\sqrt{2}\omega\beta}{2+\alpha\beta} - \beta)q \\ (-\beta - \frac{\sqrt{2}\omega\beta}{2})p & -\sqrt{2} \end{pmatrix},$$

$$Q_0 = \begin{pmatrix} -\frac{1}{4}i\alpha & \frac{\sqrt{2}}{2}iq \\ (\frac{\sqrt{2}}{2}i + \frac{\sqrt{2}}{4}i\alpha\beta)p & \frac{1}{4}i\alpha \end{pmatrix}, \qquad V_3 = \begin{pmatrix} -4\sqrt{2}\beta & \frac{2(2\beta^2-\sqrt{2}\beta^2\omega)}{2+\alpha\beta}q \\ (2\beta^2 + \sqrt{2}\beta^2\omega)p & 4\sqrt{2}\beta \end{pmatrix},$$

$$V_2 = \begin{pmatrix} -i\beta^2 qp + 4i & \frac{\beta}{2+\alpha\beta} m_2 q \\ \frac{\beta}{2} m_1 p & i\beta^2 qp - 4i \end{pmatrix}, \qquad V_1 = \begin{pmatrix} -\sqrt{2}\beta pq & \frac{1}{2(2+\alpha\beta)}(m_3 q_x + m_5 q) \\ \frac{1}{4}(m_4 p_x + m_6 p) & \sqrt{2}\beta pq \end{pmatrix},$$

$$V_0 = \begin{pmatrix} \frac{1}{8}i\alpha^2 + \frac{1}{4}i\alpha\beta qp + \frac{1}{2}ir + \frac{1}{2}ipq & \frac{\sqrt{2}}{2}q_x - m_7 q \\ (\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{4}\alpha\beta)p_x - m_8 p & -\frac{1}{8}i\alpha^2 - \frac{1}{4}i\alpha\beta qp - \frac{1}{2}ir - \frac{1}{2}iqp \end{pmatrix},$$

where

$$m_1 = -6\sqrt{2}i - \sqrt{2}i\alpha\beta - 4i\omega, \qquad m_2 = 6\sqrt{2}i + \sqrt{2}i\alpha\beta - 4i\omega,$$
$$m_3 = 4i\beta - 2\sqrt{2}i\beta\omega, \qquad m_4 = 4i\beta + 2\sqrt{2}i\beta\omega,$$
$$m_5 = 8 + 2\alpha\beta - 4qp\beta^2 + 2\sqrt{2}\beta^2\omega qp + \sqrt{2}\alpha\beta\omega,$$
$$m_6 = 4\beta qp + 2\sqrt{2}\omega qp - 2\alpha\beta + \sqrt{2}\omega\alpha\beta - 8, \qquad m_7 = \frac{\sqrt{2}}{2}i\beta qp + \frac{\sqrt{2}}{4}i\alpha,$$
$$m_8 = \frac{\sqrt{2}}{4}i\alpha\beta^2 qp + \frac{\sqrt{2}}{8}i\alpha^2\beta + \frac{\sqrt{2}}{4}i\alpha + \frac{\sqrt{2}}{2}i\beta qp, \quad \omega = \sqrt{-\alpha\beta}.$$

Here the complex number $\lambda$ is the associated spectral parameter, and $\Psi$ is the eigenfunction associated with $\lambda$ of the system. Equations (3) and (4) can be obtained by the zero-curvature equation $U_t - V_x + [U, V] = 0$ of (6a) and (6b). It is well known that some exact solutions of the mixed NLS equation (5) with $\alpha = 0$ have been constructed via DT [43–46] and the Hirota method [47, 48].

The main aim of this paper is to construct the DT to derive the rogue wave solutions of mixed NLS equation (5) by using the generalized Darboux transformation, then analyze the rogue wave through their figures. In the end, we investigate modulation instability of the mixed NLS equation and point out the connection between RW and modulation instability.

## 2 Darboux transformation

Based on procedure of Darboux transformation for AKNS system in [49], we can construct a Darboux matrix $T$ to satisfy $\Psi^{[1]} = T\Psi$, and to present the determinant representation of the $n$-fold transformation.

Assume the Darboux matrix $T$ is the form of

$$T_1 = T_1|_{\lambda=\lambda_1} = \begin{pmatrix} a_1 & 0 \\ 0 & d_1 \end{pmatrix} \lambda + \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix}, \tag{7}$$

where $a_1$, $d_1$, $a_0$, $b_0$, $c_0$, $d_0$ are undetermined functions of $(x, t)$, which will be parameterized by the eigenfunction associated with $\lambda_1$ and the seed solution $(q, p)$ for equations (3) and (4).

We consider $n$ eigenfunctions $\Psi_j$ as

$$\Psi_j = \begin{pmatrix} \phi_j \\ \varphi_j \end{pmatrix},$$

where $j = 1, 2, \ldots$, $\phi_j = \phi_j(x, t, \lambda_j)$, $\varphi_j = \varphi_j(x, t, \lambda_j)$, which is the eigenfunction of the Lax pair equations (6a)-(6b) with seed solution $(q, p)$ and spectral parameters $\lambda_j$. Then we can get the Darboux matrix through a series of calculation, and the detailed process to construct the Darboux matrix can be found consulting [41].

In fact, the DT is a special gauge transformation, we can start from a trivial solution and get the nontrivial exact solutions by this method eventually. The basic ideas of DT is to use the seed solutions and the Lax pair, we get the DT with the aid of the gauge transformation in combination with the spectral problem. To be specific, the gauge transformation is

$$\Psi^{[1]} = T_1\Psi, \tag{8}$$

where the matrix $T$ can transform the Lax pair (6a)-(6b) into a new one, possessing the same form

$$\Psi_x^{[1]} = U^{[1]}\Psi, \qquad \Psi_t^{[1]} = V^{[1]}\Psi, \tag{9}$$

where $U^{[1]}$ and $V^{[1]}$ have the same form as $U$ and $V$ but replacing $q$, $p$ with $q_1$, $p_1$. And $U^{[1]}$, $V^{[1]}$ satisfy the two equations

$$T_{1x} + T_1U = U^{[1]}T_1, \qquad T_{1t} + T_1V = V^{[1]}T_1. \tag{10}$$

From equation (10), we can construct a basic DT matrix, and it is possible to find the relationship between the new potential function $q_1$ and the initial potential $q$. Referring to the method to get Darboux matrix $T$ of literature [41], using equation (10) and comparing the coefficients of $\lambda^j$, we can get the expression of $a_0, d_0$ in equation (7) as

$$q_1 = \frac{a_1q}{d_1} - \frac{2i\beta b_0}{d_1(-\beta + \frac{(\alpha\beta+\sqrt{2}\omega)\beta}{2+\alpha\beta})}, \qquad p_1 = \frac{d_1p}{a_1} + \frac{2ic_0\beta}{a_1(-\beta - \frac{\sqrt{2}\omega\beta}{2})},$$
$$a_0 = \frac{-i(2\sqrt{2} + 4\sqrt{-\alpha\beta} - \sqrt{2}\alpha\beta)a_1}{2\beta(2+\sqrt{2}\omega)}, \qquad d_0 = \frac{-i(-2\sqrt{2} + 4\sqrt{-\alpha\beta} + \sqrt{2}\alpha\beta)a_1}{2\beta(-2+\sqrt{2}\omega)},$$

moreover, $b_0$, $c_0$ are satisfied with

$$b_{0x} = \frac{8ib_0}{\beta(2+\sqrt{2}\omega)(-2+\sqrt{2}\omega)^2}, \qquad b_{0t} = -\frac{16ib_0r}{(-2+\sqrt{2}\omega)^3(2+\sqrt{2}\omega)}, \tag{11}$$

$$c_{0x} = \frac{8ic_0}{\beta(2+\sqrt{2}\omega)(-2+\sqrt{2}\omega)^2}, \qquad c_{0t} = \frac{16ric_0}{(-2+\sqrt{2}\omega)(2+\sqrt{2}\omega)^3}. \tag{12}$$

Solving equations (11) and (12), we have

$$b_0 = C_{11}e^{m_9}, \qquad c_0 = C_{12}e^{m_{10}},$$

where

$$C_{11} = \lambda_1 - \frac{i(2\sqrt{2} + 4\omega - \sqrt{2}\alpha\beta)}{2\beta(2+\sqrt{2}\omega)},$$
$$C_{12} = \lambda_1 - \frac{i(-2\sqrt{2} + 4\omega + \sqrt{2}\alpha\beta)}{2\beta(-2+\sqrt{2}\omega)},$$

$$m_9 = \frac{-4i(-2rt\beta + \sqrt{2}r\omega\beta t + x\alpha\beta + 2\sqrt{2}\omega x - 2x)}{(\alpha\beta + 2\sqrt{2}\omega - 2)(2 + \alpha\beta)(-2 + \sqrt{2}\omega)\beta},$$
$$m_{10} = \frac{-4i(2rt\beta + \sqrt{2}r\omega\beta t - x\alpha\beta + 2\sqrt{2}\omega x + 2x)}{(-\alpha\beta + 2\sqrt{2}\omega + 2)(2 + \alpha\beta)(2 + \sqrt{2}\omega)\beta}.$$

Now, we have gotten four parameters $a_0$, $b_0$, $c_0$, $d_0$, and the next work is to calculate the remaining two variables $a_1$, $d_1$. Then from the equation $\Psi^{[1]} = T_1\Psi$, there must be one $\lambda = \lambda_1$ satisfying $T_1|_{\lambda=\lambda_1}\Psi_1 = 0$ [37], we get

$$(a_1\lambda + a_0)\phi_1 + b_0\varphi_1 = 0, \qquad c_0\phi_1 + (d_1\lambda + d_0)\varphi_1 = 0,$$

then $a_1 = -\frac{\varphi_1}{\phi_1}e^{m_9}$, $d_1 = -\frac{\phi_1}{\varphi_1}e^{m_{10}}$, now the Darboux matrix $T$ and the potential functions $q_1$, $p_1$ can be obtained explicitly,

$$q_1 = \frac{\varphi_1^2}{\phi_1^2}e^{(m_9-m_{10})}q + 2i\frac{C_{11}}{(-\beta + \frac{(\alpha\beta+\sqrt{2}\sqrt{-\alpha\beta})\beta}{2+\alpha\beta})},$$
$$p_1 = \frac{\phi_1^2}{\varphi_1^2}e^{(m_{10}-m_9)}p - 2i\frac{C_{12}}{(-\beta + \frac{\sqrt{2}\sqrt{-\alpha\beta}\beta}{2})}.$$

Now, we assume a $\widetilde{\lambda}$, and in order to get a unified $\widetilde{\lambda}$, we assume

$$\frac{C_{11}}{(-\beta + \frac{(\alpha\beta+\sqrt{2}\sqrt{-\alpha\beta})\beta}{2+\alpha\beta})} - \frac{C_{12}}{(-\beta + \frac{\sqrt{2}\sqrt{-\alpha\beta}\beta}{2})} = 0,$$

and we get

$$\alpha = 0, -\frac{(-\frac{3\sqrt{2}}{2} - i\lambda\beta + \frac{\sqrt{-14-4i\sqrt{2}\lambda\beta-4\lambda^2\beta^2}}{2})^2}{\beta}, -\frac{(-\frac{3\sqrt{2}}{2} - i\lambda\beta - \frac{\sqrt{-14-4i\sqrt{2}\lambda\beta-4\lambda^2\beta^2}}{2})^2}{\beta}.$$

In this section, we reduce the number of parameters and want one relatively simple $T$, without loss of generalization we can assume $\alpha = 0$, and finally we can get

$$T_1|_{\lambda=\lambda_1} = \begin{pmatrix} -\frac{\varphi_1}{\phi_1}e^{(i(\frac{1}{\beta}x-\frac{1}{\beta^2}t))}(\lambda - \frac{i\sqrt{2}}{2\beta}) & (-\frac{i\sqrt{2}}{2\beta} + \lambda_1)e^{(i(\frac{1}{\beta}x-\frac{1}{\beta^2}t))} \\ (-\frac{i\sqrt{2}}{2\beta} + \lambda_1)e^{(-i(\frac{1}{\beta}x-\frac{1}{\beta^2}t))} & -\frac{\phi_1}{\varphi_1}(\lambda - \frac{i\sqrt{2}}{2\beta})e^{(-i(\frac{1}{\beta}x-\frac{1}{\beta^2}t))} \end{pmatrix},$$

and the relationships between the potential functions are as follows:

$$q_1 = q\left(\frac{\varphi_1}{\phi_1}\right)^2 e^{(2i(\frac{1}{\beta}x-\frac{1}{\beta^2}t))} - 2i\frac{\varphi_1}{\phi_1}\left(-\frac{i\sqrt{2}}{2\beta} + \lambda_1\right)e^{(2i(\frac{1}{\beta}x-\frac{1}{\beta^2}t))}, \tag{13a}$$
$$p_1 = p\left(\frac{\phi_1}{\varphi_1}\right)^2 e^{(-2i(\frac{1}{\beta}x-\frac{1}{\beta^2}t))} + 2i\frac{\phi_1}{\varphi_1}\left(-\frac{i\sqrt{2}}{2\beta} + \lambda_1\right)e^{(-2i(\frac{1}{\beta}x-\frac{1}{\beta^2}t))}, \tag{13b}$$

under the reduction conditions of $q^* = p$ and $\lambda_1^* = -\lambda_1$, substituting this relation into (13a)-(13b), we have

$$q_1^* = q^*\left(\frac{\phi_1}{\varphi_1}\right)^2 e^{(-2i(\frac{1}{\beta}x-\frac{1}{\beta^2}t))} + 2i\left(\frac{\phi_1}{\varphi_1}\right)\left(-\frac{i\sqrt{2}}{2\beta} + \lambda_1\right)e^{(-2i(\frac{1}{\beta}x-\frac{1}{\beta^2}t))} = p_1,$$

and the corresponding new eigenfunction

$$\Psi_j^{[1]} = \begin{pmatrix} \frac{1}{\phi_1}\begin{vmatrix} -(\lambda_j - \frac{i\sqrt{2}}{2\beta})\phi_j & \varphi_j \\ -(\lambda_1 - \frac{i\sqrt{2}}{2\beta})\phi_1 & \varphi_1 \end{vmatrix} e^{(i(\frac{1}{\beta}x - \frac{1}{\beta^2}t))} \\ \frac{1}{\varphi_1}\begin{vmatrix} -(\lambda_j - \frac{i\sqrt{2}}{2\beta})\varphi_j & \phi_j \\ -(\lambda_j - \frac{i\sqrt{2}}{2\beta})\varphi_1 & \phi_1 \end{vmatrix} e^{(-i(\frac{1}{\beta}x - \frac{1}{\beta^2}t))} \end{pmatrix}.$$

It is straightforward to verify that when $j = 1$, the $\Psi_j^{[1]} = T_1|_{\lambda=\lambda_1}\Psi_j = 0$, this fact implies that $\lambda_1$ may not be used more than once when considering the iterations for the DT. But Matveev and Salle [2] have pointed out that a generalized DT does exist, we can use the method to solve the rational solution of equation (5).

In order to get the DT of more than two orders, we translate the spectral parameter $\widetilde{\lambda}_1 = -\frac{i\sqrt{2}}{2\beta} + \lambda$ and $\widetilde{\lambda}_j = -\frac{i\sqrt{2}}{2\beta} + \lambda_j$. We assume

$$\begin{aligned} &n = 1, \quad T_1 = P_1\widetilde{\lambda} + P_0 = \begin{pmatrix} a_1\widetilde{\lambda} & b_0 \\ c_0 & d_1\widetilde{\lambda} \end{pmatrix}, \\ &n = 2, \quad T_2 = P_2\widetilde{\lambda}^2 + P_1\widetilde{\lambda} + P_0 = \begin{pmatrix} a_2 & 0 \\ 0 & d_2 \end{pmatrix}\widetilde{\lambda}^2 + \begin{pmatrix} 0 & b_1 \\ c_1 & 0 \end{pmatrix}\widetilde{\lambda} + \begin{pmatrix} a_0 & 0 \\ 0 & d_0 \end{pmatrix}, \\ &\ldots, \end{aligned}$$

similarly $T_n = \sum_{l=0}^{n} P_l\widetilde{\lambda}^l$, where

$$\begin{aligned} &P_n = \begin{pmatrix} a_n & 0 \\ 0 & d_n \end{pmatrix} \in \left\{\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\right\}, \\ &P_{n-1} = \begin{pmatrix} 0 & b_n \\ c_n & 0 \end{pmatrix} \in \left\{\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}\right\}, \end{aligned}$$

$a, b, c, d$ are complex functions of $(x, t)$. $P_0$ is a constant matrix when $n$ is even and $P_0$ is a matrix that is known. Similar to one-fold DT, we can construct the $T_n$ and it is given in Appendix 1.

Based on the $n$-fold Darboux transformation in [49], we just consider the condition of $n$ being even, through the $n$-fold DT $T_n$, we get

$$q^{[n]} = \frac{\Omega_{n1}^2}{\Omega_{n3}^2}q - 2i\frac{\Omega_{n1}\Omega_{n2}}{\Omega_{n3}^2}, \qquad p^{[n]} = \frac{\Omega_{n3}^2}{\Omega_{n1}^2}p + 2i\frac{\Omega_{n3}\Omega_{n4}}{\Omega_{n1}^2}, \tag{14}$$

here, for $n = 2k$,

$$\Omega_{n1} = \begin{vmatrix} \widetilde{\lambda}_1^{n-1}\varphi_1 & \widetilde{\lambda}_1^{n-2}\phi_1 & \cdots & \widetilde{\lambda}_1\varphi_1 & \phi_1 \\ \widetilde{\lambda}_2^{n-1}\varphi_2 & \widetilde{\lambda}_2^{n-2}\phi_2 & \cdots & \widetilde{\lambda}_2\varphi_2 & \phi_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \widetilde{\lambda}_{n-1}^{n-1}\varphi_{n-1} & \widetilde{\lambda}_{n-1}^{n-2}\phi_{n-1} & \cdots & \widetilde{\lambda}_{n-1}\varphi_{n-1} & \phi_{n-1} \\ \widetilde{\lambda}_n^{n-1}\varphi_n & \widetilde{\lambda}_n^{n-2}\phi_n & \cdots & \widetilde{\lambda}_n\varphi_n & \phi_n \end{vmatrix},$$

$$\Omega_{n2} = \begin{vmatrix} \widetilde{\lambda}_1^n\phi_1 & \widetilde{\lambda}_1^{n-2}\phi_1 & \cdots & \widetilde{\lambda}_1\varphi_1 & \phi_1 \\ \widetilde{\lambda}_2^n\phi_2 & \widetilde{\lambda}_2^{n-2}\phi_2 & \cdots & \widetilde{\lambda}_2\varphi_2 & \phi_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \widetilde{\lambda}_{n-1}^n\phi_{n-1} & \widetilde{\lambda}_{n-1}^{n-2}\phi_{n-1} & \cdots & \widetilde{\lambda}_{n-1}\varphi_{n-1} & \phi_{n-1} \\ \widetilde{\lambda}_n^n\phi_n & \widetilde{\lambda}_n^{n-2}\phi_n & \cdots & \widetilde{\lambda}_n\varphi_n & \phi_n \end{vmatrix},$$

$$\Omega_{n3} = \begin{vmatrix} \widetilde{\lambda}_1^{n-1}\phi_1 & \widetilde{\lambda}_1^{n-2}\varphi_1 & \cdots & \widetilde{\lambda}_1\phi_1 & \varphi_1 \\ \widetilde{\lambda}_2^{n-1}\phi_2 & \widetilde{\lambda}_2^{n-2}\varphi_2 & \cdots & \widetilde{\lambda}_2\phi_2 & \varphi_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \widetilde{\lambda}_{n-1}^{n-1}\phi_{n-1} & \widetilde{\lambda}_{n-1}^{n-2}\varphi_{n-1} & \cdots & \widetilde{\lambda}_{n-1}\phi_{n-1} & \varphi_{n-1} \\ \widetilde{\lambda}_n^{n-1}\phi_n & \widetilde{\lambda}_n^{n-2}\varphi_n & \cdots & \widetilde{\lambda}_n\phi_n & \varphi_n \end{vmatrix},$$

$$\Omega_{n1} = \begin{vmatrix} \widetilde{\lambda}_1^n\varphi_1 & \widetilde{\lambda}_1^{n-2}\varphi_1 & \cdots & \widetilde{\lambda}_1\phi_1 & \varphi_1 \\ \widetilde{\lambda}_2^n\varphi_2 & \widetilde{\lambda}_2^{n-2}\varphi_2 & \cdots & \widetilde{\lambda}_2\phi_2 & \varphi_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \widetilde{\lambda}_{n-1}^n\varphi_{n-1} & \widetilde{\lambda}_{n-1}^{n-2}\varphi_{n-1} & \cdots & \widetilde{\lambda}_{n-1}\phi_{n-1} & \varphi_{n-1} \\ \widetilde{\lambda}_n^n\phi_n & \widetilde{\lambda}_n^{n-2}\varphi_n & \cdots & \widetilde{\lambda}_n\phi_n & \varphi_n \end{vmatrix},$$

under this condition of $\lambda_2 = -\lambda_1^*$, $\Psi_2 = \binom{-\varphi_1^*}{\phi_1^*}$, the two-fold form is

$$q^{[2]} = \frac{\Omega_{21}^2}{\Omega_{23}^2}q - 2i\frac{\Omega_{21}\Omega_{22}}{\Omega_{23}^2}, \tag{15}$$

at the same time $\Psi_2$ can also satisfy equations (6a)-(6b). Substituting $\Psi_1$, $\Psi_2$ into (15), we get

$$\begin{aligned}
\Omega_{21} &= \widetilde{\lambda}_1\varphi_1\phi_2 - \widetilde{\lambda}_2\phi_1\varphi_2 = -\widetilde{\lambda}_1\varphi_1\varphi_1^* + \widetilde{\lambda}_1^*\phi_1\phi_1^*, \\
\Omega_{22} &= \big(\widetilde{\lambda}_1^2 - \widetilde{\lambda}_2^2\big)\phi_1\phi_2 = -\big(\widetilde{\lambda}_1^2 - \widetilde{\lambda}_1^{*2}\big)\phi_1\varphi_1^*, \\
\Omega_{23} &= \widetilde{\lambda}_1\phi_1\varphi_2 - \widetilde{\lambda}_2\varphi_1\phi_2 = \widetilde{\lambda}_1\phi_1\phi_1^* - \widetilde{\lambda}_1^*\varphi_1\varphi_1^*,
\end{aligned}$$

then equation (14) can be written as

$$q^{[2]} = \frac{(-\widetilde{\lambda}_1|\varphi_1|^2 + \widetilde{\lambda}_1^*|\phi_1|^2)^2}{(\widetilde{\lambda}_1|\phi_1|^2 - \widetilde{\lambda}_1^*|\varphi_1|^2)^2}q - 2i\frac{-(\widetilde{\lambda}_1^2 - \widetilde{\lambda}_1^{*2})(-\widetilde{\lambda}_1|\varphi_1|^2 + \widetilde{\lambda}_1^*|\phi_1|^2)\phi_1\varphi_1^*}{(\widetilde{\lambda}_1|\phi_1|^2 - \widetilde{\lambda}_1^*|\varphi_1|^2)^2}. \tag{16}$$

Equations (14) and (16) give us the new solutions after a Darboux transformation, and we can use the results of (14) and (15) to get two-order and three-order rogue wave solutions. In the next section, we will consider the condition of the degeneration case of the Darboux matrix $T_{2k}$ to get the exact rogue wave solutions of equation (5). In the case of degeneration, the $q^{[2k]}$ can be expressed as an infinite expression which is of the form of $\frac{0}{0}$, then we can make a Taylor expansion at $\widetilde{\lambda} = \widetilde{\lambda}_0$ and get the smooth solutions of equation (5).

## 3 Solutions of rogue wave

In order to get the rational solution, we can apply the generalized Darboux transformation. We start with the seed solution of equation (5),

$$q = c_1 e^{i(a_1 x - (\alpha a_1 - a_1^2 + c_1^2 - a_1 c_1^2\beta + r)t)}. \tag{17}$$

Substituting (17) into the spectral equations (6a)-(6b), we have the corresponding solution for the spectral problem,

$$\Psi_1(f) = \begin{pmatrix} (C_2 \mathrm{e}^{-A} + C_1 \mathrm{e}^{A}) \mathrm{e}^{\frac{i}{2}(a_1 x - (\alpha a_1 - a_1{}^2 + c_1{}^2 - \beta c_1{}^2 a_1 + r)t)} \\ -2(2+\alpha\beta)(\frac{C_2 \mathrm{e}^{-A}(-\frac{i}{16}A_1 + M_2)}{M_1} + \frac{C_1 \mathrm{e}^{A}(\frac{i}{16}A_1 + M_2)}{M_1}) \mathrm{e}^{-\frac{i}{2}(a_1 x - (\alpha a_1 - a_1{}^2 + c_1{}^2 - \beta c_1{}^2 a_1 + r)t)} \end{pmatrix},$$

where

$$\begin{aligned}
A &= \frac{A_1(x + \frac{it}{8}(16\sqrt{2}\lambda + 8ic_1^2\beta + 8ia_1 + 4i\alpha + 16i\lambda^2\beta))}{16\sqrt{2}\lambda + 8ic_1^2\beta + 8ia_1 + 4i\alpha + 16i\lambda^2\beta}, \\
A_1 &= 4\big((16\lambda c_1^2\beta + 16\lambda a_1 + 8\lambda\alpha)(\sqrt{2}i - \lambda\beta) - 32\sqrt{2}i\lambda^3\beta - 8\lambda^2\alpha\beta - 32\lambda^2 + 16\lambda^4\beta^2 \\
&\quad + 4a_1^2 + 8c_1^2 + \alpha^2 + 4a_1\alpha + 4c_1^2\alpha\beta\big)^{\frac{1}{2}}\big(-2a_1 - 2c_1^2\beta - 4\lambda^2\beta + 4\sqrt{2}i\lambda - \alpha\big), \\
M_1 &= \frac{1}{4}\big(8\sqrt{2}\omega\beta^2\lambda^3 + 4\sqrt{2}\omega\beta^2 c_1^2 + 4\sqrt{2}\omega\lambda a_1\beta \\
&\quad + 2\sqrt{2}\omega\lambda\alpha\beta + 2\sqrt{2}i\alpha + 4\sqrt{2}ia_1 + 2\sqrt{2}ic_1^2\alpha\beta^2 \\
&\quad - 16i\lambda^2\omega\beta + 24\sqrt{2}i\lambda^2\beta + 2\sqrt{2}ia_1\alpha\beta + 4\sqrt{2}i\lambda^2\alpha\beta^2 + 4\sqrt{2}ic_1^2\beta \\
&\quad + \sqrt{2}i\alpha^2\beta + 16\lambda - 8\lambda c_1^2\beta^2 - 16\beta^2\lambda^3 + 4a\lambda\beta - 8a_1\beta\lambda\big)c_1, \\
M_2 &= \frac{\sqrt{2}}{2}\lambda\beta c_1^2 + 2\sqrt{2}\lambda^3\beta + \frac{i}{2}c_1^2\lambda^2\beta^2 + i\lambda^4\beta^2 - \frac{i}{4}c_1^2 a_1\beta - \frac{i}{8}c_1^2\alpha\beta \\
&\quad - \frac{i}{4}a_1\alpha - \frac{i}{16}\alpha^2 - \frac{1}{4}ia_1^2 - 2i\lambda^2.
\end{aligned}$$

Next we will simplify the result; we assume $r = 1$, $a_1 = 0$, $c_1 = 1$, $\beta = 1$, $\alpha = 0$, while $q = e^{2it}$, and $\lambda = \frac{1}{2} + \frac{i}{2} + \frac{\sqrt{2}i}{2}$ is only one zero point of the $A_1$, $f$ limit to 0, and assuming $\lambda = \frac{1}{2} + ih$, now

$$\begin{aligned}
A_1 &= \sqrt{64i\lambda\left(\lambda^6 - 2\lambda^4 - \frac{1}{8}\lambda^2 + \frac{1}{16}\right)\sqrt{2} + 2 + 16\lambda^8 - 192\lambda^6 + 60\lambda^4 + 12\lambda^2}, \\
A &= \frac{A_1(x + i(2\sqrt{2}\lambda + 2i\lambda^2 + i)t)}{4\sqrt{2}\lambda + 4i\lambda^2 + 2i}, \\
C_1 &= -\frac{h_1 + \sqrt{h_1^2 - 1}}{\sqrt{h_1^2 - 1}}, \qquad C_2 = \frac{h_1 - \sqrt{h_1^2 - 1}}{\sqrt{h_1^2 - 1}}, \\
h &= \frac{1}{2} + \frac{\sqrt{2}}{2} + f^2, \qquad h_1 = 2\sqrt{2}\lambda + 2i\lambda^2 + i.
\end{aligned}$$

We Taylor expand the vector function $\Psi_1(f)$ at $f = 0$,

$$\Psi_1(f) = \Psi_1^{[0]} + \Psi_1^{[1]}f^2 + \Psi_1^{[2]}f^3 + \cdots + \Psi_1^{[N]}f^N + \cdots,$$

where

$$\Psi_1^{[0]} = \begin{pmatrix} (\frac{1}{2} - \frac{1}{2}i)\sqrt{-1-i}e^{it} \\ (\frac{1}{2} + \frac{1}{2}i)\sqrt{-1-i}e^{-it} \end{pmatrix},$$

$$\Psi_1^{[1]} = \begin{pmatrix} (-\frac{1}{4}+\frac{1}{4}i)\sqrt{1-7i}(-x+2t+it)e^{it} \\ (-\frac{1}{20}-\frac{1}{10}i)\sqrt{1-7i}(7t-3x-4-2i+it+ix)e^{-it} \end{pmatrix}.$$

Substituting $\beta = 1$, $\Psi_1^{[0]}$ into (13a)-(13b), we get $q_1 = ie^{(-2i(-x+2t))}$ and $p_1 = -ie^{(2i(-x+2t))}$, while $q_1$ and $p_1$ are not rogue wave solutions that we prefer, so we consider two-fold DT to get the new solution $q^{[2]}$.

Assume $a_1 = \frac{\partial\varphi_1(\lambda_0+f)}{\partial f}|_{f=0}$, $b_1 = \frac{\partial\phi_1(\lambda_0+f)}{\partial f}|_{f=0}$ by a Taylor expansion, we plug $\Psi_1^{[1]}$ into equation (16),

$$q_{\mathrm{rw}}^{[1]} = \frac{(-\widetilde{\lambda}_0|a_1|^2+\widetilde{\lambda}_0^*|b_1|^2)^2}{(\widetilde{\lambda}_0|b_1|^2-\widetilde{\lambda}_0^*|a_1|^2)^2}q - 2i\frac{-(\widetilde{\lambda}_0^2-\widetilde{\lambda}_0^{*2})(-\widetilde{\lambda}_0|a_1|^2+\widetilde{\lambda}_0^*|b_1|^2)\phi_1\varphi_1^*}{(\widetilde{\lambda}_0|b_1|^2-\widetilde{\lambda}_0^*|a_1|^2)^2}, \tag{18}$$

then we obtain the first-order rational solution $q_{\mathrm{rational}}^{[1]}$, which means the one-order rogue wave solution is

$$q_{\mathrm{rw}}^{[1]} = \frac{G_1G_2}{H^2}e^{2it},$$

with

$$\begin{aligned} G_1 &= -\bigl(-3t+1+i+x+ix^2-4ixt+ix+5it^2-3it\bigr), \\ G_2 &= \bigl(5it^2-3it-1-i-4ixt+ix+ix^2+t+x\bigr), \\ H &= \bigl(3t-1+i-x+ix^2-4ixt+ix+5it^2-3it\bigr). \end{aligned}$$

The plot of this solution is shown in Figures 1 and 2. Figure 1 is the three-dimensional plot of $|q_{\mathrm{rw}}^{[1]}|$ with $r = 1$, $a_1 = 0$, $c_1 = 1$, $\beta = 1$, $\alpha = 0$, and Figure 2 is the corresponding density plot. It is shown that the first-order rogue wave solution is localized in both space and time.

Next we calculate the $q_{\mathrm{rw}}^{[2]}$, through the above procedure,

$$\begin{aligned} q_{\mathrm{rw}}^{[2]} = {} & \frac{(-\widetilde{\lambda}_0|a_2^{[2]}|^2+\widetilde{\lambda}_0^*|b_2^{[2]}|^2)^2}{(\widetilde{\lambda}_0|b_2^{[2]}|^2-\widetilde{\lambda}_0^*|a_2^{[2]}|^2)^2}q^{[2]} \\ & - 2i\frac{(\widetilde{\lambda}_0^2-\widetilde{\lambda}_0^{*2})(-\widetilde{\lambda}_0|a_2^{[2]}|^2+\widetilde{\lambda}_0^*|b_2^{[2]}|^2)b_2^{[2]}(a_2^{[2]})^*}{(\widetilde{\lambda}_0|b_2^{[2]}|^2-\widetilde{\lambda}_0^*|a_2^{[2]}|^2)^2}, \end{aligned} \tag{19}$$

**Figure 1 The three-dimensional plot of the first-order wave solution $|q_{\mathrm{rw}}^{[1]}|$ with parameters $r = 1$, $a_1 = 0$, $c_1 = 1$, $\beta = 1$, $\alpha = 0$.**

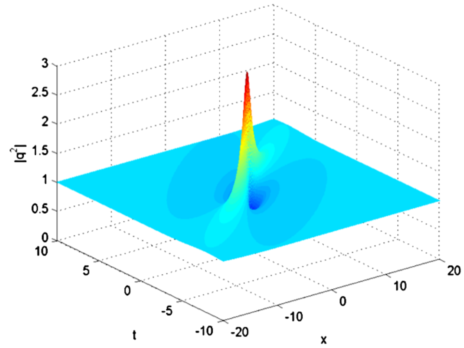


**Figure 2** The corresponding density plot of $|q_{\rm rw}^{[1]}|$.

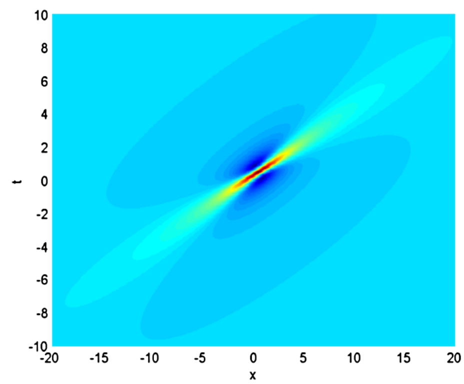
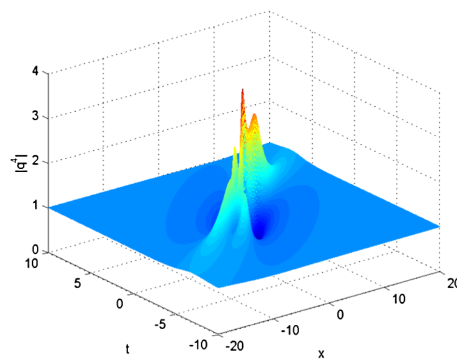



**Figure 3** The three-dimensional plot of the two-order wave solution $|q_{\rm rw}^{[2]}|$ with parameters $r = 1$, $a_1 = 0$, $c_1 = 1$, $\beta = 1$, $\alpha = 0$.




with

$$b_2^{[2]} = \frac{\begin{vmatrix} \frac{\partial^2}{\partial f^2}((\widetilde{\lambda}_0 + if^2)^2\phi_1(\lambda_0 + f)) & \frac{\partial^2}{\partial f^2}((\widetilde{\lambda}_0 + if^2)\varphi_1(\lambda_0 + f)) & \frac{\partial^2}{\partial f^2}\phi_1(\lambda_0 + f) \\ \widetilde{\lambda}_0^2 b_1 & \widetilde{\lambda}_0 a_1 & b_1 \\ -(\widetilde{\lambda}_0^2 a_1)^* & -(\widetilde{\lambda}_0 b_1)^* & -a_1^* \end{vmatrix}}{\begin{vmatrix} \widetilde{\lambda}_0 b_1 & a_1 \\ (\widetilde{\lambda}_0 a_1)^* & b_1^* \end{vmatrix}},$$

$$a_2^{[2]} = \frac{\begin{vmatrix} \frac{\partial^2}{\partial f^2}((\widetilde{\lambda}_0 + if^2)^2\varphi_1(\lambda_0 + f)) & \frac{\partial^2}{\partial f^2}((\widetilde{\lambda}_0 + if^2)\phi_1(\lambda_0 + f)) & \frac{\partial^2}{\partial f^2}\varphi_1(\lambda_0 + f) \\ \widetilde{\lambda}_0^2 a_1 & \widetilde{\lambda}_0 b_1 & a_1 \\ (\widetilde{\lambda}_0^2 b_1)^* & (\widetilde{\lambda}_0 a_1)^* & b_1^* \end{vmatrix}}{\begin{vmatrix} \widetilde{\lambda}_0 a_1 & b_1 \\ -(\widetilde{\lambda}_0 b_1)^* & -a_1^* \end{vmatrix}}.$$

Substituting $a_2^{[2]}$, $b_2^{[2]}$ into equation (19), we get the two-order rogue wave solution

$$q_{\rm rw}^{[2]} = \frac{-G_1 G_3 G_4}{H^2 H_1^2} e^{2it},$$

$G_3$, $G_4$, $H_1$ are given in Appendix 2. The plot of this solution is shown in Figures 3 and 4. Figure 3 is the three-dimensional plot of $|q_{\rm rw}^{[2]}|$ with $r = 1$, $a_1 = 0$, $c_1 = 1$, $\beta = 1$, $\alpha = 0$, and Figure 4 is the corresponding density plot.

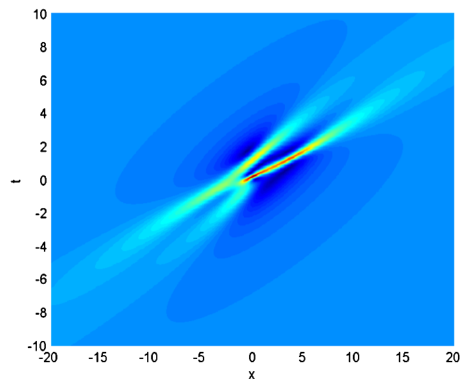



**Figure 4 The corresponding density plot of $|q_{\mathrm{rw}}^{[2]}|$.**

In the same way, the three-order solution can be given as

$$q_{\mathrm{rw}}^{[3]} = \frac{(-\widetilde{\lambda}_0|a_3^{[4]}|^2 + \widetilde{\lambda}_0^*|b_3^{[4]}|^2)^2}{(\widetilde{\lambda}_0|b_3^{[4]}|^2 - \widetilde{\lambda}_0^*|a_3^{[4]}|^2)^2} q^{[4]} - 2i \frac{(\widetilde{\lambda}_0^2 - \widetilde{\lambda}_0^{*2})(-\widetilde{\lambda}_0|a_3^{[4]}|^2 + \widetilde{\lambda}_0^*|b_3^{[4]}|^2)b_3^{[4]}(a_3^{[4]})^*}{(\widetilde{\lambda}_0|b_3^{[4]}|^2 - \widetilde{\lambda}_0^*|a_3^{[4]}|^2)^2}. \quad (20)$$

Through the Darboux transformation, the eigenfunction $\psi_j^{[n]} = T_n\psi_j$ can be given. We can give the $\psi_j^{[4]}$ as follows:

$$\psi_j^{[4]} = \frac{\frac{RM_1}{\Omega_{43}}}{\frac{RM_2}{\Omega_{41}}},$$

where

$$RM_1 = \begin{vmatrix} \widetilde{\lambda}_j^4\phi_j & \widetilde{\lambda}_j^3\varphi_j & \widetilde{\lambda}_j^2\phi_j & \widetilde{\lambda}_j\varphi_j & \phi_j \\ \widetilde{\lambda}_1^4\phi_1 & \widetilde{\lambda}_1^3\varphi_1 & \widetilde{\lambda}_1^2\phi_1 & \widetilde{\lambda}_1\varphi_1 & \phi_1 \\ \widetilde{\lambda}_2^4\phi_2 & \widetilde{\lambda}_2^3\varphi_2 & \widetilde{\lambda}_2^2\phi_2 & \widetilde{\lambda}_2\varphi_2 & \phi_2 \\ \widetilde{\lambda}_3^4\phi_3 & \widetilde{\lambda}_3^3\varphi_3 & \widetilde{\lambda}_3^2\phi_3 & \widetilde{\lambda}_3\varphi_3 & \phi_3 \\ \widetilde{\lambda}_4^4\phi_4 & \widetilde{\lambda}_4^3\varphi_4 & \widetilde{\lambda}_4^2\phi_4 & \widetilde{\lambda}_4\varphi_4 & \phi_4 \end{vmatrix},$$

$$RM_2 = \begin{vmatrix} \widetilde{\lambda}_j^4\varphi_j & \widetilde{\lambda}_j^3\phi_j & \widetilde{\lambda}_j^2\varphi_j & \widetilde{\lambda}_j\phi_j & \varphi_j \\ \widetilde{\lambda}_1^4\varphi_1 & \widetilde{\lambda}_1^3\phi_1 & \widetilde{\lambda}_1^2\varphi_1 & \widetilde{\lambda}_1\phi_1 & \varphi_1 \\ \widetilde{\lambda}_2^4\varphi_2 & \widetilde{\lambda}_2^3\phi_2 & \widetilde{\lambda}_2^2\varphi_2 & \widetilde{\lambda}_2\phi_2 & \varphi_2 \\ \widetilde{\lambda}_3^4\varphi_3 & \widetilde{\lambda}_3^3\phi_3 & \widetilde{\lambda}_3^2\varphi_3 & \widetilde{\lambda}_3\phi_3 & \varphi_3 \\ \widetilde{\lambda}_4^4\varphi_4 & \widetilde{\lambda}_4^3\phi_4 & \widetilde{\lambda}_4^2\varphi_4 & \widetilde{\lambda}_4\phi_4 & \varphi_4 \end{vmatrix},$$

and $a_3^{[4]}$, $b_3^{[4]}$ are given as follows:

$$b_3^{[4]} = \frac{\begin{vmatrix} rm_{41}^3 & rm_{32}^3 & rm_{21}^3 & rm_{12}^3 & rm_{01}^3 \\ rm_{41}^1 & rm_{32}^1 & rm_{21}^1 & rm_{12}^1 & rm_{01}^1 \\ -rm_{41}^{1*} & -rm_{32}^{1*} & -rm_{21}^{1*} & -rm_{12}^{1*} & -rm_{01}^{1*} \\ rm_{41}^2 & rm_{32}^2 & rm_{21}^2 & rm_{12}^2 & rm_{01}^2 \\ -rm_{41}^{2*} & -rm_{32}^{2*} & -rm_{21}^{2*} & -rm_{12}^{2*} & -rm_{01}^{2*} \end{vmatrix}}{\begin{vmatrix} rm_{31}^1 & rm_{22}^1 & rm_{11}^1 & rm_{02}^1 \\ -rm_{31}^{1*} & -rm_{22}^{1*} & -rm_{11}^{1*} & -rm_{02}^{1*} \\ rm_{31}^2 & rm_{22}^2 & rm_{11}^2 & rm_{02}^2 \\ -rm_{31}^{2*} & -rm_{22}^{2*} & -rm_{11}^{2*} & -rm_{02}^{2*} \end{vmatrix}},$$

**Figure 5 The three-dimensional plot of the two-order wave solution $|q_{\mathrm{rw}}^{[3]}|$ with $r = 1$, $a_1 = 0$, $c_1 = 1$, $\beta = 1$, $\alpha = 0$.**

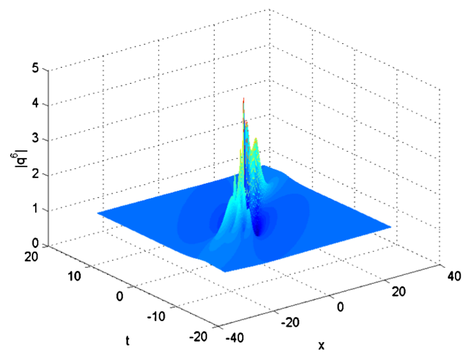
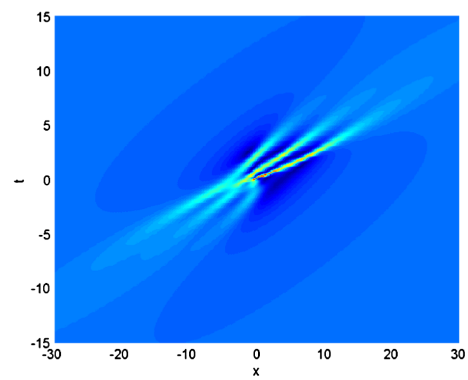



**Figure 6 The corresponding density plot of $|q_{\mathrm{rw}}^{[3]}|$.**




$$
a_3^{[4]} = \frac{\begin{vmatrix} rm_{42}^3 & rmm_{31}^3 & rm_{22}^3 & rm_{11}^3 & rm_{02}^3 \\ rm_{42}^1 & rm_{31}^1 & rm_{22}^1 & rm_{11}^1 & rm_{02}^1 \\ -rm_{42}^{1*} & -rm_{31}^{1*} & -rm_{22}^{1*} & -rm_{11}^{1*} & -rm_{02}^{1*} \\ rm_{42}^2 & rm_{31}^2 & rm_{22}^2 & rm_{11}^2 & rm_{02}^2 \\ -rm_{42}^{2*} & -rm_{31}^{2*} & -rm_{22}^{2*} & -rm_{11}^{2*} & -rm_{02}^{2*} \end{vmatrix}}{\begin{vmatrix} rm_{32}^1 & rm_{21}^1 & rm_{12}^1 & rm_{01}^1 \\ -rm_{32}^{1*} & -rm_{21}^{1*} & -rm_{12}^{1*} & -rm_{01}^{1*} \\ rm_{32}^2 & rm_{21}^2 & rm_{12}^2 & rm_{01}^2 \\ -rm_{32}^{2*} & -rm_{21}^{2*} & -rm_{12}^{2*} & -rm_{01}^{2*} \end{vmatrix}},
$$

where $\widetilde{\lambda}_2 = -\widetilde{\lambda}_1^*$, $\widetilde{\lambda}_4 = -\widetilde{\lambda}_3^*$, and the corresponding solution of the Lax pair is $(-\varphi_1^* \ \phi_1^*)^T$, and

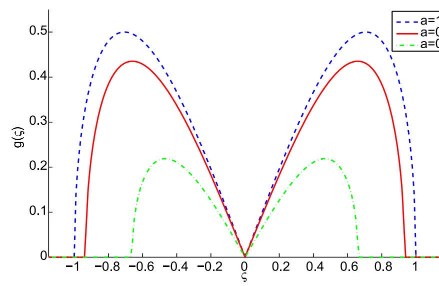
$$
rm_{j1}^l = \frac{\partial^l}{\partial f^l}\bigl(\bigl(\widetilde{\lambda}_0 + if^2\bigr)^j \phi_1\bigl(\lambda_0 + if^2\bigr)\bigr), \qquad rm_{j2}^l = \frac{\partial^l}{\partial f^l}\bigl(\bigl(\widetilde{\lambda}_0 + if^2\bigr)^j \varphi_1\bigl(\lambda_0 + if^2\bigr)\bigr).
$$

Similarly, the figures of three-order rogue wave solution can be given. Figure 5 is the three-dimensional plot of $|q_{\mathrm{rw}}^{[3]}|$ with $r = 1$, $a_1 = 0$, $c_1 = 1$, $\beta = 1$, $\alpha = 0$, and Figure 6 is the corresponding density plot.

## 4 Modulation instability

In order to discuss the modulation instability [50] of the mixed NLS equation (5), we first consider the steady state solution $q_{m1} = ae^{i\omega t}$, where $\omega = a^2 + r$. Then putting a small perturbation $U_1$ onto the steady state solution, we have $q_{m2} = (a + U_1)e^{i\omega t}$.

**Figure 7 Gain spectra of modulation instability for the parameter value $\beta = 1$, and for the different power levels $a = 1$, $a = 0.8$, $a = 0.5$.**




After the simple substitution and linearizing the equation, we have

$$iU_{1t} + U_{1xx} + i\beta a^2(2U_{1x} + \widetilde{U}_{1x}) + i\alpha U_{1x} + a^2(U_1 + \bar{U}_1) = 0.$$

Immediately following this we set

$$U_1 = \mu\cos(\varsigma x + \tau t) + i\nu\sin(\varsigma x + \tau t),$$

where $\varsigma$ is the frequency and $\tau$ is the wave number of the perturbation. Substituting the assumed solution into the linearized equation, we have a split of real and imaginary parts, namely

$$\mu(-\varsigma^2 + 2a^2) + \nu(-\tau - a^2\beta\varsigma - \alpha\varsigma) = 0,$$
$$\mu(-\tau - 3a^2\beta\varsigma - \alpha\varsigma) + \nu(-\varsigma^2) = 0.$$

According to the initial condition we have the following determinant:

$$(\tau + a^2\beta\varsigma)(\tau + 3a^2\beta\varsigma + \alpha\varsigma) - \varsigma^2(\varsigma^2 - 2a^2) = 0.$$

Solving the determinant we obtain the following dispersion relation:

$$\tau = -2a^2 - \alpha \pm \varsigma\sqrt{\varsigma^2 + a^4\beta^2 - 2a^2}.$$

Depending on the relation above, we know that if $\varsigma^2 + a^4\beta^2 - 2a^2 > 0$ the frequency $\varsigma$ is real at any value of the wave number $\tau$, whereas $\varsigma$ becomes complex. That is to say, in the case of $\varsigma^2 + a^4\beta^2 - 2a^2 < 0$ the instability region of the disturbance will grow with time exponentially. Then we consider the gain spectrum of MI. According to the above result, we have

$$g(\varsigma) = 2\,\mathrm{Im}(\tau) = 2\sqrt{(\varsigma^2 + a^4\beta^2 - 2a^2)\varsigma^2},$$

where $g(\varsigma)$ represents the gain of the frequency $\varsigma$. Figure 7 shows the gain at three different power levels.

## 5 Conclusions

In conclusion, we have constructed the one-, two-, and three-order rogue wave solutions for the general mixed NLS equation (5) by using the generalized DT. The $n$-fold DT is given by a gauge transformation, then a generalized DT is proposed through the Taylor expansion and a limit procedure. Moreover, some exact rogue wave solutions are derived explicitly. What is more, it is found from the plots of the rogue waves that the rogue waves appear suddenly and disappear quickly in the space-time framework. Finally, we give evidence for the connection between the occurrence of the rogue wave solution and the modulation instability.

## Appendix 1

In this paper, we use the $T_n$ when $n$ is even [41]; $T_n|_{n=2k}$ $(k = 1, 2, 3, \ldots)$ can be expressed as follows:

$$T_n = \begin{pmatrix} \frac{T_{11}}{\Delta_1} & \frac{T_{12}}{\Delta_1} \\ \frac{T_{13}}{\Delta_2} & \frac{T_{14}}{\Delta_2} \end{pmatrix},$$

where

$$T_{11} = \begin{vmatrix} \widetilde{\lambda}^n & 0 & \cdots & \widetilde{\lambda}^2 & 0 & 1 \\ \widetilde{\lambda}_1^n \phi_1 & \widetilde{\lambda}_1^{n-1} \varphi_1 & \cdots & \widetilde{\lambda}_1^2 \phi_1 & \widetilde{\lambda}_1 \varphi_1 & \phi_1 \\ \widetilde{\lambda}_2^n \phi_2 & \widetilde{\lambda}_2^{n-1} \varphi_2 & \cdots & \widetilde{\lambda}_2^2 \phi_2 & \widetilde{\lambda}_2 \varphi_2 & \phi_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \widetilde{\lambda}_{n-1}^n \phi_{n-1} & \widetilde{\lambda}_{n-1}^{n-1} \varphi_{n-1} & \cdots & \widetilde{\lambda}_{n-1}^2 \phi_{n-1} & \widetilde{\lambda}_{n-1} \varphi_{n-1} & \phi_{n-1} \\ \widetilde{\lambda}_n^n \phi_n & \widetilde{\lambda}_n^{n-1} \varphi_n & \cdots & \widetilde{\lambda}_n^2 \phi_n & \widetilde{\lambda}_n \varphi_n & \phi_n \end{vmatrix},$$

$$T_{12} = \begin{vmatrix} 0 & \widetilde{\lambda}^{n-1} & \cdots & 0 & \widetilde{\lambda} & 0 \\ \widetilde{\lambda}_1^n \phi_1 & \widetilde{\lambda}_1^{n-1} \varphi_1 & \cdots & \widetilde{\lambda}_1^2 \phi_1 & \widetilde{\lambda}_1 \varphi_1 & \phi_1 \\ \widetilde{\lambda}_2^n \phi_2 & \widetilde{\lambda}_2^{n-1} \varphi_2 & \cdots & \widetilde{\lambda}_2^2 \phi_2 & \widetilde{\lambda}_2 \varphi_2 & \phi_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \widetilde{\lambda}_{n-1}^n \phi_{n-1} & \widetilde{\lambda}_{n-1}^{n-1} \varphi_{n-1} & \cdots & \widetilde{\lambda}_{n-1}^2 \phi_{n-1} & \widetilde{\lambda}_{n-1} \varphi_{n-1} & \phi_{n-1} \\ \widetilde{\lambda}_n^n \phi_n & \widetilde{\lambda}_n^{n-1} \varphi_n & \cdots & \widetilde{\lambda}_n^2 \phi_n & \widetilde{\lambda}_n \varphi_n & \phi_n \end{vmatrix},$$

$$T_{21} = \begin{vmatrix} 0 & \widetilde{\lambda}^{n-1} & \cdots & 0 & \widetilde{\lambda} & 0 \\ \widetilde{\lambda}_1^n \varphi_1 & \widetilde{\lambda}_1^{n-1} \phi_1 & \cdots & \widetilde{\lambda}_1^2 \varphi_1 & \widetilde{\lambda}_1 \phi_1 & \varphi_1 \\ \widetilde{\lambda}_2^n \varphi_2 & \widetilde{\lambda}_2^{n-1} \phi_2 & \cdots & \widetilde{\lambda}_2^2 \varphi_2 & \widetilde{\lambda}_2 \phi_2 & \varphi_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \widetilde{\lambda}_{n-1}^n \varphi_{n-1} & \widetilde{\lambda}_{n-1}^{n-1} \phi_{n-1} & \cdots & \widetilde{\lambda}_{n-1}^2 \varphi_{n-1} & \widetilde{\lambda}_{n-1} \phi_{n-1} & \varphi_{n-1} \\ \widetilde{\lambda}_n^n \varphi_n & \widetilde{\lambda}_n^{n-1} \phi_n & \cdots & \widetilde{\lambda}_n^2 \varphi_n & \widetilde{\lambda}_n \phi_n & \varphi_n \end{vmatrix},$$

$$T_{22} = \begin{vmatrix} \widetilde{\lambda}^n & 0 & \cdots & \widetilde{\lambda}^2 & 0 & 1 \\ \widetilde{\lambda}_1^n \varphi & \widetilde{\lambda}_1^{n-1} \phi_1 & \cdots & \widetilde{\lambda}_1^2 \varphi_1 & \widetilde{\lambda}_1 \phi_1 & \varphi_1 \\ \widetilde{\lambda}_2^n \varphi_2 & \widetilde{\lambda}_2^{n-1} \phi_2 & \cdots & \widetilde{\lambda}_2^2 \varphi_2 & \widetilde{\lambda}_2 \phi_2 & \varphi_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \widetilde{\lambda}_{n-1}^n \varphi_{n-1} & \widetilde{\lambda}_{n-1}^{n-1} \phi_{n-1} & \cdots & \widetilde{\lambda}_{n-1}^2 \varphi_{n-1} & \widetilde{\lambda}_{n-1} \phi_{n-1} & \varphi_{n-1} \\ \widetilde{\lambda}_n^n \varphi_n & \widetilde{\lambda}_n^{n-1} \phi_n & \cdots & \widetilde{\lambda}_n^2 \varphi_n & \widetilde{\lambda}_n \phi_n & \varphi_n \end{vmatrix},$$

$$\Delta_1 = \begin{vmatrix} \widetilde{\lambda}_1^{n-1}\phi_1 & \widetilde{\lambda}_1^{n-2}\varphi_1 & \cdots & \widetilde{\lambda}_1\phi_1 & \varphi_1 \\ \widetilde{\lambda}_2^{n-1}\phi_2 & \widetilde{\lambda}_2^{n-2}\varphi_2 & \cdots & \widetilde{\lambda}_2\phi_2 & \varphi_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \widetilde{\lambda}_{n-1}^{n-1}\phi_{n-1} & \widetilde{\lambda}_{n-1}^{n-2}\varphi_{n-1} & \cdots & \widetilde{\lambda}_{n-1}\phi_{n-1} & \varphi_{n-1} \\ \widetilde{\lambda}_n^{n-1}\phi_n & \widetilde{\lambda}_n^{n-2}\varphi_n & \cdots & \widetilde{\lambda}_n\phi_n & \varphi_n \end{vmatrix},$$

$$\Delta_2 = \begin{vmatrix} \widetilde{\lambda}_1^{n-1}\varphi_1 & \widetilde{\lambda}_1^{n-2}\phi_1 & \cdots & \widetilde{\lambda}_1\varphi_1 & \phi_1 \\ \widetilde{\lambda}_2^{n-1}\varphi_2 & \widetilde{\lambda}_2^{n-2}\phi_2 & \cdots & \widetilde{\lambda}_2\varphi_1 & \phi_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \widetilde{\lambda}_{n-1}^{n-1}\varphi_2 & \widetilde{\lambda}_{n-1}^{n-2}\phi_{n-1} & \cdots & \widetilde{\lambda}_{n-1}\varphi_1 & \phi_{n-1} \\ \widetilde{\lambda}_n^{n-1}\varphi_2 & \widetilde{\lambda}_n^{n-2}\phi_n & \cdots & \widetilde{\lambda}_n\varphi_1 & \phi_n \end{vmatrix}.$$

## Appendix 2

The rational solution of a two-order rogue wave is

$$\begin{aligned} G_3 ={}& -162t + 54x - 162it - 360tx + 72x^2 + 432t^2 + 1{,}152t^2x - 504tx^2 + 72x^3 \\ & - 864t^3 + 60x^4 + 2{,}052t^4 - 3{,}120t^3x + 1{,}872t^2x^2 - 528tx^3 + 24x^5 - 2{,}196t^5 \\ & - 276x^4t + 4{,}104t^4x - 3{,}192t^3x^2 + 1{,}296t^2x^3 + 198x^5t^2 + 2{,}050t^6x - 2{,}490t^5x^2 \\ & + 1{,}734t^4x^3 - 3{,}660t^5x + 3{,}624t^4x^2 - 108x^5t + 642x^4t^2 - 1{,}976t^3x^3 - 30x^6t \\ & + 8x^6 + 1{,}600t^6 - 750t^7 + 2x^7 + ix^8 + 54ix + 666it^2 + 90ix^2 - 1{,}224it^3 + 72ix^3 \\ & + 48ix^4 + 1{,}752it^4 + 18ix^5 - 1{,}314it^5 + 10ix^6 + 1{,}490it^6 - 1{,}500it^7 + 4ix^7 + 18i \\ & + 18 + 625it^8 - 432itx + 1{,}224it^2x - 504itx^2 + 1{,}512it^2x^2 - 2{,}448it^3x - 432itx^3 \\ & - 2{,}148it^3x^2 + 2{,}541it^4x + 948it^2x^3 - 1{,}492ix^4t^3 - 210itx^4 - 132ix^5t + 726ix^4t^2 \\ & - 3{,}492it^5x + 3{,}678it^4x^2 - 2{,}152it^3x^3 - 60i^6t + 396ix^5t^2 + 4{,}100it^6x \\ & - 4{,}980it^5x^2 + 3{,}468it^4x^3 + 1{,}366ix^4t^4 - 496ix^5t^3 + 116ix^6t^2 \\ & - 16ix^7t - 16ix^7t - 2{,}000it^7x + 2{,}900it^6x^2 - 2{,}480it^5x^3, \\ G_4 ={}& -288t - 25{,}002it^4x - 6{,}768it^2x^2 + 36i + 15{,}828it^3x^2 - 864xt + 36x^2 + 2{,}340t^2 \\ & - 2{,}256ix^4t^2 + 5{,}940t^2x, +72x^3 - 7{,}872t^3 + 78x^4 + 13{,}950t^4 - 15{,}816t^3x \\ & + 6{,}732t^2x^2 - 1{,}224tx^3 + 42x^5 - 15{,}006t^5 - 654x^4t + 22{,}410t^4x - 13{,}596t^3x^2 \\ & + 4{,}164t^2x^3 - 1{,}638x^5t^2 - 15{,}602t^6x + 21{,}030t^5x^2 + 30{,}520t^7x - 43{,}628t^6x^2 \\ & + 152x^7t - 1{,}324x^6t^2 - 18{,}986t^4x^4 - 15{,}438t^4x^3 + 6{,}376x^5t^3 + 6{,}622x^4t^3 \\ & - 13{,}596t^5x + 9{,}552t^4x^2 - 156x^5t + 888x^4t^2 - 3{,}640t^3x^3 + 210x^6t + 9{,}375t^9 \\ & - 9{,}575t^8 - 5x^9 - 7x^8 + 16x^6 + 7{,}976t^6 + 4{,}890t^7 - 10x^7 - 3{,}125t^{10} - x^{10} \\ & - 120ix^7t + 1{,}260it^6x^3 - 24{,}420it^6x - 144ix^2 + 300it^2x^7 + 324ix^5t + 1{,}848itx^3 \\ & - 33{,}125t^8x + 53{,}500t^7x^2 - 51{,}700t^6x^3 + 32{,}890t^5x^4 - 14{,}270t^4x^5 + 4{,}220t^3x^6 \\ & - 820t^2x^7 + 95tx^8 + 1{,}040t^3x^7 + 12{,}500t^9x + 660ix^6t^2 + 13{,}800it^7x \\ & - 23{,}125t^8x^2 + 2{,}600t^7x^3 - 19{,}650t^6x^4 + 10{,}424t^5x^5 - 393 - t^4x^6 - 185t^2x^8 \end{aligned}$$

$$
\begin{aligned}
&+ 20tx^9 + 32{,}580it^5x - 23{,}736it^4x^2 + 9{,}480ix^3t^3 - 3{,}342it^5x^4 - 144it - 432it^2 \\
&+ 1{,}656itx^2 - 5{,}676it^2x^3 + 1{,}182itx^4 - 144ix^4t^3 + 1{,}830ix^4t^4 - 1{,}800ix^5t^3 \\
&- 144ix^6t + 564ix^5t^2 + 17{,}808it^5x^2 - 5{,}436it^4x^3 + 12{,}984it^3x - 11{,}340it^6x^2 \\
&+ 2{,}904it^4x^3 - 4{,}125it^8x + 2{,}700it^7x^2 + 2{,}610it^4x^5 - 1{,}140it^3x^6 - 45itx^8 \\
&+ 504itx - 4{,}032it^2x + 3{,}984it^3 - 264ix^3 - 216ix^4 - 10{,}968it^4 - 1{,}368tx^2 \\
&- 114ix^5 + 17{,}478it^5 - 24ix^6 - 19{,}200it^6 + 13{,}200it^7 + 12ix^7 \\
&+ 9ix^8 - 6{,}375it^8 + 1{,}875it^9 + 3ix^9 + 36{,}232t^5x, \\
H_1 ={}& 162t - 54x - 162it + 360xt - 72x^2 - 432t^2 - 1{,}152t^2x + 504tx^2 - 72x^3 + 864t^3 \\
&- 60x^4 - 2{,}052t^4 + 3{,}120xt^3 - 18 - 1{,}872x^2t^2 + 528x^3t - 24x^5 + 2{,}196t^5 \\
&+ 276x^4t - 4{,}104xt^4 + 3{,}192x^2t^3 - 1{,}296x^3t^2 - 198x^5t^2 - 2{,}050xt^6 + 2{,}490x^2t^5 \\
&- 1{,}734x^3t^4 + 746x^4t^3 + 3{,}660xt^5 - 3{,}624x^2t^4 + 108x^5t - 624x^4t^2 + 1{,}976x^3t^3 \\
&+ 30x^6t - 8x^6 - 1{,}600t^6 + 750t^7 + 18i - 2x^7 + ix^8 + 54ix + 666it^2 + 90ix^2 \\
&- 1{,}224it^3 + 72ix^3 + 48ix^4 + 1{,}752it^4 + 18ix^5 - 1{,}314it^5 + 10ix^6 + 1{,}490ix^6 \\
&- 1{,}500it^7 + 4ix^7 + 625it^8 - 432ixt + 1{,}224ixt^2 - 504ix^2t + 1{,}512ix^2t^2 \\
&- 2{,}448ixt^3 - 432ix^3t - 2{,}148ix^2t^3 + 2{,}541ixt^4 + 948ix^3t^3 - 210ix^4t \\
&- 132ix^5t + 726ix^5t^2 - 3{,}492it^5x + 3{,}678ix^2t^4 - 2{,}125it^3x^3 - 1{,}492ix^4t^3 \\
&- 60ix^6t + 396ix^5t^2 + 4{,}100ixt^6 - 4{,}980ix^2t^5 + 3{,}468ix^3t^4 + 1{,}336ix^4t^4 \\
&- 496ix^5t^3 + 116ix^6t^2 - 16ix^7t - 2{,}000it^7x + 2{,}900ix^2t^6 - 2{,}480ix^3t^6.
\end{aligned}
$$

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
All authors have made equal contributions. All authors have read and approved the final manuscript.

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