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The eigenvalue problem for a coupled system of singular p-Laplacian differential equations involving fractional differential-integral conditions

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Abstract

In this paper, we deal with a coupled system of singular p-Laplacian differential equations involving fractional differential-integral conditions. By employing Schauder's fixed point theorem and the upper and lower solution method, we establish an eigenvalue interval for the existence of positive solutions. As an application an example is presented to illustrate the main results.

MSC: 34B15

Keywords: upper and lower solutions; p-Laplacian operator; eigenvalue; fractional differential equation

1 Introduction

In this paper, we consider the following system of nonlinear fractional differential equations with different fractional derivatives:

$$\begin{cases} -D^{\beta_1}(\varphi_{p_1}(-D^{\alpha_1}u_1))(t) = \lambda f_1(u_1(t), D^{\gamma_1}u_1(t), D^{\gamma_2}u_2(t)), \quad 0 < t < 1, \\ -D^{\beta_2}(\varphi_{p_2}(-D^{\alpha_2}u_2))(t) = f_2(t, u_1(t)), \\ D^{\alpha_i}u_i(0) = D^{\alpha_i}u_i(1) = 0, \\ D^{\gamma_i}u_i(0) = 0, \qquad D^{\alpha_i - 1}u_i(1) = \xi_i I^{\omega_i}(D^{\gamma_i}u_i(\eta_i)), \quad i = 1, 2, \end{cases} \tag{1.1}$$

where $D^{\alpha_i}, D^{\beta_i}, D^{\gamma_i}$ $(i = 1, 2)$ are the standard Riemannn-Liouville fractional derivatives, I^{ω_i} is the Riemann-Liouville fractional integral, φ_{p_i} is the p-Laplacian operator defined by $\varphi_{p_i}(s) = |s|^{p_i-2}s, p_i > 2$ $(i = 1, 2)$, and the nonlinearity $f_1(x, y, z)$ may be singular at $x = 0, y = 0, z = 0$.

Throughout this paper, we always suppose that:

(s_0) $0 < \gamma_i \le 1 < \alpha_i < \beta_i < 2, \alpha_1 - \gamma_1 > 1, \alpha_2 - \gamma_2 > 1, \omega_i > 0, \xi_i > 0, \eta_i \in [0, 1]$ $(i = 1, 2)$.

(s_1) $\Gamma(\alpha_i - \gamma_i + \omega_i) > \xi_i \eta_i^{\alpha_i - \gamma_i + \omega_i - 1}$ $(i = 1, 2)$.

(s_2) Let q_i satisfies the relation $\frac{1}{q_i} + \frac{1}{p_i} = 1$, where p_i is given by (1.1), then $1 < q_i < 2$.

Fractional calculus provides an excellent tool for describing the hereditary properties of various materials and processes. Concerning the development of theory, method and application of fractional calculus, we refer the reader to the recent papers [1–8].

On the other hand, the study of coupled systems involving fractional differential equations is also important as such systems occur in various problems of applied nature. So considerable work has been done to study the existence result for them nowadays [9–12]. The authors got the existence solutions by the method of the fixed point theorem, the coincidence degree theorem, or Schauder's fixed point theorem.

The theory of upper and lower solutions is well known to be an effective method to deal with the existence of solutions for the boundary value problems of the fractional differential equations. In [13] the authors used the method of upper and lower solutions and investigated the existence of solutions for initial value problems. By the same method some people got the solutions of boundary value problems for fractional differential equations, such as [14, 15]. To the best of our knowledge, only few papers considered the existence of solutions by using the method of upper and lower solutions for boundary value problems with fractional coupled systems.

The aim of this paper is to deal with the eigenvalue problem for a coupled system of fractional differential equations involving differential-integral conditions. The novelty of this paper is that the nonlinear terms f_1, f_2 in the system (1.1) involve different unknown functions $u_1(t)$, $u_2(t)$ and their Riemann-Liouville fractional derivatives with different orders, and $f_1(x,y,z)$ may be singular at $x = 0, y = 0, z = 0$. We establish an eigenvalue interval for the existence of positive solutions by Schauder's fixed point theorem and the upper and lower solutions method.

2 Preliminaries and lemmas

Lemma 2.1 ([16]) *Let $h_i \in L^1(0,1)$, then the problem*

$$\begin{cases} -D^{\beta_i} v_i(t) = h_i(t), & 0 < t < 1, \\ v_i(0) = v_i(1) = 0, \end{cases}$$

has the unique solution $v_i(t) = \int_0^1 G(\beta_i, t, s) h_i(s)\, ds$ $(i = 1, 2)$, where

$$G(\beta_i, t, s) = \begin{cases} \frac{1}{\Gamma(\beta_i)} [t(1-s)]^{\beta_i - 1}, & t \leq s, \\ \frac{1}{\Gamma(\beta_i)} \{[t(1-s)]^{\beta_i - 1} - (t-s)^{\beta_i - 1}\}, & s \leq t. \end{cases}$$

Lemma 2.2 ([17]) *Let $h_i \in L^1(0,1)$, then the fractional integral boundary value problem*

$$\begin{cases} -D^{\alpha_i - \gamma_i} v_i(t) = h_i(t), & 0 < t < 1, \\ v_i(0) = 0, \qquad D^{\alpha_i - \gamma_i - 1} v_i(1) = \xi_i I^{\omega_i} v_i(\eta_i), & i = 1, 2, \end{cases}$$

has a unique solution $v_i(t) = \int_0^1 H_i(t,s) h_i(s)\, ds$, where

$$H_i(t,s) = \begin{cases} \frac{[\Gamma(\alpha_i-\gamma_i+\omega_i)-\xi_i(\eta_i-s)^{\alpha_i-\gamma_i+\omega_i-1}]t^{\alpha_i-\gamma_i-1}-[\Gamma(\alpha_i-\gamma_i+\omega_i)-\xi_i\eta_i^{\alpha_i-\gamma_i+\omega_i-1}](t-s)^{\alpha_i-\gamma_i-1}}{\Delta_i}, & s \leq t, s \leq \eta_i; \\ \frac{[\Gamma(\alpha_i-\gamma_i+\omega_i)-\xi_i(\eta_i-s)^{\alpha_i-\gamma_i+\omega_i-1}]t^{\alpha_i-\gamma_i-1}}{\Delta_i}, & t \leq s \leq \eta_i; \\ \frac{\Gamma(\alpha_i-\gamma_i+\omega_i)[t^{\alpha_i-\gamma_i-1}-(t-s)^{\alpha_i-\gamma_i-1}]+\xi_i\eta_i^{\alpha_i-\gamma_i+\omega_i-1}(t-s)^{\alpha_i-\gamma_i-1}}{\Delta_i}, & \eta_i \leq s \leq t; \\ \frac{\Gamma(\alpha_i-\gamma_i+\omega_i)t^{\alpha_i-\gamma_i-1}}{\Delta_i}, & s \geq t, s \geq \eta_i, \end{cases}$$

and $\Delta_i = \Gamma(\alpha_i - \gamma_i)[\Gamma(\alpha_i - \gamma_i + \omega_i) - \xi_i \eta_i^{\alpha_i - \gamma_i + \omega_i - 1}]$.

Lemma 2.3 ([17, 18]) *The functions $G(\beta_i, t, s)$ and $H_i(t, s)$ have the following properties*:

(1) $G(\beta_i, t, s) > 0$, $H_i(t, s) > 0$, *for* $t, s \in (0, 1)$.

(2)
$$\frac{t^{\beta_i-1}(1-t)s(1-s)^{\beta_i-1}}{\Gamma(\beta_i)} \le G(\beta_i, t, s) \le \frac{\beta_i - 1}{\Gamma(\beta_i)} t^{\beta_i-1}(1-t), \quad \text{for } t, s \in [0, 1].$$

(3)
$$e_i t^{\alpha_i-\gamma_i-1}\big[1-(1-s)^{\alpha_i-\gamma_i-1}\big] \le H_i(t, s) \le d_i t^{\alpha_i-\gamma_i-1}, \quad \text{for } t, s \in [0, 1],$$

where $d_i = \frac{1}{\Delta_i}[\Gamma(\alpha_i - \gamma_i + \omega_i) + \xi_i \eta_i^{\alpha_i-\gamma_i+\omega_i-1}]$, $e_i = \Gamma(\alpha_i - \gamma_i)$.

Proof From [18], we can see that $G(\beta_i, t, s) > 0$ and (2) hold.

In the following, we will prove (3).

For $s \le t, s \le \eta_i$,

$$\begin{aligned}
H_i(t, s) &= \frac{1}{\Delta_i}\Big\{\big[\Gamma(\alpha_i - \gamma_i + \omega_i) - \xi_i(\eta_i - s)^{\alpha_i-\gamma_i+\omega_i-1}\big]t^{\alpha_i-\gamma_i-1} \\
&\quad - \big[\Gamma(\alpha_i - \gamma_i + \omega_i) - \xi_i \eta_i^{\alpha_i-\gamma_i+\omega_i-1}\big](t-s)^{\alpha_i-\gamma_i-1}\Big\} \\
&\ge \frac{1}{\Delta_i}\big[\Gamma(\alpha_i - \gamma_i + \omega_i) - \xi_i \eta_i^{\alpha_i-\gamma_i+\omega_i-1}\big]\big[t^{\alpha_i-\gamma_i-1} - (t-s)^{\alpha_i-\gamma_i-1}\big] \\
&\ge \frac{1}{\Delta_i}\big[\Gamma(\alpha_i - \gamma_i + \omega_i) - \xi_i \eta_i^{\alpha_i-\gamma_i+\omega_i-1}\big]\big[t^{\alpha_i-\gamma_i-1} - (t-ts)^{\alpha_i-\gamma_i-1}\big] \\
&= \frac{1}{\Delta_i}\big[\Gamma(\alpha_i - \gamma_i + \omega_i) - \xi_i \eta_i^{\alpha_i-\gamma_i+\omega_i-1}\big]t^{\alpha_i-\gamma_i-1}\big[1-(1-s)^{\alpha_i-\gamma_i-1}\big] \\
&= \Gamma(\alpha_i - \gamma_i)t^{\alpha_i-\gamma_i-1}\big[1-(1-s)^{\alpha_i-\gamma_i-1}\big] \\
&= e_i t^{\alpha_i-\gamma_i-1}\big[1-(1-s)^{\alpha_i-\gamma_i-1}\big], \\
H_i(t, s) &= \frac{1}{\Delta_i}\big[\Gamma(\alpha_i - \gamma_i + \omega_i) - \xi_i(\eta_i - s)^{\alpha_i-\gamma_i+\omega_i-1}\big]t^{\alpha_i-\gamma_i-1} \\
&\quad - \big[\Gamma(\alpha_i - \gamma_i + \omega_i) - \xi_i \eta_i^{\alpha_i-\gamma_i+\omega_i-1}\big](t-s)^{\alpha_i-\gamma_i-1} \\
&\le \frac{1}{\Delta_i}\big[\Gamma(\alpha_i - \gamma_i + \omega_i)t^{\alpha_i-\gamma_i-1} + \xi_i \eta_i^{\alpha_i-\gamma_i+\omega_i-1}(t-s)^{\alpha_i-\gamma_i-1}\big] \\
&\le \frac{1}{\Delta_i}\big[\Gamma(\alpha_i - \gamma_i + \omega_i) + \xi_i \eta_i^{\alpha_i-\gamma_i+\omega_i-1}\big]t^{\alpha_i-\gamma_i-1} \\
&= d_i t^{\alpha_i-\gamma_i-1}.
\end{aligned}$$

For $t \le s \le \eta_i$,

$$\begin{aligned}
H_i(t, s) &= \frac{1}{\Delta_i}\big[\Gamma(\alpha_i - \gamma_i + \omega_i) - \xi_i(\eta_i - s)^{\alpha_i-\gamma_i+\omega_i-1}\big]t^{\alpha_i-\gamma_i-1} \\
&\ge \frac{1}{\Delta_i}\big[\Gamma(\alpha_i - \gamma_i + \omega_i) - \xi_i \eta_i^{\alpha_i-\gamma_i+\omega_i-1}\big]t^{\alpha_i-\gamma_i-1} \\
&\ge \frac{1}{\Delta_i}\big[\Gamma(\alpha_i - \gamma_i + \omega_i) - \xi_i \eta_i^{\alpha_i-\gamma_i+\omega_i-1}\big]t^{\alpha_i-\gamma_i-1}\big[1-(1-s)^{\alpha_i-\gamma_i-1}\big] \\
&= \Gamma(\alpha_i - \gamma_i)t^{\alpha_i-\gamma_i-1}\big[1-(1-s)^{\alpha_i-\gamma_i-1}\big] \\
&= e_i t^{\alpha_i-\gamma_i-1}\big[1-(1-s)^{\alpha_i-\gamma_i-1}\big],
\end{aligned}$$

$$
\begin{aligned}
H_i(t,s) &= \frac{1}{\Delta_i}\bigl[\Gamma(\alpha_i-\gamma_i+\omega_i)-\xi_i(\eta_i-s)^{\alpha_i-\gamma_i+\omega_i-1}\bigr]t^{\alpha_i-\gamma_i-1} \\
&\leq \frac{1}{\Delta_i}\bigl[\Gamma(\alpha_i-\gamma_i+\omega_i)t^{\alpha_i-\gamma_i-1}+\xi_i\eta_i^{\alpha_i-\gamma_i+\omega_i-1}t^{\alpha_i-\gamma_i-1}\bigr] \\
&= \frac{1}{\Delta_i}\bigl[\Gamma(\alpha_i-\gamma_i+\omega_i)+\xi_i\eta_i^{\alpha_i-\gamma_i+\omega_i-1}\bigr]t^{\alpha_i-\gamma_i-1} \\
&= d_i t^{\alpha_i-\gamma_i-1}.
\end{aligned}
$$

For $\eta_i \leq s \leq t$,

$$
\begin{aligned}
H_i(t,s) &= \frac{1}{\Delta_i}\bigl\{\Gamma(\alpha_i-\gamma_i+\omega_i)\bigl[t^{\alpha_i-\gamma_i-1}-(t-s)^{\alpha_i-\gamma_i-1}\bigr]+\xi_i\eta_i^{\alpha_i-\gamma_i+\omega_i-1}(t-s)^{\alpha_i-\gamma_i-1}\bigr\} \\
&\geq \frac{1}{\Delta_i}\bigl\{\bigl[\Gamma(\alpha_i-\gamma_i+\omega_i)-\xi_i\eta_i^{\alpha_i-\gamma_i+\omega_i-1}\bigr]t^{\alpha_i-\gamma_i-1} \\
&\quad -\bigl[\Gamma(\alpha_i-\gamma_i+\omega_i)-\xi_i\eta_i^{\alpha_i-\gamma_i+\omega_i-1}\bigr](t-s)^{\alpha_i-\gamma_i-1}\bigr\} \\
&= \frac{1}{\Delta_i}\bigl[\Gamma(\alpha_i-\gamma_i+\omega_i)-\xi_i\eta_i^{\alpha_i-\gamma_i+\omega_i-1}\bigr]\bigl[t^{\alpha_i-\gamma_i-1}-(t-s)^{\alpha_i-\gamma_i-1}\bigr] \\
&\geq \Gamma(\alpha_i-\gamma_i)t^{\alpha_i-\gamma_i-1}\bigl[1-(1-s)^{\alpha_i-\gamma_i-1}\bigr] \\
&= e_i t^{\alpha_i-\gamma_i-1}\bigl[1-(1-s)^{\alpha_i-\gamma_i-1}\bigr], \\
H_i(t,s) &= \frac{1}{\Delta_i}\bigl\{\Gamma(\alpha_i-\gamma_i+\omega_i)\bigl[t^{\alpha_i-\gamma_i-1}-(t-s)^{\alpha_i-\gamma_i-1}\bigr]+\xi_i\eta_i^{\alpha_i-\gamma_i+\omega_i-1}(t-s)^{\alpha_i-\gamma_i-1}\bigr\} \\
&\leq \frac{1}{\Delta_i}\bigl[\Gamma(\alpha_i-\gamma_i+\omega_i)t^{\alpha_i-\gamma_i-1}+\xi_i\eta_i^{\alpha_i-\gamma_i+\omega_i-1}(t-s)^{\alpha_i-\gamma_i-1}\bigr] \\
&\leq \frac{1}{\Delta_i}\bigl[\Gamma(\alpha_i-\gamma_i+\omega_i)+\xi_i\eta_i^{\alpha_i-\gamma_i+\omega_i-1}\bigr]t^{\alpha_i-\gamma_i-1} \\
&= d_i t^{\alpha_i-\gamma_i-1}.
\end{aligned}
$$

For $s \geq t, s \geq \eta_i$,

$$
\begin{aligned}
H_i(t,s) &= \frac{1}{\Delta_i}\Gamma(\alpha_i-\gamma_i+\omega_i)t^{\alpha_i-\gamma_i-1} \\
&\geq \frac{1}{\Delta_i}\bigl[\Gamma(\alpha_i-\gamma_i+\omega_i)-\xi_i\eta_i^{\alpha_i-\gamma_i+\omega_i-1}\bigr]t^{\alpha_i-\gamma_i-1}\bigl[1-(1-s)^{\alpha_i-\gamma_i-1}\bigr] \\
&= e_i t^{\alpha_i-\gamma_i-1}\bigl[1-(1-s)^{\alpha_i-\gamma_i-1}\bigr], \\
H_i(t,s) &= \frac{1}{\Delta_i}\Gamma(\alpha_i-\gamma_i+\omega_i)t^{\alpha_i-\gamma_i-1} \\
&\leq \frac{1}{\Delta_i}\bigl[\Gamma(\alpha_i-\gamma_i+\omega_i)t^{\alpha_i-\gamma_i-1}+\xi_i\eta_i^{\alpha_i-\gamma_i+\omega_i-1}t^{\alpha_i-\gamma_i-1}\bigr] \\
&= \frac{1}{\Delta_i}\bigl[\Gamma(\alpha_i-\gamma_i+\omega_i)+\xi_i\eta_i^{\alpha_i-\gamma_i+\omega_i-1}\bigr]t^{\alpha_i-\gamma_i-1} \\
&= d_i t^{\alpha_i-\gamma_i-1}.
\end{aligned}
$$

From the above, the proof of (3) is completed. Clearly $H_i(t,s) > 0$ for $(t,s) \in (0,1)$, since (3) holds. □

Lemma 2.4 ([17]) *Let $h_1 \in L^1(0,1)$, if* (s_0)-(s_2) *hold, then the fractional boundary value problem*

$$\begin{cases} -D^{\beta_1}(\varphi_{p_1}(-D^{\alpha_1-\gamma_1}v_1))(t) = h_1(t), \\ D^{\alpha_1-\gamma_1}v_1(0) = D^{\alpha_1-\gamma_1}v_1(1) = 0, \\ v_1(0) = 0, \qquad D^{\alpha_1-\gamma_1-1}v_1(1) = \xi_1 I^{\omega_1}(v_1(\eta_1)), \end{cases} \tag{2.1}$$

has the unique positive solution

$$v_1(t) = \int_0^1 H_1(t,s)\left(\int_0^1 G(\beta_1,s,\tau)h_1(\tau)\,d\tau\right)^{q_1-1} ds. \tag{2.2}$$

Now let us consider the following modified problem of the BVP (1.1):

$$\begin{cases} -D^{\beta_1}(\varphi_{p_1}(-D^{\alpha_1-\gamma_1}v_1))(t) = \lambda f_1(I^{\gamma_1}v_1(t), v_1(t), v_2(t)), \\ -D^{\beta_2}(\varphi_{p_2}(-D^{\alpha_2-\gamma_2}v_2))(t) = f_2(t, I^{\gamma_1}v_1(t)), \\ D^{\alpha_i-\gamma_i}v_i(0) = D^{\alpha_i-\gamma_i}v_i(1) = 0, \\ v_i(0) = 0, \qquad D^{\alpha_i-\gamma_i-1}v_i(1) = \xi_i I^{\omega_i}(v_i(\eta_i)), \quad i = 1,2. \end{cases} \tag{2.3}$$

Lemma 2.5 *Let $u_i(t) = I^{\gamma_i}v_i(t)$, $v_i(t) \in C[0,1]$ $(i = 1,2)$. Then* (1.1) *can be transformed into* (2.3). *Moveover, if $(v_1(t), v_2(t)) \in C[0,1] \times C[0,1]$ is a positive solution of the problem* (2.3), *then $(I^{\gamma_1}v_1(t), I^{\gamma_2}v_2(t))$ is a positive solution of the problem* (1.1).

Proof Let $u_i(t) = I^{\gamma_i}v_i(t), v_i(t) \in C[0,1]$, by the definition of the Riemannn-Liouville fractional derivatives and integrals, we obtain

$$D^{\alpha_i}u_i(t) = D^{\alpha_i-\gamma_i}v_i(t), \quad D^{\alpha_i+1}u_i(t) = D^{\alpha_i-\gamma_i+1}v_i(t), \quad D^{\alpha_i-1}u_i(t) = D^{\alpha_i-\gamma_i-1}v_i(t). \tag{2.4}$$

Thus by applying (2.4), the BVP (1.1) reduces to the modified boundary value problem (2.3).

Consequently, if $(v_1(t), v_2(t)) \in C[0,1] \times C[0,1]$ is a positive solution of the problem (2.3), then $(I^{\gamma_1}v_1(t), I^{\gamma_2}v_2(t))$ is a positive solution of the problem (1.1).

It is well know that $(v_1, v_2) \in C[0,1] \times C[0,1]$ is a solution of system (2.3), if and only if $(v_1, v_2) \in C[0,1]\times \in C[0,1]$ is a solution of the following nonlinear integral equation system:

$$\begin{cases} v_1(t) = \lambda^{q_1-1}\int_0^1 H_1(t,s)(\int_0^1 G(\beta_1,s,\tau)f_1(I^{\gamma_1}v_1(\tau), v_1(\tau), v_2(\tau))\,d\tau)^{q_1-1}\,ds, \\ v_2(t) = \int_0^1 H_2(t,s)(\int_0^1 G(\beta_2,s,\tau)f_2(\tau, I^{\gamma_1}v_1(\tau))\,d\tau)^{q_2-1}\,ds. \end{cases} \tag{2.5}$$

Now define an operator

$$(Av_1)(t) = \int_0^1 H_2(t,s)\left(\int_0^1 G(\beta_2,s,\tau)f_2\bigl(\tau, I^{\gamma_1}v_1(\tau)\bigr)\,d\tau\right)^{q_2-1} ds.$$

Then the integral system (2.5) is equivalent to the following nonlinear integral-differential equation:

$$\begin{cases} -D^{\beta_1}(\varphi_{p_1}(-D^{\alpha_1-\gamma_1}v_1))(t) = \lambda f_1(I^{\gamma_1}v_1(t), v_1(t), Av_1(t)), \\ D^{\alpha_1-\gamma_1}v_1(0) = D^{\alpha_1-\gamma_1}v_1(1) = 0, \\ v_1(0) = 0, \qquad D^{\alpha_1-\gamma_1-1}v_1(1) = \xi_1 I^{\omega_1}(v_1(\eta_1)), \end{cases} \tag{2.6}$$

i.e. the operator equation

$$v_1(t) = \lambda^{q_1-1} \int_0^1 H_1(t,s)\left(\int_0^1 G(\beta_1,s,\tau)f_1\bigl(I^{\gamma_1}v_1(\tau),v_1(\tau),(Av_1)(t)\bigr)\,d\tau\right)^{q_1-1} ds. \qquad \square$$

Definition 2.1 A continuous function $\Psi(t)$ is called a lower solution of the problem (2.6) if it is satisfies

$$\begin{cases} -D^{\beta_1}(\varphi_{p_1}(-D^{\alpha_1-\gamma_1}\Psi))(t) \leq \lambda f_1(I^{\gamma_1}\Psi(t),\Psi(t),A\Psi(t)), \\ D^{\alpha_1-\gamma_1}\Psi(0) \geq 0, \qquad D^{\alpha_1-\gamma_1}\Psi(1) \geq 0, \\ \Psi(0) \geq 0, \qquad D^{\alpha_1-\gamma_1-1}\Psi(1) \geq \xi_1 I^{\omega_1}(\Psi(\eta_1)), \end{cases}$$

where

$$(A\Psi)(t) = \int_0^1 H_2(t,s)\left(\int_0^1 G(\beta_2,s,\tau)f_2\bigl(\tau,I^{\gamma_1}\Psi(\tau)\bigr)\,d\tau\right)^{q_2-1} ds.$$

Definition 2.2 A continuous function $\Phi(t)$ is called an upper solution of the problem (2.6) if it is satisfies

$$\begin{cases} -D^{\beta_1}(\varphi_{p_1}(-D^{\alpha_1-\gamma_1}\Phi))(t) \geq \lambda f_1(I^{\gamma_1}\Phi(t),\Phi(t),A\Phi(t)), \\ D^{\alpha_1-\gamma_1}\Phi(0) \leq 0, \qquad D^{\alpha_1-\gamma_1}\Phi(1) \leq 0, \\ \Phi(0) \leq 0, \qquad D^{\alpha_1-\gamma_1-1}\Phi(1) \leq \xi_1 I^{\omega_1}(\Phi(\eta_1)), \end{cases}$$

where

$$(A\Phi)(t) = \int_0^1 H_2(t,s)\left(\int_0^1 G(\beta_2,s,\tau)f_2\bigl(\tau,I^{\gamma_1}\Phi(\tau)\bigr)\,d\tau\right)^{q_2-1} ds.$$

Lemma 2.6 (Maximal principle) *If $v_1 \in C([0,1],R)$ satisfies*

$$v_1(0) = 0, \qquad D^{\alpha_1-\gamma_1-1}v_1(1) = \xi_1 I^{\omega_1}\bigl(v_1(\eta_1)\bigr)$$

and $-D^{\alpha_1-\gamma_1}v_1(t) \geq 0$ for any $t \in [0,1]$, then $v_1(t) \geq 0, t \in [0,1]$

Proof By Lemma 2.3, the conclusion is obvious, we omit the proof. $\square$

3 Main results

To establish the existence of a solution to the boundary value problem (1.1), we need to make the following assumptions.

(H$_1$) $f_1(x,y,z) : (0,+\infty)^3 \to [0,+\infty]$ is continuous and non-increasing in $x,y,z > 0$, respectively, and for all $r \in (0,1)$, there exists a constant $\varepsilon > 0$, such that, for any $(x,y,z) \in (0,+\infty)^3$, we have

$$f_1(rx,ry,rz) \leq r^{-\varepsilon} f_1(x,y,z).$$

(H$_2$) $f_2(t,x) : [0,1] \times [0,+\infty) \to [0,+\infty]$ is continuous and non-decreasing in $x > 0$, and there exists a constant $0 < \sigma < \frac{1}{q_2-1}$, such that, for any $r \in (0,1)$, $(t,x) \in [0,1] \times [0,+\infty)$, we have

$$f_2(t, rx) \geq r^\sigma f_2(t, x).$$

Remark For $r \geq 1$, and $x, y, z > 0$, we have

$$f_1(rx, ry, rz) \geq r^{-\varepsilon} f_1(x, y, z), \tag{3.1}$$

$$f_2(t, rx) \leq r^\sigma f_2(t, x). \tag{3.2}$$

Theorem 3.1 *Suppose* (H_1) *and* (H_2) *hold, and the following condition is satisfied*:

(H_3) $f_1(1,1,1) \neq 0$, *and*

$$0 < \int_0^1 f_1\left(\frac{\Gamma(\alpha_1 - \gamma_1)}{\Gamma(\alpha_1)} t^{\alpha_1 - 1}, t^{\alpha_1 - \gamma_1 - 1}, b t^{\alpha_2 - \gamma_2 - 1}\right) dt < +\infty,$$

where

$$\begin{aligned} b &= e_2 \int_0^1 \left[1 - (1-s)^{\alpha_2 - \gamma_2 - 1}\right] s^{(\beta_2 - 1)(q_2 - 1)} (1-s)^{q_2 - 1}\, ds \\ &\quad \times \left(\frac{1}{\Gamma(\beta_2)} \int_0^1 \tau (1-\tau)^{\beta_2 - 1} f_2\left(\tau, \frac{\Gamma(\alpha_1 - \gamma_1)}{\Gamma(\alpha_1)} \tau^{\alpha_1 - 1}\right) d\tau\right)^{q_2 - 1}, \\ e_2 &= \Gamma(\alpha_2 - \gamma_2). \end{aligned}$$

Then there exists a constant $\lambda^* > 0$ *such that for any* $\lambda \in (\lambda^*, +\infty)$, *the BVP* (1.1) *has at least one positive solution* $(u_1(t), u_2(t))$, *and, moreover, there exist two constants* $0 < l < 1$ *and* $L > 1$ *such that*

$$\begin{aligned} &l \frac{\Gamma(\alpha_1 - \gamma_1)}{\Gamma(\alpha_1)} t^{\alpha_1 - 1} \leq u_1(t) \leq L \frac{\Gamma(\alpha_1 - \gamma_1)}{\Gamma(\alpha_1)} t^{\alpha_1 - 1}, \\ &b l^{\sigma(q_2 - 1)} \frac{\Gamma(\alpha_2 - \gamma_2)}{\Gamma(\alpha_2)} t^{\alpha_2 - 1} \leq u_2(t) \leq L^{\sigma(q_2 - 1)} a \frac{\Gamma(\alpha_2 - \gamma_2)}{\Gamma(\alpha_2)} t^{\alpha_2 - 1}, \end{aligned}$$

where

$$\begin{aligned} a &= d_2 \left(\frac{\beta_2 - 1}{\Gamma(\beta_2)} \int_0^1 f_2\left(\tau, \frac{\Gamma(\alpha_1 - \gamma_1)}{\Gamma(\alpha_1)} \tau^{\alpha_1 - 1}\right) d\tau\right)^{q_2 - 1}, \\ d_2 &= \frac{1}{\Delta_2}\left[\Gamma(\alpha_2 - \gamma_2 + \omega_2) + \xi_2 \eta_2^{\alpha_2 - \gamma_2 + \omega_2 - 1}\right], \\ \Delta_2 &= \Gamma(\alpha_2 - \gamma_2)\left[\Gamma(\alpha_2 - \gamma_2 + \omega_2) - \xi_2 \eta_2^{\alpha_2 - \gamma_2 + \omega_2 - 1}\right]. \end{aligned}$$

Proof Let $E = C[0,1]$, and define a subset P of E as follows: $P = \{v_1(t) \in E : \text{there exists a constant } 0 < l < 1 \text{ such that } l t^{\alpha_1 - \gamma_1 - 1} \leq v_1(t) \leq l^{-1} t^{\alpha_1 - \gamma_1 - 1}, t \in [0,1]\}$. Clearly, P is a non-empty set, since $t^{\alpha_1 - \gamma_1 - 1} \in P$. Also

$$I^{\gamma_1} t^{\alpha_1 - \gamma_1 - 1} = \frac{1}{\Gamma(\gamma_1)} \int_0^t (t-s)^{\gamma_1 - 1} s^{\alpha_1 - \gamma_1 - 1}\, ds = \frac{\Gamma(\alpha_1 - \gamma_1)}{\Gamma(\alpha_1)} t^{\alpha_1 - 1}.$$

Now define the operator T_λ in E

$$(T_\lambda \nu_1)(t) = \lambda^{q_1-1}\int_0^1 H_1(t,s)\left(\int_0^1 G(\beta_1,s,\tau)f_1\big(I^{\gamma_1}\nu_1(\tau),\nu_1(\tau),A\nu_1(\tau)\big)\,d\tau\right)^{q_1-1} ds, \quad (3.3)$$

where

$$A\nu_1(t) = \int_0^1 H_2(t,s)\left(\int_0^1 G(\beta_2,s,\tau)f_2\big(\tau,I^{\gamma_1}\nu_1(\tau)\big)\,d\tau\right)^{q_2-1} ds.$$

We assert that T_λ is well defined and $T_\lambda(P)\subset P$. In fact, for any $\nu_1(t)\in P$, there exists a positive number $0<l_{\nu_1}<1$ such that $l_{\nu_1}t^{\alpha_1-\gamma_1-1}\le \nu_1(t)\le l_{\nu_1}^{-1}t^{\alpha_1-\gamma_1-1}, t\in[0,1]$. It follows from Lemma 2.3 and (H_2) that

$$A\nu_1(t) = \int_0^1 H_2(t,s)\left(\int_0^1 G(\beta_2,s,\tau)f_2\big(\tau,I^{\gamma_1}\nu_1(\tau)\big)\,d\tau\right)^{q_2-1} ds,$$

$$\begin{aligned}
&\int_0^1 G(\beta_2,s,\tau)f_2\big(\tau,I^{\gamma_1}\nu_1(\tau)\big)\,d\tau\\
&\quad\le \int_0^1 G(\beta_2,s,\tau)f_2\left(\tau,l_{\nu_1}^{-1}\frac{\Gamma(\alpha_1-\gamma_1)}{\Gamma(\alpha_1)}\tau^{\alpha_1-1}\right)d\tau\\
&\quad\le l_{\nu_1}^{-\sigma}\frac{\beta_2-1}{\Gamma(\beta_2)}\int_0^1 s^{\beta_2-1}(1-s)f_2\left(\tau,\frac{\Gamma(\alpha_1-\gamma_1)}{\Gamma(\alpha_1)}\tau^{\alpha_1-1}\right)d\tau\\
&\quad\le l_{\nu_1}^{-\sigma}\frac{\beta_2-1}{\Gamma(\beta_2)}\int_0^1 f_2\left(\tau,\frac{\Gamma(\alpha_1-\gamma_1)}{\Gamma(\alpha_1)}\tau^{\alpha_1-1}\right)d\tau,
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1 G(\beta_2,s,\tau)f_2\big(\tau,I^{\gamma_1}\nu_1(\tau)\big)\,d\tau\\
&\quad\ge \int_0^1 G(\beta_2,s,\tau)f_2\left(\tau,l_{\nu_1}\frac{\Gamma(\alpha_1-\gamma_1)}{\Gamma(\alpha_1)}\tau^{\alpha_1-1}\right)d\tau\\
&\quad\ge l_{\nu_1}^{\sigma}\int_0^1 G(\beta_2,s,\tau)f_2\left(\tau,\frac{\Gamma(\alpha_1-\gamma_1)}{\Gamma(\alpha_1)}\tau^{\alpha_1-1}\right)d\tau\\
&\quad\ge l_{\nu_1}^{\sigma}\frac{s^{\beta_2-1}(1-s)}{\Gamma(\beta_2)}\int_0^1 \tau(1-\tau)^{\beta_2-1}f_2\left(\tau,\frac{\Gamma(\alpha_1-\gamma_1)}{\Gamma(\alpha_1)}\tau^{\alpha_1-1}\right)d\tau.
\end{aligned}$$

Then,

$$\begin{aligned}
A\nu_1(t) &\le \int_0^1 H_2(t,s)\left(l_{\nu_1}^{-\sigma}\frac{\beta_2-1}{\Gamma(\beta_2)}\int_0^1 f_2\left(\tau,\frac{\Gamma(\alpha_1-\gamma_1)}{\Gamma(\alpha_1)}\tau^{\alpha_1-1}\right)d\tau\right)^{q_2-1} ds\\
&\le d_2 t^{\alpha_2-\gamma_2-1}l_{\nu_1}^{-\sigma(q_2-1)}\left(\frac{\beta_2-1}{\Gamma(\beta_2)}\int_0^1 f_2\left(\tau,\frac{\Gamma(\alpha_1-\gamma_1)}{\Gamma(\alpha_1)}\tau^{\alpha_1-1}\right)d\tau\right)^{q_2-1}\\
&= a l_{\nu_1}^{-\sigma(q_2-1)}t^{\alpha_2-\gamma_2-1}, \qquad (3.4)
\end{aligned}$$

where

$$a = d_2\left(\frac{\beta_2-1}{\Gamma(\beta_2)}\int_0^1 f_2\left(\tau,\frac{\Gamma(\alpha_1-\gamma_1)}{\Gamma(\alpha_1)}\tau^{\alpha_1-1}\right)d\tau\right)^{q_2-1}$$

and

$$\begin{aligned} Av_1(t) &\geq \int_0^1 H_2(t,s)\left(l_{v_1}^{\sigma}\frac{s^{\beta_2-1}(1-s)}{\Gamma(\beta_2)}\int_0^1 \tau(1-\tau)^{\beta_2-1}f_2\left(\tau,\frac{\Gamma(\alpha_1-\gamma_1)}{\Gamma(\alpha_1)}t^{\alpha_1-1}\right)d\tau\right)^{q_2-1}ds \\ &\geq e_2 t^{\alpha_2-\gamma_2-1}l_{v_1}^{\sigma(q_2-1)}\int_0^1 s^{(\beta_2-1)(q_2-1)}(1-s)^{q_2-1}\left[1-(1-s)^{\alpha_2-\gamma_2-1}\right]ds \\ &\quad\times\left(\int_0^1 \frac{1}{\Gamma(\beta_2)}\tau(1-\tau)^{\beta_2-1}f_2\left(\tau,\frac{\Gamma(\alpha_1-\gamma_1)}{\Gamma(\alpha_1)}\tau^{\alpha_1-1}\right)d\tau\right)^{q_2-1} \\ &= bl_{v_1}^{\sigma(q_2-1)}t^{\alpha_2-\gamma_2-1}, \end{aligned} \tag{3.5}$$

where

$$\begin{aligned} b &= e_2\int_0^1\left[1-(1-s)^{\alpha_2-\gamma_2-1}\right]s^{(\beta_2-1)(q_2-1)}(1-s)^{q_2-1}\,ds \\ &\quad\times\left(\frac{1}{\Gamma(\beta_2)}\int_0^1\tau(1-\tau)^{\beta_2-1}f_2\left(\tau,\frac{\Gamma(\alpha_1-\gamma_1)}{\Gamma(\alpha_1)}\tau^{(\alpha_1-1)}\right)d\tau\right)^{q_2-1}. \end{aligned}$$

Since $0<\sigma<\frac{1}{q_2-1}$, and by Lemma 2.3 and (H_1), (H_3), we also have

$$\begin{aligned} (T_\lambda v_1)(t) &= \lambda^{q_1-1}\int_0^1 H_1(t,s)\left(\int_0^1 G(\beta_1,s,\tau)f_1\left(I^{\gamma_1}v_1(\tau),v_1(\tau),Av_1(\tau)\right)d\tau\right)^{q_1-1}ds \\ &\leq \lambda^{q_1-1}\int_0^1 H_1(t,s)\left(\int_0^1 G(\beta_1,s,\tau)\right. \\ &\qquad\left.\times f_1\left(l_{v_1}\frac{\Gamma(\alpha_1-\gamma_1)}{\Gamma(\alpha_1)}\tau^{\alpha_1-1},l_{v_1}\tau^{\alpha_1-\gamma_1-1},bl_{v_1}^{\sigma(q_2-1)}\tau^{\alpha_2-\gamma_2-1}\right)d\tau\right)^{q_1-1}ds \\ &\leq \lambda^{q_1-1}\int_0^1 H_1(t,s)\left(\int_0^1 G(\beta_1,s,\tau)\right. \\ &\qquad\left.\times f_1\left(l_{v_1}\frac{\Gamma(\alpha_1-\gamma_1)}{\Gamma(\alpha_1)}\tau^{\alpha_1-1},l_{v_1}\tau^{\alpha_1-\gamma_1-1},bl_{v_1}\tau^{\alpha_2-\gamma_2-1}\right)d\tau\right)^{q_1-1}ds \\ &\leq \lambda^{q_1-1}l_{v_1}^{-\varepsilon(q_1-1)}\int_0^1 H_1(t,s)\left(\int_0^1\frac{\beta_1-1}{\Gamma(\beta_1)}s^{\beta_1-1}(1-s)\right. \\ &\qquad\left.\times f_1\left(\frac{\Gamma(\alpha_1-\gamma_1)}{\Gamma(\alpha_1)}\tau^{\alpha_1-1},\tau^{\alpha_1-\gamma_1-1},b\tau^{\alpha_2-\gamma_2-1}\right)d\tau\right)^{q_1-1}ds \\ &\leq \lambda^{q_1-1}l_{v_1}^{-\varepsilon(q_1-1)}d_1t^{\alpha_1-\gamma_1-1}\left(\int_0^1 s^{(\beta_1-1)(q_1-1)}(1-s)^{q_1-1}\,ds\right) \\ &\qquad\times\left(\frac{\beta_1-1}{\Gamma(\beta_1)}\int_0^1 f_1\left(\frac{\Gamma(\alpha_1-\gamma_1)}{\Gamma(\alpha_1)}\tau^{\alpha_1-1},\tau^{\alpha_1-\gamma_1-1},b\tau^{\alpha_2-\gamma_2-1}\right)d\tau\right)^{q_1-1} \\ &\leq \lambda^{q_1-1}l_{v_1}^{-\varepsilon(q_1-1)} \\ &\qquad\times d_1\left(\frac{\beta_1-1}{\Gamma(\beta_1)}\int_0^1 f_1\left(\frac{\Gamma(\alpha_1-\gamma_1)}{\Gamma(\alpha_1)}\tau^{\alpha_1-1},\tau^{\alpha_1-\gamma_1-1},b\tau^{\alpha_2-\gamma_2-1}\right)d\tau\right)^{q_1-1}t^{\alpha_1-\gamma_1-1} \\ &< +\infty. \end{aligned} \tag{3.6}$$

On the other hand, as $0 < \sigma < \frac{1}{q_2-1}$, from Lemma 2.3 and (3.1), we have

$$\begin{aligned}
(T_\lambda \nu_1)(t) &= \lambda^{q_1-1}\int_0^1 H_1(t,s)\biggl(\int_0^1 G(\beta_1,s,\tau)f_1\bigl(I^{\gamma_1}\nu_1(\tau),\nu_1(\tau),A\nu_1(\tau)\bigr)\,d\tau\biggr)^{q_1-1}\,ds \\
&\geq \lambda^{q_1-1}\int_0^1 H_1(t,s)\biggl(\int_0^1 G(\beta_1,s,\tau) \\
&\quad\times f_1\biggl(l_{\nu_1}^{-1}\frac{\Gamma(\alpha_1-\gamma_1)}{\Gamma(\alpha_1)}\tau^{\alpha_1-1},l_{\nu_1}^{-1}\tau^{\alpha_1-\gamma_1-1},al_{\nu_1}^{-\sigma(q_2-1)}\tau^{\alpha_2-\gamma_2-1}\biggr)\,d\tau\biggr)^{q_1-1}\,ds \\
&\geq \lambda^{q_1-1}\int_0^1 H_1(t,s)\biggl(\int_0^1 G(\beta_1,s,\tau) \\
&\quad\times f_1\biggl(l_{\nu_1}^{-1}\frac{\Gamma(\alpha_1-\gamma_1)}{\Gamma(\alpha_1)}\tau^{\alpha_1-1},l_{\nu_1}^{-1}\tau^{\alpha_1-\gamma_1-1},al_{\nu_1}^{-1}\tau^{\alpha_2-\gamma_2-1}\biggr)\,d\tau\biggr)^{q_1-1}\,ds \\
&\geq \lambda^{q_1-1}l_{\nu_1}^{\varepsilon(q_1-1)}\int_0^1 H_1(t,s)\biggl(\frac{s^{\beta_1-1}(1-s)}{\Gamma(\beta_1)}\int_0^1 \tau(1-\tau)^{\beta_1-1} \\
&\quad\times f_1\biggl(\frac{\Gamma(\alpha_1-\gamma_1)}{\Gamma(\alpha_1)}\tau^{\alpha_1-1},\tau^{\alpha_1-\gamma_1-1},a\tau^{\alpha_2-\gamma_2-1}\biggr)\,d\tau\biggr)^{q_1-1}\,ds \\
&\geq \lambda^{q_1-1}l_{\nu_1}^{\varepsilon(q_1-1)}e_1t^{\alpha_1-\gamma_1-1}\int_0^1\bigl[1-(1-s)^{\alpha_1-\gamma_1-1}\bigr]s^{(\beta_1-1)(q_1-1)}(1-s)^{q_1-1}\,ds \\
&\quad\times\biggl(\frac{1}{\Gamma(\beta_1)}\int_0^1 \tau(1-\tau)^{\beta_1-1}f_1\biggl(\frac{\Gamma(\alpha_1-\gamma_1)}{\Gamma(\alpha_1)}\tau^{\alpha_1-1},\tau^{\alpha_1-\gamma_1-1},a\tau^{\alpha_2-\gamma_2-1}\biggr)\,d\tau\biggr)^{q_1-1} \\
&= k\lambda^{q_1-1}l_{\nu_1}^{\varepsilon(q_1-1)}t^{\alpha_1-\gamma_1-1}\biggl(\frac{1}{\Gamma(\beta_1)}\int_0^1 \tau(1-\tau)^{\beta_1-1} \\
&\quad\times f_1\biggl(\frac{\Gamma(\alpha_1-\gamma_1)}{\Gamma(\alpha_1)}\tau^{\alpha_1-1},\tau^{\alpha_1-\gamma_1-1},a\tau^{\alpha_2-\gamma_2-1}\biggr)\,d\tau\biggr)^{q_1-1},
\end{aligned} \tag{3.7}$$

where

$$k=\int_0^1\bigl[1-(1-s)^{\alpha_1-\gamma_1-1}\bigr]s^{(\beta_1-1)(q_1-1)}(1-s)^{q_1-1}\,ds.$$

Choose

$$\begin{aligned}
\widetilde{l_{\nu_1}} = \min\biggl\{&\frac{1}{2},\biggl\{\lambda^{q_1-1}l_{\nu_1}^{-\varepsilon(q_1-1)}d_1\biggl(\frac{\beta_1-1}{\Gamma(\beta_1)}\int_0^1 f_1\biggl(\frac{\Gamma(\alpha_1-\gamma_1)}{\Gamma(\alpha_1)}\tau^{\alpha_1-1}, \\
&\tau^{\alpha_1-\gamma_1-1},b\tau^{\alpha_2-\gamma_2-1}\biggr)\,d\tau\biggr)^{q_1-1}\biggr\}^{-1}, \\
&k\lambda^{q_1-1}l_{\nu_1}^{\varepsilon(q_1-1)}\biggl(\frac{1}{\Gamma(\beta_1)}\int_0^1 \tau(1-\tau)^{\beta_1-1}f_1\biggl(\frac{\Gamma(\alpha_1-\gamma_1)}{\Gamma(\alpha_1)}\tau^{\alpha_1-1}, \\
&\tau^{\alpha_1-\gamma_1-1},a\tau^{\alpha_2-\gamma_2-1}\biggr)\,d\tau\biggr)^{q_1-1}\biggr\}.
\end{aligned} \tag{3.8}$$

Then it follows from (3.4)-(3.8) that

$$\widetilde{l_{\nu_1}}t^{\alpha_1-\gamma_1-1}\leq T_\lambda\nu_1(t)\leq \widetilde{l_{\nu_1}}^{-1}t^{\alpha_1-\gamma_1-1}.$$

This implies that T_λ is well defined and $T_\lambda(P) \subset P$. Furthermore, comparing (3.3) and (2.2), the right hand side of (3.3) is exactly the same as the right hand of (2.2), if $h_1(t)$ in (2.1) is taken as $\lambda f_1(I^{\gamma_1}\nu_1(t), \nu_1(t), A\nu_1(t))$. Hence as the left hand side of (2.2), i.e. $\nu_1(t)$ satisfies equation (2.1) according to Lemma 2.4, the left hand side of (3.3), i.e. $T_\lambda \nu_1(t)$ must also satisfy equation (2.1) with $h_1(t)$ replace by $\lambda f_1(I^{\gamma_1}\nu_1(t), \nu_1(t), A\nu_1(t))$, namely

$$\begin{cases} -D^{\beta_1}(\varphi_{p_1}(-D^{\alpha_1-\gamma_1}(T_\lambda \nu_1))(t) = \lambda f_1(I^{\gamma_1}\nu_1(t), \nu_1(t), A\nu_1(t)), \\ D^{\alpha_1-\gamma_1}(T_\lambda \nu_1)(0) = D^{\alpha_1-\gamma_1}(T_\lambda \nu_1)(1) = 0, \\ (T_\lambda \nu_1)(0) = 0, \qquad D^{\alpha_1-\gamma_1-1}(T_\lambda \nu_1)(1) = \xi_1 I^{\omega_1}(T_\lambda \nu_1)(\eta_1), \end{cases} \tag{3.9}$$

where

$$(A\nu_1)(t) = \int_0^1 H_2(t,s)\left(\int_0^1 G(\beta_2, s, \tau) f_2\big(\tau, I^{\gamma_1}\nu_1(\tau)\big)\, d\tau\right)^{q_2-1} ds.$$

Next, we shall find the upper and lower solutions of (1.1). First of all, let

$$e(t) = \int_0^1 H_1(t,s)\left(\int_0^1 G(\beta_1, s, \tau) f_1\left(\frac{\Gamma(\alpha_1-\gamma_1)}{\Gamma(\alpha_1)}\tau^{\alpha_1-1}, \tau^{\alpha_1-\gamma_1-1}, A\tau^{\alpha_1-\gamma_1-1}\right) d\tau\right)^{q_2-1} ds,$$

where

$$At^{\alpha_1-\gamma_1-1} = \int_0^1 H_2(t,s)\left(\int_0^1 G(\beta_2, s, \tau) f_2\left(\tau, \frac{\Gamma(\alpha_1-\gamma_1)}{\Gamma(\alpha_1)}\tau^{\alpha_1-1}\right) d\tau\right)^{q_2-1} ds.$$

Similar to (3.4) and (3.5), the following inequalities are also valid:

$$At^{\alpha_1-\gamma_1-1} \geq bt^{\alpha_2-\gamma_2-1}$$

and

$$At^{\alpha_1-\gamma_1-1} \leq at^{\alpha_2-\gamma_2-1}.$$

By Lemma 2.3, (H_1), and (3.7), we also have

$$\begin{aligned} e(t) &\geq \int_0^1 H_1(t,s)\left(\int_0^1 G(\beta_1, s, \tau) f_1\left(\frac{\Gamma(\alpha_1-\gamma_1)}{\Gamma(\alpha_1)}\tau^{\alpha_1-1}, \tau^{\alpha_1-\gamma_1-1}, a\tau^{\alpha_2-\gamma_2-1}\right) d\tau\right)^{q_2-1} ds \\ &\geq kt^{\alpha_1-\gamma_1-1}\left(\frac{1}{\Gamma(\beta_1)}\int_0^1 \tau(1-\tau)^{\beta_1-1} f_1\left(\frac{\Gamma(\alpha_1-\gamma_1)}{\Gamma(\alpha_1)}\tau^{\alpha_1-1},\right.\right. \\ &\quad \left.\left. \tau^{\alpha_1-\gamma_1-1}, a\tau^{\alpha_2-\gamma_2-1}\right) d\tau\right)^{q_1-1}, \end{aligned}$$

and consequently there exists a constant $\lambda_1 \geq 1$ such that

$$\lambda_1^{q_1-1} e(t) \geq t^{\alpha_1-\gamma_1-1}, \quad \forall t \in [0,1]. \tag{3.10}$$

On the other hand by (H_1) and (H_2), we know that A is increasing and T_λ is decreasing, and thus for $\lambda > \lambda_1$, from (3.6) we have

$$\begin{aligned}
&\int_0^1 H_1(t,s)\biggl(\int_0^1 G(\beta_1,s,\tau)f_1\bigl(I^{\gamma_1}\lambda^{q_1-1}e(\tau),\lambda^{q_1-1}e(\tau),A\lambda^{q_1-1}e(\tau)\bigr)\,d\tau\biggr)^{q_1-1}\,ds\\
&\quad\le \int_0^1 H_1(t,s)\biggl(\int_0^1 G(\beta_1,s,\tau)f_1\bigl(I^{\gamma_1}\lambda_1^{q_1-1}e(\tau),\lambda_1^{q_1-1}e(\tau),A\lambda_1^{q_1-1}e(\tau)\bigr)\,d\tau\biggr)^{q_1-1}\,ds\\
&\quad\le \int_0^1 H_1(t,s)\biggl(\int_0^1 G(\beta_1,s,\tau)f_1\bigl(I^{\gamma_1}\tau^{\alpha_1-\gamma_1-1},\tau^{\alpha_1-\gamma_1-1},A\tau^{\alpha_1-\gamma_1-1}\bigr)\,d\tau\biggr)^{q_1-1}\,ds\\
&\quad\le \int_0^1 H_1(t,s)\biggl(\int_0^1 G(\beta_1,s,\tau)f_1\biggl(\frac{\Gamma(\alpha_1-\gamma_1)}{\Gamma(\alpha_1)}\tau^{\alpha_1-1},\tau^{\alpha_1-\gamma_1-1},b\tau^{\alpha_2-\gamma_2-1}\biggr)\,d\tau\biggr)^{q_1-1}\,ds\\
&\quad\le d_1\biggl(\frac{\beta_1-1}{\Gamma(\beta_1)}\int_0^1 f_1\biggl(\frac{\Gamma(\alpha_1-\gamma_1)}{\Gamma(\alpha_1)}\tau^{\alpha_1-1},\tau^{\alpha_1-\gamma_1-1},b\tau^{\alpha_2-\gamma_2-1}\biggr)\,d\tau\biggr)^{q_1-1}\,ds\\
&\quad< +\infty.
\end{aligned}\tag{3.11}$$

Applying (3.2) and $0 < \sigma < \frac{1}{q_2-1}$, for any $t \in [0,1]$, we have

$$\begin{aligned}
A(\lambda^*)^{q_1-1}e(t) &= \int_0^1 H_2(t,s)\biggl(\int_0^1 G(\beta_2,s,\tau)f_2\bigl(\tau,(\lambda^*)^{q_1-1}I^{\gamma_1}e(\tau)\bigr)\,d\tau\biggr)^{q_2-1}\,ds\\
&\le (\lambda^*)^{\sigma(q_1-1)(q_2-1)}\int_0^1 H_2(t,s)\biggl(\int_0^1 G(\beta_2,s,\tau)f_2\bigl(\tau,I^{\gamma_1}e(\tau)\bigr)\,d\tau\biggr)^{q_2-1}\,ds\\
&\le (\lambda^*)^{(q_1-1)}\int_0^1 H_2(t,s)\biggl(\int_0^1 G(\beta_2,s,\tau)f_2\bigl(\tau,I^{\gamma_1}e(\tau)\bigr)\,d\tau\biggr)^{q_2-1}\,ds\\
&= (\lambda^*)^{(q_1-1)}Ae(t).
\end{aligned}\tag{3.12}$$

Let

$$C_1 = \max_{0\le t\le 1} e(t), \qquad C_2 = \max_{0\le t\le 1} Ae(t), \qquad C = \max\{2, C_1, C_2\},$$

then we have

$$I^{\gamma_1}e(t) \le \frac{C_1}{\Gamma(\gamma_1)} \le C_1 \le C, \qquad Ae(t) \le C_2 \le C, \qquad e(t) \le C_1 \le C. \tag{3.13}$$

Now, take

$$\lambda^* > \biggl\{\lambda_1, \biggl[\frac{C^\varepsilon}{f_1^{q_1-1}(1,1,1)k[\frac{1}{\Gamma(\beta_1)}\int_0^1 \tau(1-\tau)^{\beta_1-1}\,d\tau]^{q_1-1}}\biggr]^{\frac{1}{(q_1-1)[1-\varepsilon(q_1-1)]}}\biggr\}.$$

Then by (3.12), (3.13), and (H_1), we have

$$\begin{aligned}
&(\lambda^*)^{q_1-1}\bigl(f_1\bigl(I^{\gamma_1}(\lambda^*)^{q_1-1}e(\tau),(\lambda^*)^{q_1-1}e(\tau),A(\lambda^*)^{q_1-1}e(\tau)\bigr)\bigr)^{q_1-1}\\
&\quad\ge (\lambda^*)^{q_1-1}\bigl(f_1\bigl((\lambda^*)^{q_1-1}I^{\gamma_1}e(\tau),(\lambda^*)^{q_1-1}e(\tau),(\lambda^*)^{q_1-1}Ae(\tau)\bigr)\bigr)^{q_1-1}\\
&\quad\ge (\lambda^*)^{q_1-1}(\lambda^*)^{-\varepsilon(q_1-1)^2}\bigl(f_1\bigl(I^{\gamma_1}e(\tau),e(\tau),Ae(\tau)\bigr)\bigr)^{q_1-1}
\end{aligned}$$

$$\begin{aligned}&\geq (\lambda^*)^{(q_1-1)[1-\varepsilon(q_1-1)]}\big(f_1(C,C,C)\big)^{q_1-1}\\&\geq (\lambda^*)^{(q_1-1)[1-\varepsilon(q_1-1)]}C^{-\varepsilon}f_1^{q_1-1}(1,1,1).\end{aligned}\tag{3.14}$$

Consequently, (3.7) and (3.14) yield

$$\begin{aligned}&(\lambda^*)^{q_1-1}\int_0^1 H_1(t,s)\bigg(\int_0^1 G(\beta_1,s,\tau)f_1\big(I^{\gamma_1}(\lambda^*)^{q_1-1}e(\tau),\\&\quad(\lambda^*)^{q_1-1}e(\tau),A(\lambda^*)^{q_1-1}e(\tau)\,d\tau\big)\bigg)^{q_1-1}ds\\&\geq (\lambda^*)^{(q_1-1)[1-\varepsilon(q_1-1)]}C^{-\varepsilon}f_1^{q_1-1}(1,1,1)\int_0^1 H_1(t,s)\bigg(\int_0^1 G(\beta_1,s,\tau)\,d\tau\bigg)^{q_1-1}ds\\&\geq (\lambda^*)^{(q_1-1)[1-\varepsilon(q_1-1)]}C^{-\varepsilon}f_1^{q_1-1}(1,1,1)kt^{\alpha_1-\gamma_1-1}\bigg[\frac{1}{\Gamma(\beta_1)}\int_0^1\tau(1-\tau)^{\beta_1-1}\,d\tau\bigg]^{q_1-1}\\&\geq t^{\alpha_1-\gamma_1-1}.\end{aligned}\tag{3.15}$$

Let

$$\begin{aligned}&\Phi(t)=(\lambda^*)^{q_1-1}e(t),\\&\Psi(t)=(\lambda^*)^{q_1-1}\int_0^1 H_1(t,s)\bigg(\int_0^1 G(\beta_1,s,\tau)f_1\big(I^{\gamma_1}(\lambda^*)^{q_1-1}e(\tau),\\&\qquad(\lambda^*)^{q_1-1}e(\tau),A(\lambda^*)^{q_1-1}e(\tau)\big)\,d\tau\bigg)^{q_1-1}ds,\end{aligned}$$

then

$$\Phi(t)=T_{\lambda^*}\big(t^{\alpha_1-\gamma_1-1}\big),\qquad \Psi(t)=T_{\lambda^*}\big(\Phi(t)\big).\tag{3.16}$$

It follows from the monotonicity of A, f_1, and (3.10), (3.15), that for any $t\in[0,1]$

$$\begin{aligned}&\Phi(t)=(\lambda^*)^{q_1-1}\int_0^1 H_1(t,s)\bigg(\int_0^1 G(\beta_1,s,\tau)f_1\bigg(\frac{\Gamma(\alpha_1-\gamma_1)}{\Gamma(\alpha_1)}\tau^{\alpha_1-1},\\&\qquad\tau^{\alpha_1-\gamma_1-1},A\tau^{\alpha_1-\gamma_1-1}\bigg)\,d\tau\bigg)^{q_1-1}ds\\&\quad\geq\lambda_1 e(t)\geq t^{\alpha_1-\gamma_1-1},\\&\Psi(t)=(\lambda^*)^{q_1-1}\int_0^1 H_1(t,s)\bigg(\int_0^1 G(\beta_1,s,\tau)f_1\big(I^{\gamma_1}(\lambda^*)^{q_1-1}e(\tau),\\&\qquad(\lambda^*)^{q_1-1}e(\tau),A(\lambda^*)^{q_1-1}e(\tau)\big)\,d\tau\bigg)^{q_1-1}ds\\&\quad\geq t^{\alpha_1-\gamma_1-1}.\end{aligned}\tag{3.17}$$

Moveover, by (3.9) and (3.16), we know

$$\begin{aligned}&D^{\alpha_1-\gamma_1}\Phi(0)=D^{\alpha_1-\gamma_1}\Phi(1)=0,\qquad\Phi(0)=0,\qquad D^{\alpha_1-\gamma_1-1}\Phi(1)=\xi_1 I^{\omega_1}\Phi(\eta_1),\\&D^{\alpha_1-\gamma_1}\Psi(0)=D^{\alpha_1-\gamma_1}\Psi(1)=0,\qquad\Psi(0)=0,\qquad D^{\alpha_1-\gamma_1-1}\Psi(1)=\xi_1 I^{\omega_1}\Psi(\eta_1).\end{aligned}\tag{3.18}$$

Proceeding as in (3.6)-(3.8), we get that $\Phi(t), \Psi(t) \in P$. By (3.16) and (3.17), we have

$$t^{\alpha_1-\gamma_1-1} \leq \Psi(t) = (T_{\lambda^*}\Phi)(t), \qquad t^{\alpha_1-\gamma_1-1} \leq \Phi(t), \quad \forall t \in [0,1], \tag{3.19}$$

which implies

$$\begin{aligned} \Psi(t) &= (T_{\lambda^*}\Phi)(t) \\ &= (\lambda^*)^{q_1-1} \int_0^1 H_1(t,s)\biggl(\int_0^1 G(\beta_1,s,\tau) f_1\bigl(I^{\gamma_1}(\lambda^*)^{q_1-1}e(\tau), \\ &\quad (\lambda^*)^{q_1-1}e(\tau), A(\lambda^*)^{q_1-1}e(\tau)\bigr)\,d\tau\biggr)^{q_1-1} ds \\ &\leq (\lambda^*)^{q_1-1} \int_0^1 H_1(t,s)\biggl(\int_0^1 G(\beta_1,s,\tau) f_1\bigl(I^{\gamma_1}\tau^{\alpha_1-\gamma_1-1}, \tau^{\alpha_1-\gamma_1-1}, A\tau^{\alpha_1-\gamma_1-1}\bigr)\,d\tau\biggr)^{q_1-1} ds \\ &= \Phi(t). \end{aligned} \tag{3.20}$$

Thus, by (3.9), (3.16), (3.17), and (3.20)

$$\begin{aligned} &D^{\beta_1}\bigl(\varphi_{p_1}\bigl(-D^{\alpha_1-\gamma_1}\Psi\bigr)\bigr)(t) + \lambda^* f_1\bigl(I^{\gamma_1}\Psi(t), \Psi(t), A\Psi(t)\bigr) \\ &\quad = D^{\beta_1}\bigl(\varphi_{p_1}\bigl(-D^{\alpha_1-\gamma_1}(T_{\lambda^*}\Phi)\bigr)\bigr)(t) + \lambda^* f_1(I^{\gamma_1}\Psi(t), \Psi(t), A\Psi(t) \\ &\quad \geq -\lambda^* f_1\bigl(I^{\gamma_1}\Phi(t), \Phi(t), A\Phi(t)\bigr) + \lambda^* f_1\bigl(I^{\gamma_1}\Phi(t), \Phi(t), A\Phi(t)\bigr) = 0, \end{aligned} \tag{3.21}$$

$$\begin{aligned} &D^{\beta_1}\bigl(\varphi_{p_1}\bigl(-D^{\alpha_1-\gamma_1}\Phi\bigr)\bigr)(t) + \lambda^* f_1\bigl(I^{\gamma_1}\Phi(t), \Phi(t), A\Phi(t)\bigr) \\ &\quad = D^{\beta_1}(\varphi_{p_1}\bigl(-D^{\alpha_1-\gamma_1}\bigl(T_{\lambda^*}\bigl(t^{\alpha_1-\gamma_1-1}\bigr)\bigr)\bigr) + \lambda^* f_1\bigl(I^{\gamma_1}\Phi(t), \Phi(t), \Phi(t)\bigr) \\ &\quad \leq -\lambda^* f_1\bigl(I^{\gamma_1}t^{\alpha_1-\gamma_1-1}, t^{\alpha_1-\gamma_1-1}, At^{\alpha_1-\gamma_1-1}\bigr) \\ &\qquad + \lambda^* f_1\bigl(I^{\gamma_1}t^{\alpha_1-\gamma_1-1}, t^{\alpha_1-\gamma_1-1}, At^{\alpha_1-\gamma_1-1}\bigr) = 0. \end{aligned} \tag{3.22}$$

It follows from (3.18) and (3.21)-(3.22) that $\Psi(t), \Phi(t)$ are upper and lower solutions of BVP (2.6), and that $\Psi(t), \Phi(t) \in P$. Now let us define a function

$$F(\nu_1) = \begin{cases} f_1(I^{\gamma_1}\Psi(t), \Psi(t), A\Psi(t)), & \nu_1 < \Psi(t), \\ f_1(I^{\gamma_1}\nu_1(t), \nu_1(t), A\nu_1(t)), & \Psi(t) \leq \nu_1 \leq \Phi(t), \\ f_1(I^{\gamma_1}\Phi(t), \Phi(t), A\Phi(t)), & \nu_1 > \Phi(t). \end{cases}$$

Clearly, $F : [0,+\infty] \to [0,+\infty]$ is continuous.

We now show that the fractional boundary value problem

$$\begin{cases} -D^{\beta_1}(\varphi_{p_1}(-D^{\alpha_1-\gamma_1}\nu_1))(t) = \lambda^* F(\nu_1), \\ D^{\alpha_1-\gamma_1}\nu_1(0) = D^{\alpha_1-\gamma_1}\nu_1(1) = 0, \\ \nu_1(0) = 0, \qquad D^{\alpha_1-\gamma_1-1}\nu_1(1) = \xi_1 I^{\omega_1}\nu_1(\eta_1), \end{cases} \tag{3.23}$$

has a positive solution. Define the operator D_{λ^*} by

$$D_{\lambda^*}\nu_1(t) = (\lambda^*)^{q_1-1} \int_0^1 H_1(t,s)\biggl(\int_0^1 G(\beta_1,s,\tau) F\bigl(\nu_1(\tau)\,d\tau\bigr)\biggr)^{q_1-1} ds.$$

Then $D_{\lambda^*} : C[0,1] \to C[0,1]$, and a fixed point of the operator D_{λ^*} is a solution of the BVP (3.23). On the other hand, from the definition of F and the fact that the function $f_1(x,y,z)$ is non-increasing in x,y,z respectively, and A is non-decreasing, we obtain $f_1(I^{\gamma_1}\Phi(t),\Phi(t),A\Phi(t)) \le F(v_1(t)) \le f_1(I^{\gamma_1}\Psi(t),\Psi(t),A\Psi(t))$, provided that $\Psi(t) \le v_1(t) \le \Phi(t)$, $F(v_1(t)) = f_1(I^{\gamma_1}\Psi(t),\Psi(t),A\Psi(t))$, provided that $v_1(t) < \Psi(t)$, and $F(v_1(t)) = f_1(I^{\gamma_1}\Phi(t),\Phi(t),A\Phi(t))$, provided that $v_1(t) > \Phi(t)$. So we have

$$f_1\big(I^{\gamma_1}\Phi(t),\Phi(t),A\Phi(t)\big) \le F\big(v_1(t)\big) \le f_1\big(I^{\gamma_1}\Psi(t),\Psi(t),A\Psi(t)\big), \quad \forall v_1(t)\in E.$$

Furthermore, since $\Psi(t) \ge t^{\alpha_1-\gamma_1-1}$, we have

$$\begin{aligned} f_1\big(I^{\gamma_1}\Phi(t),\Phi(t),A\Phi(t)\big) &\le F\big(v_1(t)\big) \\ &\le f_1\big(I^{\gamma_1}t^{\alpha_1-\gamma_1-1},t^{\alpha_1-\gamma_1-1},At^{\alpha_1-\gamma_1-1}\big), \quad \forall v_1(t)\in E. \end{aligned} \tag{3.24}$$

It follows from (3.11), for any $v_1(t)\in E$

$$\begin{aligned} D_{\lambda^*}v_1(t) &= \big(\lambda^*\big)^{q_1-1}\int_0^1 H_1(t,s)\bigg(\int_0^1 G(\beta_1,s,\tau)F\big(v_1(\tau)\,d\tau\big)\bigg)^{q_1-1} ds \\ &\le \big(\lambda^*\big)^{q_1-1}\int_0^1 H_1(t,s)\bigg(\int_0^1 G(\beta_1,s,\tau)f_1\big(I^{\gamma_1}\tau^{\alpha_1-\gamma_1-1}, \\ &\quad \tau^{\alpha_1-\gamma_1-1},A\tau^{\alpha_1-\gamma_1-1}\big)\,d\tau\bigg)^{q_1-1} ds \\ &\le \big(\lambda^*\big)^{q_1-1}d_1\bigg(\frac{\beta_1-1}{\Gamma(\beta_1)}\int_0^1 f_1\bigg(\frac{\Gamma(\alpha_1-\gamma_1)}{\Gamma(\alpha_1)}\tau^{\alpha_1-1},\tau^{\alpha_1-\gamma_1-1},b\tau^{\alpha_2-\gamma_2-1}\bigg)\,d\tau\bigg)^{q_1-1} ds \\ &< +\infty, \end{aligned} \tag{3.25}$$

namely, the operator D_{λ^*} is uniformly bounded.

On the other hand, let $\Omega \subset E$ be bounded. As the function $H_1(t,s), G(\beta_1,t,s)$ is uniformly continuous on $[0,1]\times[0,1]$, $D_{\lambda^*}(\Omega)$ is equicontinuous. By the Arzela-Ascoli theorem, we have $D_{\lambda^*}: E \to E$ is completely continuous. Thus by using the Schauder fixed point theorem, D_{λ^*} has at least one fixed point x such the $x = D_{\lambda^*}x$.

Now we prove

$$\Psi(t) \le x(t) \le \Phi(t), \quad t\in[0,1].$$

Since x is a fixed point of D_{λ^*}, by (3.18) and (3.23), we have

$$\begin{aligned} &D^{\alpha_1-\gamma_1}x(0) = D^{\alpha_1-\gamma_1}x(1) = 0, \qquad x(0) = 0, \qquad D^{\alpha_1-\gamma_1-1}x(1) = \xi_1 I^{\omega_1}\big(x(\eta_1)\big), \\ &D^{\alpha_1-\gamma_1}\Phi(0) = D^{\alpha_1-\gamma_1}\Phi(1) = 0, \qquad \Phi(0) = 0, \qquad D^{\alpha_1-\gamma_1-1}\Phi(1) = \xi_1 I^{\omega_1}\big(\Phi(\eta_1)\big). \end{aligned} \tag{3.26}$$

From (3.9), (3.16), (3.24), and noting that x is a fixed point of D_{λ^*}, we also have

$$\begin{aligned} &D^{\beta_1}\big(\varphi_{p_1}\big(-D^{\alpha_1-\gamma_1}\Phi\big)\big)(t) - D^{\beta_1}\big(\varphi_{p_1}\big(-D^{\alpha_1-\gamma_1}x\big)\big)(t) \\ &\quad = -\lambda^* f_1\big(I^{\gamma_1}t^{\alpha_1-\gamma_1-1},t^{\alpha_1-\gamma_1-1},At^{\alpha_1-\gamma_1-1}\big) + \lambda^* F(x(t) \le 0. \end{aligned}$$

Let $z(t) = \varphi_{p_1}(-D^{\alpha_1-\gamma_1}\Phi)(t) - \varphi_{p_1}(-D^{\alpha_1-\gamma_1}x)(t)$, then

$$\begin{aligned}
&D^{\beta_1}z(t) = D^{\beta_1}\bigl(\varphi_{p_1}\bigl(-D^{\alpha_1-\gamma_1}\Phi(t)\bigr)\bigr) - D^{\beta_1}\bigl(\varphi_{p_1}\bigl(-D^{\alpha_1-\gamma_1}x(t)\bigr)\bigr) \le 0,\\
&z(0) = \varphi_{p_1}\bigl(-D^{\alpha_1-\gamma_1}\Phi(0)\bigr) - \varphi_{p_1}\bigl(-D^{\alpha_1-\gamma_1}x(0)\bigr) = 0,\\
&z(1) = \varphi_{p_1}\bigl(-D^{\alpha_1-\gamma_1}\Phi(1)\bigr) - \varphi_{p_1}\bigl(-D^{\alpha_1-\gamma_1}x(1)\bigr) = 0.
\end{aligned}$$

In view of Lemmas 2.1 and 2.3, we obtain

$$z(t) \ge 0,$$

i.e.

$$\varphi_{p_1}\bigl(-D^{\alpha_1-\gamma_1}\Phi(t)\bigr) - \varphi_{p_1}\bigl(-D^{\alpha_1-\gamma_1}x(t)\bigr) \ge 0,$$

Noticing that φ_{p_1} is monotone increasing, we have

$$-D^{\alpha_1-\gamma_1}\Phi(t) \ge -D^{\alpha_1-\gamma_1}x(t),$$

i.e.

$$-D^{\alpha_1-\gamma_1}\bigl(\Phi(t) - x(t)\bigr) \ge 0.$$

It follows from Lemma 2.6 and (3.26)

$$\Phi(t) - x(t) \ge 0.$$

Then we have $x(t) \le \Phi(t)$ on $[0,1]$. In the same way we also have $x(t) \ge \Psi(t)$ on $[0,1]$. So

$$\Psi(t) \le x(t) \le \Phi(t). \tag{3.27}$$

Consequently, $F(x(t)) = f_1(I^{\gamma_1}x(t), x(t), Ax(t)), t \in [0,1]$. Hence $x(t)$ is a positive solution of the problem (2.6). Finally, by (3.27) and $\Phi, \Psi \in P$, we have

$$l_\Psi t^{\alpha_1-\gamma_1-1} \le \Psi(t) \le x(t) \le \Phi(t) \le l_\Phi^{-1} t^{\alpha_1-\gamma_1-1}.$$

Then by Lemmas 2.5

$$\begin{cases} u_1(t) = I^{\gamma_1}x(t),\\ u_2(t) = I^{\gamma_1}v_2(t),\end{cases}$$

where $v_2(t) = \int_0^1 H_2(t,s)(\int_0^1 G(\beta_2,s,\tau)f_2(\tau, I^{\gamma_1}x(\tau))\,d\tau)^{q_2-1}\,ds$ is the unique positive solution of system (1.1).

Since the process is similar to (3.4) and (3.5) we obtain

$$\begin{aligned}
v_2(t) &= \int_0^1 H_2(t,s)\biggl(\int_0^1 G(\beta_2,s,\tau)f_2\bigl(\tau, I^{\gamma_1}x(\tau)\bigr)\,d\tau\biggr)^{q_2-1}\,ds\\
&\le \int_0^1 H_2(t,s)\biggl(\int_0^1 G(\beta_2,s,\tau)f_2\biggl(\tau, l_\Phi^{-1}\frac{\Gamma(\alpha_1-\gamma_1)}{\Gamma(\alpha_1)}\tau^{\alpha_1-1}\biggr)\,d\tau\biggr)^{q_2-1}\,ds\\
&\le a l_\Phi^{-\sigma(q_2-1)} t^{\alpha_2-\gamma_2-1}
\end{aligned}$$

and

$$\begin{aligned} v_2(t) &= \int_0^1 H_2(t,s)\left(\int_0^1 G(\beta_2,s,\tau)f_2\big(\tau,I^{\gamma_1}x(\tau)\big)\,d\tau\right)^{q_2-1} ds \\ &\geq \int_0^1 H_2(t,s)\left(\int_0^1 G(\beta_2,s,\tau)f_2\left(\tau,l_\Psi\frac{\Gamma(\alpha_1-\gamma_1)}{\Gamma(\alpha_1)}\tau^{\alpha_1-1}\right)d\tau\right)^{q_2-1} ds \\ &\geq b l_\Psi^{\sigma(q_2-1)} t^{\alpha_2-\gamma_2-1}, \end{aligned}$$

i.e.

$$b l_\Psi^{\sigma(q_2-1)}\frac{\Gamma(\alpha_2-\gamma_2)}{\Gamma(\alpha_2)}t^{\alpha_2-1} \leq u_2(t) = I^{\gamma_1}v_2(t) \leq a l_\Phi^{-\sigma(q_2-1)}\frac{\Gamma(\alpha_2-\gamma_2)}{\Gamma(\alpha_2)}t^{\alpha_2-1}. \tag{3.28}$$

□

Example Consider the following boundary value problem:

$$\begin{cases} -D^{\frac{5}{2}}(\varphi_3(-D^{\frac{4}{3}}u_1))(t) = \lambda(u_1^{-\frac{2}{9}}(t) + [D^{\frac{1}{6}}u_1(t)]^{-\frac{1}{2}} + [D^{\frac{1}{4}}u_2(t)]^{-\frac{1}{8}}), \\ -D^{\frac{11}{4}}(\varphi_4(-D^{\frac{3}{2}}u_2))(t) = (t^2+1)u_1^{\frac{1}{7}}(t), \\ D^{\frac{4}{3}}u_1(0) = D^{\frac{4}{3}}u_1(1) = 0, D^{\frac{1}{6}}u_1(0) = 0, D^{\frac{1}{3}}u_1(1) = 2I^{\frac{5}{6}}(D^{\frac{1}{6}}u_1(\frac{1}{3})), \\ D^{\frac{3}{2}}u_2(0) = D^{\frac{3}{2}}u_2(1) = 0, D^{\frac{1}{4}}u_2(0) = 0, D^{\frac{1}{2}}u_2(1) = 5I^{\frac{7}{4}}(D^{\frac{1}{4}}u_2(\frac{1}{2})). \end{cases} \tag{3.29}$$

Let $\alpha_1 = \frac{4}{3}, \alpha_2 = \frac{3}{2}, \beta_1 = \frac{5}{2}, \beta_2 = \frac{11}{4}, \gamma_1 = \frac{1}{6}, \gamma_2 = \frac{1}{4}, p_1 = 3, p_2 = 4, \omega_1 = \frac{5}{6}, \omega_2 = \frac{7}{4}, \xi_1 = 2, \xi_2 = 5, \eta_1 = \frac{1}{3}, \eta_2 = \frac{1}{2}$.

First, we have

$$\Gamma(\alpha_1-\gamma_1+\omega_1) = \Gamma(2) = 1 > \xi_1\eta_1^{\alpha_1-\gamma_1+\omega_1-1} = 2\left(\frac{1}{3}\right),$$

$$\Gamma(\alpha_2-\gamma_2+\omega_2) = \Gamma(3) = 2 > \xi_2\eta_2^{\alpha_2-\gamma_2+\omega_2-1} = 5\left(\frac{1}{2}\right)^2,$$

and $q_1 = \frac{3}{2}, q_2 = \frac{4}{3}$, then (s_0), (s_1), and (s_2) hold.

Second, let

$$f_1(x,y,z) = x^{-\frac{2}{9}} + y^{-\frac{1}{2}} + z^{-\frac{1}{8}}, \qquad f_2(t,x) = \big(t^2+1\big)x^{\frac{1}{7}}, \qquad \sigma = \frac{1}{3} < \frac{1}{q_2-1} = 3,$$

and for all $r \in (0,1)$, $(x,y,z) \in (0,+\infty)^3$, $(t,x) \in (0,1)\times(0,+\infty)$,

$$\begin{aligned} &f_1(rx,ry,rz) = r^{-\frac{2}{9}}x^{-\frac{2}{9}} + r^{-\frac{1}{2}}y^{-\frac{1}{2}} + r^{-\frac{1}{8}}z^{-\frac{1}{8}} \leq r^{-\frac{1}{2}}f_1(x,y,z), \\ &f_2(t,rx) = \big(t^2+1\big)r^{\frac{1}{7}}x^{\frac{1}{7}} \geq r^{\frac{1}{3}}f_2(t,x), \end{aligned}$$

which implies that (H_1), (H_2) hold. On the other hand, by direct calculation, we have $f_1(1,1,1) = 3 \neq 0$, and then

$$\begin{aligned} b &= e_2\int_0^1 \big[1-(1-s)^{\alpha_2-\gamma_2-1}\big]s^{(\beta_2-1)(q_2-1)}(1-s)^{q_2-1}\,ds \\ &\quad\times\left(\frac{1}{\Gamma(\beta_2)}\int_0^1 \tau(1-\tau)^{\beta_2-1}f_2\left(\tau,\frac{\Gamma(\alpha_1-\gamma_1)}{\Gamma(\alpha_1)}\tau^{\alpha_1-1}\right)d\tau\right)^{q_2-1} \end{aligned}$$

$$= \Gamma\left(\frac{5}{4}\right)\int_0^1 \left[1-(1-s)^{\frac{1}{4}}\right]s^{\frac{7}{4}\frac{1}{3}}(1-s)^{\frac{1}{3}}\,ds$$
$$\times\left(\frac{1}{\Gamma(\frac{11}{4})}\int_0^1 \tau(1-\tau)^{\frac{7}{4}}\left(\tau^2+1\right)\left[\frac{\Gamma(\frac{7}{6})}{\Gamma(\frac{4}{3})}\right]^{\frac{1}{7}}\tau^{\frac{1}{21}}\,d\tau\right)^{\frac{1}{3}} > 0.$$

Thus

$$\int_0^1 f_1\left(\frac{\Gamma(\alpha_1-\gamma_1)}{\Gamma(\alpha_1)}t^{\alpha_1-1}, t^{\alpha_1-\gamma_1-1}, bt^{\alpha_2-\gamma_2-1}\right)dt$$
$$= \int_0^1 \left\{\left[\frac{\Gamma(\frac{7}{6})}{\Gamma(\frac{4}{3})}t^{\frac{1}{3}}\right]^{-\frac{2}{9}} + \left[t^{\frac{1}{6}}\right]^{-\frac{1}{2}} + \left[bt^{\frac{1}{4}}\right]^{-\frac{1}{8}}\right\}dt$$
$$= \int_0^1 \left\{\left[\frac{\Gamma(\frac{7}{6})}{\Gamma(\frac{4}{3})}\right]^{-\frac{2}{9}}t^{-\frac{2}{27}} + t^{-\frac{1}{12}} + b^{-\frac{1}{8}}t^{-\frac{1}{32}}\right\}dt < +\infty.$$

Hence, (H_3) holds. Then by Theorem 3.1 there exists a constant $\lambda^* > 0$ such that for any $\lambda \in (\lambda^*, +\infty)$, the BVP (1.1) has at least one positive solution $(u_1(t), u_2(t))$.

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The author declares to have no competing interests.

Author's contributions
The whole work was carried out by the author.

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