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Analytical solutions to fractional evolution equations with almost sectorial operators

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Abstract
In this paper, with the aid of functional analysis, for almost sectorial operators and some fixed point theorems, we study the existence and uniqueness of mild solutions to fractional neutral evolution equations with almost sectorial operators. We also show that mild solutions can become strong and classical solutions under appropriate assumptions. Finally, we present an example to illustrate the applicability of our results.

Keywords: fractional neutral evolution equations; almost sectorial operators; fixed point theorems; existence and uniqueness; mild solutions; strong solutions and classical solutions

1 Introduction

Throughout this paper, by $(X, \|\cdot\|)$ we denote a Banach space. As usual, for a linear operator A, $D(A)$, $R(A)$, and $\sigma(A)$ stand for the domain, range, and spectrum of A, respectively. Moreover, $\mathcal{L}(X)$ denotes the space of all bounded linear operators on X.

A sectorial operator is a linear operator A in a Banach space whose spectrum lies in a closed sector $S_\omega = \{z \in \mathbb{C}\backslash\{0\} \mid |\arg z| \leq \mu\} \cup \{0\}$ for some $0 \leq \omega < \pi$ and whose resolvent $(z-A)^{-1}$ satisfies the estimate

$$\left\|(z-A)^{-1}\right\| \leq M|z|^{-1} \quad \text{for all } z \notin S_\omega. \tag{1.1}$$

Several elliptic differential operators considered in the spaces of continuous functions or Lebesgue spaces belong to the class of sectorial operators. Therefore, many PDEs with elliptic operators can be transformed into evolution equations with sectorial operators in a Banach space; for example, see [1–3].

In 1972, Wahl [4] first pointed out that the resolvent estimates of elliptic differential operators considered in spaces of regular functions, such as the spaces of Hölder continuous functions, do not satisfy estimate (1.1). However, such operators satisfy the following estimate for some $-1 < \gamma < 0$:

$$\left\|(z-A)^{-1}\right\| \leq M|z|^{\gamma} \quad \text{for all } z \notin S_\omega.$$

Now we recall the following definition [4–8].

Definition 1.1 Let $-1 < \gamma < 0$ and $0 \leq \omega < \pi$. By $\Theta_\omega^\gamma(X)$ we denote the set of all closed linear operators $A : D(A) \subseteq X \to X$ such that

(i) the spectral set $\sigma(A)$ of A is in the sector $S_\mu = \{z \in \mathbb{C}\backslash\{0\} \mid |\arg z| \leq \mu\} \cup \{0\}$, that is, $\sigma(A) \subseteq S_\mu$;

(ii) for every $\omega < \mu < \pi$, there exists a constant $C_\mu > 0$ such that

$$\left\|(z - A)^{-1}\right\| \leq C_\mu |z|^\gamma \quad \text{for all } z \notin S_\mu.$$

A linear operator is called an almost sectorial operator on X if it belongs to $\Theta_\omega^\gamma(X)$. Concerning the relationship between sectorial and almost sectorial operators, it has been found that a sectorial operator is an almost sectorial operator, but the converse is not true [4, 5, 7–10]. Some recent results on almost sectorial operators can be found in [5, 7–15].

The use of fractional calculus in the mathematical modeling of engineering and physical problems has become increasingly popular in recent years. Examples include material sciences, mechanics, wave propagation, signal processing, system identification, and so on. In consequence, the topic of fractional (ordinary, partial, functional) differential equations has developed into a hot research area; for example, see [16–42]. Wang et al. [6] studied a fractional-order Cauchy problem with almost sectorial operators. In [43], the author discussed mild solutions for abstract fractional differential equations with almost sectorial operators and infinite delay. More recently, in [14], the authors investigated fractional Cauchy problems with almost sectorial operators. Fractional functional differential equations are used to describe anomalous diffusion processes with memory or hereditary properties. For details and some recent results on functional fractional differential equations, we refer the reader to a series of papers [25–28, 30, 31, 34, 36, 39, 44, 45]. To the best of our knowledge, the study of fractional neutral evolution equations (FNEEs) with almost sectorial operators is yet to be initiated. The aim of this paper is to investigate the existence and uniqueness of solutions of FNEEs.

The rest of the paper is organized as follows. In Section 2, we introduce some notation, definitions, and basic properties about fractional derivatives and functional analysis associated with almost sectorial operators. In Section 3, we prove the existence and uniqueness of mild solutions to FNEEs with almost sectorial operators. Under some suitable assumptions, we also show that mild solutions correspond to strong and classical solutions. Section 4 contains an example for illustration of our results, and we conclude our work in Section 5.

2 Preliminaries

Here, we recall some preliminary material related to our work [16, 19–21].

2.1 Fractional integrals and derivatives

In this subsection, we give some basic definitions and properties of the fractional integral and derivatives.

Definition 2.1 The Riemann-Liouville integral ${}_aI_t^\alpha x$ and the Riemann-Liouville fractional derivative ${}_aD_t^\alpha x$ are respectively defined as

$$\left({}_aI_t^\alpha x\right)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} x(\tau)\, d\tau, \quad t > a,$$

and

$$\left({}_aD_t^\alpha x\right)(t) = \left(D^{m+1}{}_aI_t^{m+1-\alpha} x\right)(t) = \frac{1}{\Gamma(m+1-\alpha)}\frac{d^{m+1}}{dt^{m+1}}\int_a^t (t-\tau)^{m-\alpha}x(\tau)\,d\tau, \quad t > a,$$

provided that the right-hand side is pointwise defined on $[a,\infty)$. Here, Γ denotes the gamma function, and $m \in \mathbb{N}$.

Definition 2.2 The Caputo fractional derivative ${}_a^CD_t^\alpha x$ is defined via the Riemann-Liouville derivative as

$$\left({}_a^CD_t^\alpha x\right)(t) = {}_aD_t^\alpha\left(x(t) - \sum_{k=0}^{m}\frac{x^{(k)}(a)}{k!}(t-a)^k\right), \quad t > a, m \le \alpha < m+1, m \in \mathbb{N}.$$

Note that if $x^{(k)}(a) = 0, k = 0,1,\ldots,m$, then $({}_a^CD_t^\alpha x)(t)$ coincides with $({}_aD_t^\alpha x)(t)$.

Now we enlist some properties of the Riemann-Liouville fractional integral and the Caputo derivative (see [16, 19–21]).

Proposition 2.1 *Let $\alpha,\beta > 0$. Then the following properties hold*:

(i) $({}_aI_t^\alpha\,{}_aI_t^\beta x)(t) = ({}_aI_t^{\alpha+\beta}x)(t)$ *for all* $x \in L^1(a,b)$;

(ii) $({}_aI_t^\alpha(x*y))(t) = (({}_aI_t^\alpha x)*y)(t)$ *for all* $x,y \in L^1(a,b)$, *where* $*$ *denotes convolution*;

(iii) *The Caputo derivative* ${}_a^CD_t^\alpha$ *is a left inverse of* ${}_aI_t^\alpha$, *that is*, $({}_a^CD_t^\alpha\,{}_aI_t^\alpha x)(t) = x(t)$ *for all* $x \in L^1(a,b)$; *in general, it is not a right inverse. In fact, for* $m \le \alpha < m+1$, $m \in \mathbb{N}$, *and* $x \in C^{m+1}([a,b])$, *we have* $({}_aI_t^\alpha\,{}_a^CD_t^\alpha x)(t) = x(t) - \sum_{k=0}^m \frac{x^{(k)}(a)}{k!}(t-a)^k$.

2.2 Special functions

Here we present some basic definitions and properties of two special functions that we need in the sequel (see [16, 19, 20, 29]).

Definition 2.3 The two-parameter Mittag-Leffler function is defined by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k\alpha+\beta)}, \quad \alpha,\beta > 0, z \in \mathbb{C}.$$

In particular, if $\beta = 1$, then $E_{\alpha,\beta}$ coincides with the one-parameter Mittag-Leffler function $E_\alpha(z)$, that is, $E_{\alpha,1}(z) = E_\alpha(z)$. If $\alpha = \beta = 1$, then $E_{1,1}(z) = e^z$.

Definition 2.4 The Wright-type function is defined by

$$\Phi_\alpha(z) = \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{(-z)^k}{(k-1)!}\Gamma(k\alpha)\sin(k\pi\alpha), \quad 0 < \alpha < 1, z \in \mathbb{C}.$$

The following properties of the Wright-type function (cf. [29]) are useful in establishing the definition of mild solutions to FNEEs with almost sectorial operators.

Proposition 2.2 *Let* $-1 < \nu < \infty$, $\lambda > 0$. *Then the following properties hold*:

(i) $\Phi_\alpha(t) \ge 0$ *for all* $t > 0$;

(ii) $\int_0^\infty \frac{\alpha}{t^{\alpha+1}}\Phi_\alpha(\frac{1}{t^\alpha})e^{-\lambda t}\,dt = e^{-\lambda^\alpha}$;

(iii) $\int_0^\infty \Phi_\alpha(t)t^\nu\,dt = \frac{\Gamma(1+\nu)}{\Gamma(1+\alpha\nu)}$; *in particular*, $\int_0^\infty \Phi_\alpha(t)\,dt = 1$;
(iv) $\int_0^\infty \Phi_\alpha(t)e^{-zt}\,dt = E_\alpha(-z)$, $z \in \mathbb{C}$;
(v) $\int_0^\infty \alpha t\Phi_\alpha(t)e^{-zt}\,dt = E_{\alpha,\alpha}(-z)$, $z \in \mathbb{C}$.

2.3 Complex powers

In this subsection, we give the definition and properties of complex powers of almost sectorial operators, which will be used in the next section. For more details, see [5, 6].

We define the path $\Upsilon_\theta := \{\mathbb{R}_+e^{i\theta}\} \cup \{\mathbb{R}_+e^{-i\theta}\}$ $(0 < \theta < \pi)$ oriented so that S_θ^0 lies to the left of Υ_θ. In the forthcoming analysis, we write Θ_ω^γ instead of $\Theta_\omega^\gamma(X)$. Let $\beta \in \mathbb{C}$ and $A \in \Theta_\omega^\gamma$ with $-1 < \gamma < 0$ and $0 < \omega < \pi/2$. Then the complex power A^β of A is defined by

$$A^\beta := z^\beta(A) = \frac{1}{2\pi i}\int_{\Upsilon_\theta} z^\beta(z-A)^{-1}\,dz, \quad z \in \mathbb{C}\backslash(-\infty,0].$$

Now we list some properties of A^β (see [5]).

Proposition 2.3 *Let $A \in \Theta_\omega^\gamma$ with $-1 < \gamma < 0$ and $0 < \omega < \pi/2$. Then, for all $\alpha, \beta \in \mathbb{C}$, the following statements hold.*
(i) *The operator A^β is closed.*
(ii) $A^\alpha A^\beta \subseteq A^{\alpha+\beta}$. *Moreover, if A^β is bounded, then $A^\alpha A^\beta = A^{\alpha+\beta}$.*
(iii) *A^β is injective, and $(A^\beta)^{-1} = A^{-\beta}$.*
(iv) $A^n = \underbrace{A\cdots A}_{n\ \text{times}}$ *for all $n \in \mathbb{N}$ and $A^0 = I$.*
(v) *If $\operatorname{Re}(\beta) > 1+\gamma$, then $A^{-\beta}$ is bounded.*

Based on Proposition 2.3, we now prove a lemma, which will be used in the next section.

Lemma 2.1 *Let $A \in \Theta_\omega^\gamma$ with $-1 < \gamma < 0$ and $0 < \omega < \pi/2$, and let $\operatorname{Re}(\beta) > 1+\gamma$. Then $A^{1-\beta}A^\beta = A^\beta A^{1-\beta} = A$.*

Proof Since $\operatorname{Re}(\beta) > 1+\gamma$, by Proposition 2.3(ii), (v) we get $A^{\beta-1}A^{-\beta} = A^{-1}$. On the other hand, by Proposition 2.3(iii) we can verify that $(A^{\beta-1}A^{-\beta})^{-1} = A^\beta A^{1-\beta}$. Therefore, we have $A^\beta A^{1-\beta} = A$, and the proof is completed. □

It is worth noting that Proposition 2.3(v) implies that the operator $A^{-\beta}$ belongs to $\mathcal{L}(X)$ whenever $\operatorname{Re}(\beta) > 1+\gamma$. So, in this situation, the linear space $X^\beta := D(A^\beta)$ $(\operatorname{Re}(\beta) > 1+\gamma)$ is a Banach space with the graph norm $\|x\|_\beta = \|A^\beta x\|$, $x \in X^\beta$. Of particular interest is that these spaces X^β will provide the basic topology for analyzing the solutions of FNEEs with almost sectorial operators.

2.4 Properties of the operators $\mathcal{S}_\alpha(t)$ and $\mathcal{P}_\alpha(t)$

We introduce families of operators $\{T(t)\}_{t\in S^0_{\pi/2-\omega}}$, $\{\mathcal{S}_\alpha(t)\}_{t\in S^0_{\pi/2-\omega}}$, and $\{\mathcal{P}_\alpha(t)\}_{t\in S^0_{\pi/2-\omega}}$ associated with the operator A as follows:

$$\begin{aligned}
T(t) &:= e^{-tz}(A) = \frac{1}{2\pi i}\int_{\Upsilon_\theta} e^{-tz}(z-A)^{-1}\,dz, \quad t \in S^0_{\pi/2-\omega}, z \in \mathbb{C}\backslash(-\infty,0],\\
\mathcal{S}_\alpha(t) &:= E_\alpha\big(-zt^\alpha\big)(A) = \frac{1}{2\pi i}\int_{\Upsilon_\theta} E_\alpha\big(-zt^\alpha\big)(z-A)^{-1}\,dz,\\
&\quad t \in S^0_{\pi/2-\omega}, z \in \mathbb{C}\backslash(-\infty,0],
\end{aligned} \tag{2.1}$$

$$\mathcal{P}_\alpha(t) := E_{\alpha,\alpha}\left(-zt^\alpha\right)(A) = \frac{1}{2\pi i}\int_{\Upsilon_\theta} E_{\alpha,\alpha}\left(-zt^\alpha\right)(z-A)^{-1}\,dz,$$
$$t \in S^0_{\pi/2-\omega}, z \in \mathbb{C}\backslash(-\infty,0].$$

These operators appear in representations of solutions for FNEEs with almost sectorial operators. Further details about these operators can be found in [5, 6].

Notice that the operator family $\{T(t)\}_{t\in S^0_{\pi/2-\omega}}$ is a semigroup in view of the semigroup property: $T(s+t) = T(s)T(t)$ for all $s,t \in S^0_{\pi/2-\omega}$. Also, the operator $T(t)$ can characterize the resolvent $(z+A)^{-1}$ of $-A$ as

$$(z+A)^{-1} = \int_0^\infty e^{-zt}T(t)\,dt, \quad z \in \mathbb{C}, \operatorname{Re}(z) > 0. \tag{2.2}$$

From (2.1) and (2.2) it follows that there is a one-to-one correspondence between A and the semigroup $T(t)$. By Proposition 2.2(iv), (v) and definition (2.1) we can obtain the third property, that is, the operators $\mathcal{S}_\alpha(t)$ and $\mathcal{P}_\alpha(t)$ can be represented by $T(t)$ as

$$\mathcal{S}_\alpha(t)x = \int_0^\infty \Phi_\alpha(s)T\left(st^\alpha\right)x\,ds, \quad t \in S^0_{\pi/2-\omega}, x \in D\left(\mathcal{S}_\alpha(t)\right), \tag{2.3}$$

$$\mathcal{P}_\alpha(t)x = \int_0^\infty \alpha s\Phi_\alpha(s)T\left(st^\alpha\right)x\,ds, \quad t \in S^0_{\pi/2-\omega}, x \in D\left(\mathcal{P}_\alpha(t)\right). \tag{2.4}$$

For the reader's convenience, we recall some more properties of the operator T in the following proposition.

Proposition 2.4 *Let $A \in \Theta^\gamma_\omega$ with $-1 < \gamma < 0$ and $0 < \omega < \pi/2$. Then the following properties hold.*

(i) *$T(t)$ is analytic in $S^0_{\pi/2-\omega}$, and $\frac{d^nT(t)}{dt^n} = (-A)^nT(t)$, $t \in S^0_{\pi/2-\omega}$, $n \in \mathbb{N}$.*
(ii) *There is a constant $C_0 = C_0(\gamma)$ such that $\|T(t)\| \leq C_0t^{-\gamma-1}$, $t > 0$.*
(iii) *The range $R(T(t))$ of $T(t)$, $t \in S^0_{\pi/2-\omega}$, is contained in $D(A^\infty)$. In particular, $R(T(t)) \subseteq D(A^\beta)$ for all $\beta \in \mathbb{C}$ with $\operatorname{Re}(\beta) > 0$, $A^\beta T(t)x = \frac{1}{2\pi i}\int_{\Upsilon_\theta} z^\beta e^{-tz}(z-A)^{-1}\,dz$ for all $x \in X$, and there exists a constant $C^* = C^*(\beta,\gamma) > 0$ such that $\|A^\beta T(t)\| \leq C^*t^{-\gamma-\operatorname{Re}(\beta)-1}$ for all $t > 0$.*
(iv) *If $\beta > 1+\gamma$, then $D(A^\beta) \subseteq \Sigma_T = \{x \in X \mid \lim_{t\to 0^+} T(t)x = x\}$.*

In the following, we describe the properties of the operators $\mathcal{S}_\alpha(t)$ and $\mathcal{P}_\alpha(t)$ [5, 6].

Proposition 2.5 *Let $A \in \Theta^\gamma_\omega$ with $-1 < \gamma < 0$ and $0 < \omega < \pi/2$. The following statements hold.*

(i) *For each fixed $t \in S^0_{\pi/2-\omega}$, $\mathcal{S}_\alpha(t)$ and $\mathcal{P}_\alpha(t)$ are linear and bounded operators on X. Moreover, there exist constants $C_s = C(\alpha,\gamma) > 0$ and $C_p = C(\alpha,\gamma) > 0$ such that for all $t > 0$, $\|\mathcal{S}_\alpha(t)\| \leq C_st^{-\alpha(1+\gamma)}$ and $\|\mathcal{P}_\alpha(t)\| \leq C_pt^{-\alpha(1+\gamma)}$.*
(ii) *For $t > 0$, $\mathcal{S}_\alpha(t)$ and $\mathcal{P}_\alpha(t)$ are continuous in the uniform operator topology. Moreover, for every $r > 0$, the continuity is uniform on $[r,\infty)$.*
(iii) *For each fixed $t \in S^0_{\pi/2-\omega}$ and all $x \in D(A)$, $(\mathcal{S}_\alpha(t) - I)x = \int_0^t -s^{\alpha-1}A\mathcal{P}_\alpha(s)x\,ds$.*
(iv) *For all $x \in D(A)$ and $t > 0$, ${}^C_0D^\alpha_t\mathcal{S}_\alpha(t)x = -A\mathcal{S}_\alpha(t)x$.*
(v) *For all $t > 0$, $\mathcal{S}_\alpha(t) = {}_0I^\alpha_t(t^{\alpha-1}\mathcal{P}_\alpha(t))$.*
(vi) *Let $\beta > 1+\gamma$. For all $x \in D(A^\beta)$, $\lim_{t\to 0^+}\mathcal{S}_\alpha(t)x = x$.*

Proposition 2.6 *Let $A \in \Theta_\omega^\gamma$ with $-1 < \gamma < 0$ and $0 < \omega < \pi/2$, and let $0 < \beta < 1 - \gamma$. Then*

(i) *the range $R(\mathcal{P}_\alpha(t))$ of $\mathcal{P}_\alpha(t)$ for $t > 0$ is contained in $D(A^\beta)$;*

(ii) *$\mathcal{S}'_\alpha(t)x = -t^{\alpha-1}A\mathcal{P}_\alpha(t)x$, and $\mathcal{S}'_\alpha(t)x$ for $x \in D(A)$ is locally integrable on $(0,\infty)$;*

(iii) *for all $x \in D(A)$ and $t > 0$, $\|A\mathcal{S}_\alpha(t)x\| \le Ct^{-\alpha(1+\gamma)}\|Ax\|$, where C is a constant depending on γ, α.*

Lemma 2.2 *Let $A \in \Theta_\omega^\gamma$ with $-1 < \gamma < 0$ and $0 < \omega < \pi/2$, and let $0 < \beta < 1 - \gamma$. Then, for each fixed $t \in S^0_{\pi/2-\omega}$, $\mathcal{P}_\alpha(t)$ is a bounded linear operator on X^β. Moreover, there exists a positive constant C_0 such that for all $t > 0$,*

$$\left\|A^\beta \mathcal{P}_\alpha(t)x\right\| \le \alpha C_0 \frac{\Gamma(1-\gamma-\beta)}{\Gamma(1-\alpha(\gamma+\beta))} t^{-\alpha(\gamma+\beta+1)}\|x\|.$$

Proof By relation (2.4), Proposition 2.2(iii), and Proposition 2.4(iii) we get

$$\begin{aligned}\left\|A^\beta \mathcal{S}_\alpha(t)x\right\| &\le \int_0^\infty \alpha s\Psi_\alpha(s)\left\|A^\beta T\left(st^\alpha\right)\right\| ds\|x\| \\ &\le \alpha C_0 t^{-\alpha(\gamma+\beta+1)}\int_0^\infty s^{-\gamma-\beta}\Psi_\alpha(s)\,ds\|x\| \\ &= \alpha C_0 \frac{\Gamma(1-\gamma-\beta)}{\Gamma(1-\alpha(\gamma+\beta))} t^{-\alpha(\gamma+\beta+1)}\|x\|.\end{aligned}$$

The proof is completed. □

3 Main results

Consider a problem of fractional neutral evolution equations (FNEEs) with almost sectorial operator given by

$$\begin{cases} {}^C_0D^\alpha_t(x(t)-g(t,x_t)) + Ax(t) = f(t,x_t), & 0 < \alpha < 1, t \in [0,T], \\ x(t) = \varphi(t), & t \in [-h,0], \end{cases} \tag{3.1}$$

where $h, T > 0$, A is an almost sectorial operator, that is, $A \in \Theta_\omega^\gamma$ $(-1 < \gamma < 0, 0 < \omega < \pi/2)$, $f(t,x_t)$, $g(t,x_t) : [0,T] \times C([-h,0],X) \to X$ are given functions, $\varphi(t) : [-h,0] \to X$ is an initial function, and x_t is defined by $x_t(s) = x(t+s)$ for $s \in [-h,0]$.

To study problem (3.1), we need the following assumption.

(H) $x(t) \in C([-h,T],X)$, $x(t) \in D(A)$ for all $t \in [0,T]$, $Ax \in L^1((0,T),X)$, $f(t,x_t) \in L^1((0,T),X)$, and there exists a constant β such that $\beta > 1+\gamma$ and $A^\beta g(t,x_t) \in L^1((0,T),X)$.

To define a mild solution of (3.1), we prove the following lemma.

Lemma 3.1 *Assume that condition* (H) *holds and $x(t)$ satisfies problem* (3.1). *Then, for every $\varphi(t) \in C([-h,0],X^\beta)$, $x(t)$ satisfies the integral equation*

$$x(t) = \begin{cases} \mathcal{S}_\alpha(t)(\varphi(0) - g(0,x_0)) + g(t,x_t) - \int_0^t (t-s)^{\alpha-1}A\mathcal{P}_\alpha(t-s)g(s,x_s)\,ds \\ \quad + \int_0^t (t-s)^{\alpha-1}\mathcal{P}_\alpha(t-s)f(s,x_s)\,ds, \quad t \in [0,T], \\ \varphi(t), \quad t \in [-h,0]. \end{cases}$$

Proof In view of condition (H), we know that (Theorem 3.1 in [16]) problem (3.1) is equivalent to the fractional integral equation

$$\begin{cases} x(t) = \varphi(0) - g(0,x_0) + g(t,x_t) - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}Ax(s)\,ds \\ \qquad + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s,x_s)\,ds, \quad t\in[0,T], \\ x(t) = \varphi(t), \quad t\in[-h,0]. \end{cases}$$

Applying the Laplace transform to this integral equation, we get

$$X(s) = s^{\alpha-1}\left(s^\alpha + A\right)^{-1}\left(\varphi(0) - g(0,x_0)\right) + s^\alpha\left(s^\alpha + A\right)^{-1}G(s) + \left(s^\alpha + A\right)^{-1}F(s), \tag{3.2}$$

where

$$X(s) = \int_0^\infty e^{-st}x(t)\,dt, \qquad G(s) = \int_0^\infty e^{-st}g(t,x_t)\,dt, \qquad F(s) = \int_0^\infty e^{-st}f(t,x_t)\,dt.$$

Using (2.2), integration by parts, and Proposition 2.4(i), we have

$$\begin{aligned} & s^\alpha\left(s^\alpha + A\right)^{-1}G(s) \\ &\quad = s^\alpha\int_0^\infty\int_0^\infty e^{-s^\alpha t}T(t)e^{-s\tau}g(\tau,x_\tau)\,dt\,d\tau \\ &\quad = s^\alpha\int_0^\infty\int_0^\infty e^{-s^\alpha\lambda^\alpha}T\left(\lambda^\alpha\right)e^{-s\tau}g(\tau,x_\tau)\alpha\lambda^{\alpha-1}\,d\lambda\,d\tau \\ &\quad = \int_0^\infty\left(\int_0^\infty -T\left(\lambda^\alpha\right)e^{-s\tau}g(\tau,x_\tau)\,d\tau\right)de^{-(s\lambda)^\alpha} \\ &\quad = \left(e^{-(s\lambda)^\alpha}\int_0^\infty -T\left(\lambda^\alpha\right)e^{-s\tau}g(\tau,x_\tau)\,d\tau\right)\Bigg|_{\lambda=0}^\infty \\ &\qquad - \int_0^\infty\int_0^\infty \alpha\lambda^{\alpha-1}AT\left(\lambda^\alpha\right)e^{-(s\lambda)^\alpha}e^{-s\tau}g(\tau,x_\tau)\,d\tau\,d\lambda \\ &\quad = \int_0^\infty e^{-s\tau}g(\tau,x_\tau)\,d\tau - \int_0^\infty\int_0^\infty \alpha\lambda^{\alpha-1}AT\left(\lambda^\alpha\right)e^{-(s\lambda)^\alpha}e^{-s\tau}g(\tau,x_\tau)\,d\tau\,d\lambda. \end{aligned} \tag{3.3}$$

Furthermore, by using Proposition 2.2(ii) and relation (2.4) the right-hand side of (3.3) can be written as

$$\begin{aligned} & \int_0^\infty e^{-s\tau}g(\tau,x_\tau)\,d\tau - \int_0^\infty\int_0^\infty \alpha\lambda^{\alpha-1}AT\left(\lambda^\alpha\right)e^{-(s\lambda)^\alpha}e^{-s\tau}g(\tau,x_\tau)\,d\tau\,d\lambda \\ &\quad = \int_0^\infty e^{-s\tau}g(\tau,x_\tau)\,d\tau - \int_0^\infty\int_0^\infty\int_0^\infty \alpha\lambda^{\alpha-1}AT\left(\lambda^\alpha\right)\frac{\alpha}{\theta^{\alpha+1}} \\ &\qquad \times \Psi_\alpha\left(\frac{1}{\theta^\alpha}\right)e^{-s\lambda\theta}e^{-s\tau}g(\tau,x_\tau)\,d\theta\,d\tau\,d\lambda \\ &\quad = \int_0^\infty e^{-s\tau}g(\tau,x_\tau)\,d\tau \\ &\qquad - \int_0^\infty\int_\tau^\infty\int_0^\infty \alpha\left(\frac{\omega-\tau}{\theta}\right)^{\alpha-1}AT\left(\left(\frac{\omega-\tau}{\theta}\right)^\alpha\right)\frac{\alpha}{\theta^{\alpha+2}} \\ &\qquad \times \Psi_\alpha\left(\frac{1}{\theta^\alpha}\right)e^{-s\omega}g(\tau,x_\tau)\,d\theta\,d\omega\,d\tau \end{aligned}$$

$$
\begin{aligned}
&= \int_0^\infty e^{-s\tau} g(\tau, x_\tau)\, d\tau - \int_0^\infty \int_\tau^\infty \int_0^\infty \alpha(\omega-\tau)^{\alpha-1} u A T\bigl(u(\omega-\tau)^\alpha\bigr) \\
&\quad \times \Psi_\alpha(u) e^{-s\omega} g(\tau, x_\tau)\, du\, d\omega\, d\tau \\
&= \int_0^\infty e^{-s\tau} g(\tau, x_\tau)\, d\tau - \int_0^\infty \int_\tau^\infty (\omega-\tau)^{\alpha-1} A\mathcal{P}_\alpha(\omega-\tau) e^{-s\omega} g(\tau, x_\tau)\, d\omega\, d\tau \\
&= \int_0^\infty e^{-s\tau} g(\tau, x_\tau)\, d\tau - \int_0^\infty e^{-s\omega} \left(\int_0^\omega (\omega-\tau)^{\alpha-1} A\mathcal{P}_\alpha(\omega-\tau) g(\tau, x_\tau)\, d\tau \right) d\omega. \qquad (3.4)
\end{aligned}
$$

By a similar argument we get

$$
\begin{aligned}
&s^{\alpha-1}\bigl(s^\alpha + A\bigr)^{-1}\bigl(\varphi(0) - g(0, x_0)\bigr) + \bigl(s^\alpha + A\bigr)^{-1} F(s) \\
&\quad = \int_0^\infty e^{-st} \mathcal{S}_\alpha(t)\bigl(\varphi(0) - g(0, x_0)\bigr)\, dt \\
&\qquad + \int_0^\infty e^{-st} \left(\int_0^t (t-\tau)^{\alpha-1} \mathcal{P}_\alpha(t-\tau) f(\tau, x_\tau)\, d\tau \right) dt. \qquad (3.5)
\end{aligned}
$$

Finally, combining (3.2)-(3.5), we conclude

$$
\begin{aligned}
x(t) &= \mathcal{S}_\alpha(t)\bigl(\varphi(0) - g(0, x_0)\bigr) + g(t, x_t) - \int_0^t (t-s)^{\alpha-1} A\mathcal{P}_\alpha(t-s) g(s, x_s)\, ds \\
&\quad + \int_0^t (t-s)^{\alpha-1} \mathcal{P}_\alpha(t-s) f(s, x_s)\, ds.
\end{aligned}
$$

The proof is completed. □

By Lemma 3.1 we define a mild solution to problem (3.1) as follows.

Definition 3.1 By a mild solution to problem (3.1) on the interval $[-h, T]$ we mean a function $x(t) \in C([-h, T], X)$ satisfying

$$
x(t) = \begin{cases} \mathcal{S}_\alpha(t)(\varphi(0) - g(0, x_0)) + g(t, x_t) - \int_0^t (t-s)^{\alpha-1} A\mathcal{P}_\alpha(t-s) g(s, x_s)\, ds \\ \quad + \int_0^t (t-s)^{\alpha-1} \mathcal{P}_\alpha(t-s) f(s, x_s)\, ds, \quad t \in [0, T], \\ \varphi(t), \quad t \in [-h, 0]. \end{cases}
$$

In the sequel, we use $|w|_0 = \max_{s\in[-h,0]} \|w(s)\|$, where $\|\cdot\|$ is an arbitrary norm in X.

To study the existence and uniqueness of a mild solution to problem (3.1), we require the following assumptions.

(H_1) The resolvent $(\lambda I + A)^{-1}$ of $-A$ is compact for every $\lambda > 0$.

(H_2) The function $g(t, x_t) : [0, T] \times C([-h, 0], X) \to D(A^\beta)$ is a continuous function with respect to $t \in [0, T]$, and there exists a positive constant M_g such that for any $x_t \in C([-h, 0], X)$, $A^\beta g(t, x_t)$ is strongly measurable and satisfies the inequality

$$
\bigl\|A^\beta g(t, x_t)\bigr\| \le M_g\bigl(1 + |x_t|_0\bigr),
$$

and there exist positive constants L_g and θ_1 with $\theta_1 > \alpha(1 + \gamma)$ such that for any $t, s \in [0, T]$ and $x_t, y_s \in C([-h, 0], X)$, $A^\beta g(t, x_t)$ satisfies the Lipschitz condition

$$
\bigl\|A^\beta g(t, x_t) - A^\beta g(s, y_s)\bigr\| \le L_g\bigl(|t-s|^{\theta_1} + |x_t - y_s|_0\bigr).
$$

(H_3) For almost all $t \in [0,T]$, the function $f(t,x_t):[0,T]\times C([-h,0],X)\to X$ is continuous; for each $x_t \in C([-h,0],X)$, $f(t,x_t)$ is strongly measurable; and there exists a function $m(t)\in L^p((0,T),\mathbb{R}^+)$ with $p > -\frac{1}{\alpha\gamma}$ such that $\|f(t,x_t)\| \le m(t)$ for all $t\in[0,T]$ and $x_t \in C([-h,0],X)$.

We now show the existence of a mild solution to problem (3.1) via Krasnoselskii's fixed point theorem.

Theorem 3.1 *Let $A \in \Theta_\omega^\gamma$ with $-1<\gamma<0$ and $0<\omega<\pi/2$, and $\varphi(t)\in C([-h,0],X^\beta)$ for $\beta > 1+\gamma$. Assume that conditions* (H_1)-(H_3) *hold. Then there exists T_0 such that problem* (3.1) *has a mild solution on the interval* $[-h,T_0]$.

Proof For any fixed $r>0$, we set

$$B = \Bigl\{x(t)\in C\bigl([-h,T],X\bigr): x(t)=\varphi(t), t\in[-h,0]; \max_{t\in[0,T]}\bigl\|x(t)-\varphi(0)\bigr\| \le r\Bigr\}.$$

Obviously, B is a closed convex subset of $C([-h,T],X)$. Choose $T_0\in(0,T]$ such that

$$\begin{aligned} &K + r^* M_g\bigl|A^{-\beta}\bigr| + \frac{\alpha r^* C_0 M_g \Gamma(\beta-\gamma)}{\Gamma(1-\alpha(\gamma+1-\beta))}\cdot\frac{T_0^{-\alpha(\gamma+1-\beta)}}{-\alpha(\gamma+1-\beta)} \\ &\quad + C_p\left(\frac{T_0^{1-q(1+\alpha\gamma)}}{1-q(1+\alpha\gamma)}\right)^{\frac{1}{q}}\|m\|_{L^p((0,T),\mathbb{R}^+)} \le r \end{aligned}$$

and

$$L_g\bigl|A^{-\beta}\bigr| + \frac{\alpha C_0 L_g\Gamma(\beta-\gamma)}{\Gamma(1-\alpha(\gamma+1-\beta))}\cdot\frac{T_0^{-\alpha(\gamma+1-\beta)}}{-\alpha(\gamma+1-\beta)} < 1 \tag{3.6}$$

with $q=p/(p-1)$ and

$$\begin{aligned} &r^* = 1+\max\Bigl\{r+\bigl\|\varphi(0)\bigr\|, \max_{t\in[-h,0]}\bigl\|\varphi(t)\bigr\|\Bigr\}, \\ &K = \max_{t\in[0,T_0]}\bigl\|\mathcal{S}_\alpha(t)\bigl(\varphi(0)-g(0,x_0)\bigr)-\varphi(0)\bigr\|. \end{aligned} \tag{3.7}$$

Now we consider two operators F_1 and F_2 on $C([-h,T],X)$:

$$(F_1x)(t) = \begin{cases} \mathcal{S}_\alpha(t)(\varphi(0)-g(0,x_0)) + g(t,x_t) \\ \quad + \int_0^t (t-s)^{\alpha-1}A\mathcal{P}_\alpha(t-s)g(s,x_s)\,ds, & t\in[0,T], \\ \varphi(t), & t\in[-h,0], \end{cases}$$

and

$$(F_2x)(t) = \begin{cases} \int_0^t (t-s)^{\alpha-1}\mathcal{P}_\alpha(t-s)f(s,x_s)\,ds, & t\in[0,T], \\ 0, & t\in[-h,0]. \end{cases}$$

Obviously, $x(t)$ is a mild solution to equation (3.1) if and only if the operator equation $F_1x+F_2x=x$ has a solution $x\in B$. Therefore, the existence of a mild solution is equivalent to the existence of a function $x \in B$ such that $F_1x + F_2x = x$. To prove the latter, we divide the proof into four steps.

Step 1. The operators F_1 and F_2 map the set B into $C([-h,T],X)$, respectively. First, we show that for any fixed $x(t) \in B$, $(F_1x)(t)$ is continuous for all $t \in [-h,T]$. It is clear that $(F_1x)(t)$ is continuous for $t \in [-h,0)$. For the case $t = 0$, we have

$$\begin{aligned}\big\|(F_1x)(t) - (F_1x)(0)\big\| \leq{}& \big\|\big(\mathcal{S}_\alpha(t) - I\big)\varphi(0)\big\| + \big\|g(t,x_t) - \mathcal{S}_\alpha(t)g(0,x_0)\big\| \\ &+ \left\|\int_0^t (t-s)^{\alpha-1}A\mathcal{P}_\alpha(t-s)g(s,x_s)\,ds\right\|.\end{aligned}$$

Noting that $\varphi(0), g(t,x_t) \in X^\beta$, by Proposition 2.5(vi), we get $\|(\mathcal{S}_\alpha(t) - I)\varphi(0)\| \to 0$ and $\|g(t,x_t) - \mathcal{S}_\alpha(t)g(0,x_0)\| \to 0$ as $t \to 0^+$, respectively. On the other hand, by Lemmas 2.1 and 2.2 we obtain

$$\begin{aligned}&\left\|\int_0^t (t-s)^{\alpha-1}A\mathcal{P}_\alpha(t-s)g(s,x_s)\,ds\right\| \\ &\quad\leq r^*M_g\int_0^t (t-s)^{\alpha-1}\big\|A^{1-\beta}\mathcal{P}_\alpha(t-s)\big\|\,ds \\ &\quad\leq \frac{\alpha r^*C_0M_g\Gamma(\beta-\gamma)}{\Gamma(1-\alpha(\gamma+1-\beta))}\int_0^t (t-s)^{\alpha-1}(t-s)^{-\alpha(\gamma-\beta+2)}\,ds \\ &\quad= \frac{\alpha r^*C_0M_g\Gamma(\beta-\gamma)}{\Gamma(1-\alpha(\gamma+1-\beta))}\cdot\frac{t^{-\alpha(\gamma-\beta+1)}}{-\alpha(\gamma-\beta+1)}.\end{aligned}$$

This shows that $\int_0^t (t-s)^{\alpha-1}A\mathcal{P}_\alpha(t-s)g(s,x_s)\,ds \to 0$ as $t \to 0^+$. So, $(F_1x)(t)$ is continuous at $t = 0$. For the case $0 < t_1 < t_2 \leq T$, we have

$$\begin{aligned}\big\|(F_1x)(t_1) - (F_1x)(t_2)\big\| \leq{}& \big\|\big(\mathcal{S}_\alpha(t_1) - \mathcal{S}_\alpha(t_2)\big)\big(x(0) - g(0,x_0)\big)\big\| + \big\|g(t_1,x_{t_1}) - g(t_2,x_{t_2})\big\| \\ &+ \left\|\int_0^{t_1}(t_1-s)^{\alpha-1}A\mathcal{P}_\alpha(t_1-s)g(s,x_s)\,ds\right. \\ &\left.- \int_0^{t_2}(t_2-s)^{\alpha-1}A\mathcal{P}_\alpha(t_2-s)g(s,x_s)\,ds\right\| = I_1 + I_2 + I_3,\end{aligned}$$

where

$$\begin{aligned}I_1 &= \big\|\big(\mathcal{S}_\alpha(t_1) - \mathcal{S}_\alpha(t_2)\big)\big(x(0) - g(0,x_0)\big)\big\|, \qquad I_2 = \big\|g(t_1,x_{t_1}) - g(t_2,x_{t_2})\big\|, \\ I_3 &= \left\|\int_0^{t_1}(t_1-s)^{\alpha-1}A\mathcal{P}_\alpha(t_1-s)g(s,x_s)\,ds - \int_0^{t_2}(t_2-s)^{\alpha-1}A\mathcal{P}_\alpha(t_2-s)g(s,x_s)\,ds\right\|.\end{aligned}$$

By condition (H_1) and Proposition 2.5(vi) we have $I_1 \to 0$ and $I_2 \to 0$ as $t_1 \to t_2$. Now it remains to show that, in this case, $I_3 \to 0$ as $t_1 \to t_2$. Using condition (H_2) and Lemmas 2.1 and 2.2, we have

$$\begin{aligned}I_3 &= \left\|\int_0^{t_1}s^{\alpha-1}A\mathcal{P}_\alpha(s)g(t_1-s,x_{t_1-s})\,ds - \int_0^{t_2}s^{\alpha-1}A\mathcal{P}_\alpha(s)g(t_2-s,x_{t_2-s})\,ds\right\| \\ &\leq \left\|\int_0^{t_1}s^{\alpha-1}A\mathcal{P}_\alpha(s)\big(g(t_1-s,x_{t_1-s}) - g(t_2-s,x_{t_2-s})\big)\,ds\right\|\end{aligned}$$

$$
\begin{aligned}
&+\left\|\int_{t_1}^{t_2} s^{\alpha-1} A\mathcal{P}_\alpha(s) g(t_2-s, x_{t_2-s})\,ds\right\| \\
&\le \int_0^{t_1} s^{\alpha-1}\left\|A^{1-\beta}\mathcal{P}_\alpha(s)\right\|\left\|A^\beta g(t_1-s, x_{t_1-s}) - A^\beta g(t_2-s, x_{t_2-s})\right\| ds \\
&\quad+\int_{t_1}^{t_2} s^{\alpha-1}\left\|A^{1-\beta}\mathcal{P}_\alpha(s)\right\|\left\|A^\beta g(t_2-s, x_{t_2-s})\right\| ds \\
&\le \frac{\alpha C_0 L_g \Gamma(\beta-\gamma)}{\Gamma(1-\alpha(\gamma+1-\beta))}\int_0^{t_1} s^{-\alpha\gamma+\alpha\beta-\alpha-1}\big(|t_1-t_2|^{\theta_1} + |x_{t_1-s}-x_{t_2-s}|_0\big)\,ds \\
&\quad+\frac{\alpha r^* C_0 M_g \Gamma(\beta-\gamma)}{\Gamma(1-\alpha(\gamma+1-\beta))}\int_{t_1}^{t_2} s^{-\alpha\gamma+\alpha\beta-\alpha-1}\,ds \\
&\le \frac{\alpha C_0 L_g \Gamma(\beta-\gamma)}{\Gamma(1-\alpha(\gamma+1-\beta))}\cdot\frac{t_1^{-\alpha(\gamma-\beta+1)}}{-\alpha(\gamma-\beta+1)}\Big(|t_1-t_2|^{\theta_1} + \max_{s\in[0,t_1]}|x_{t_1-s}-x_{t_2-s}|_0\Big) \\
&\quad+\frac{\alpha r^* C_0 M_g \Gamma(\beta-\gamma)}{\Gamma(1-\alpha(\gamma+1-\beta))}\cdot\frac{t_2^{-\alpha(\gamma-\beta+1)} - t_1^{-\alpha(\gamma-\beta+1)}}{-\alpha(\gamma-\beta+1)}.
\end{aligned}
$$

Moreover, since $x(t)$ is continuous and $\beta > 1+\gamma$, we have $I_3 \to 0$ as $t_1 \to t_2$. It follows that $(F_1x)(t)$ is continuous for all $t \in (0,T]$. Hence, the operator F_1 maps the set B into $C([-h,T],X)$.

Next, we show that for any fixed $x(t) \in B$, $(F_2x)(t)$ is continuous for all $t \in [-h,T]$. Obviously, $(F_2x)(t)$ is continuous for any $t \in [-h,0)$. For the case $t = 0$, by the Hölder inequality we get

$$
\begin{aligned}
\left\|(F_2x)(t) - (F_2x)(0)\right\| &\le \left(\int_0^t \big((t-s)^{\alpha-1}\left\|\mathcal{P}_\alpha(t-s)\right\|\big)^q\,ds\right)^{\frac{1}{q}}\|m\|_{L^p((0,T),\mathbb{R}^+)} \\
&\le \frac{t^{1-q(1+\alpha\gamma)}}{1-q(1+\alpha\gamma)}\|m\|_{L^p((0,T),\mathbb{R}^+)}
\end{aligned}
$$

with $0 < q = \frac{p}{p-1} < \frac{1}{1+\alpha\gamma}$. This shows that $\|(F_2x)(t) - (F_2x)(0)\| \to 0$ as $t \to 0^+$. Thus, it follows that the function $(F_2x)(t)$ is continuous at $t = 0$. For $0 < t_1 < t_2 \le T$, we have

$$
\begin{aligned}
\left\|(F_2x)(t_1) - (F_2x)(t_2)\right\| &= \left\|\int_0^{t_1}(t_1-s)^{\alpha-1}\mathcal{P}_\alpha(t_1-s)f(s,x_s)\,ds\right. \\
&\quad\left.-\int_0^{t_2}(t_2-s)^{\alpha-1}\mathcal{P}_\alpha(t_2-s)f(s,x_s)\,ds\right\| \\
&\le \left\|\int_0^{t_1}(t_1-s)^{\alpha-1}\big(\mathcal{P}_\alpha(t_1-s)-\mathcal{P}_\alpha(t_2-s)\big)f(s,x_s)\,ds\right\| \\
&\quad+\left\|\int_0^{t_1}\big((t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}\big)\mathcal{P}_\alpha(t_2-s)f(s,x_s)\,ds\right\| \\
&\quad+\left\|\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\mathcal{P}_\alpha(t_2-s)f(s,x_s)\,ds\right\| = I_1 + I_2 + I_3,
\end{aligned}
$$

where

$$
I_1 = \left\|\int_0^{t_1}(t_1-s)^{\alpha-1}\big(\mathcal{P}_\alpha(t_1-s)-\mathcal{P}_\alpha(t_2-s)\big)f(s,x_s)\,ds\right\|,
$$

$$\begin{aligned} I_2 &= \left\| \int_0^{t_1} \bigl((t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1}\bigr)\mathcal{P}_\alpha(t_2-s)f(s,x_s)\,ds \right\|, \\ I_3 &= \left\| \int_{t_1}^{t_2} (t_2-s)^{\alpha-1}\mathcal{P}_\alpha(t_2-s)f(s,x_s)\,ds \right\|. \end{aligned}$$

For I_1, we choose a small $\delta > 0$ and get

$$\begin{aligned} I_1 &\le \int_0^{t_1-\delta} (t_1-s)^{\alpha-1} \bigl\|\mathcal{P}_\alpha(t_1-s) - \mathcal{P}_\alpha(t_2-s)\bigr\| \bigl\|f(s,x_s)\bigr\|\,ds \\ &\quad + \int_{t_1-\delta}^{t_1} (t_1-s)^{\alpha-1} \bigl\|\mathcal{P}_\alpha(t_1-s) - \mathcal{P}_\alpha(t_2-s)\bigr\| \bigl\|f(s,x_s)\bigr\|\,ds \\ &\le \max_{s\in[0,t_1-\delta]} \bigl\|\mathcal{P}_\alpha(t_1-s) - \mathcal{P}_\alpha(t_2-s)\bigr\| \left(\int_0^{t_1-\delta} (t_1-s)^{q(\alpha-1)}\,ds \right)^{\frac{1}{q}} \|m\|_{L^p((0,T),\mathbb{R}^+)} \\ &\quad + \int_{t_1-\delta}^{t_1} (t_1-s)^{\alpha-1}\bigl(C_p(t_1-s)^{-\alpha(1+\gamma)} + C_p(t_2-s)^{-\alpha(1+\gamma)}\bigr) \bigl\|f(s,x_s)\bigr\|\,ds \\ &\le \max_{s\in[0,t_1-\delta]} \bigl\|\mathcal{P}_\alpha(t_1-s) - \mathcal{P}_\alpha(t_2-s)\bigr\| \left(\int_0^{t_1-\delta} (t_1-s)^{q(\alpha-1)}\,ds \right)^{\frac{1}{q}} \|m\|_{L^p((0,T),\mathbb{R}^+)} \\ &\quad + 2C_p \left(\int_{t_1-\delta}^{t_1} (t_1-s)^{-q(\alpha\gamma+1)}\,ds \right)^{\frac{1}{q}} \|m\|_{L^p((0,T),\mathbb{R}^+)} \\ &= \max_{s\in[0,t_1-\delta]} \bigl\|\mathcal{P}_\alpha(t_1-s) - \mathcal{P}_\alpha(t_2-s)\bigr\| \left(\frac{t_1^{q(\alpha-1)+1} - \delta^{q(\alpha-1)+1}}{q(\alpha-1)+1} \right)^{\frac{1}{q}} \|m\|_{L^p((0,T),\mathbb{R}^+)} \\ &\quad + 2C_p \left(\frac{\delta^{1-q(\alpha\gamma+1)}}{1-q(\alpha\gamma+1)} \right)^{\frac{1}{q}} \|m\|_{L^p((0,T),\mathbb{R}^+)}. \end{aligned}$$

This, together with Proposition 2.5(ii), leads to $I_1 \to 0$ as $t_1 \to t_2$ and $\delta \to 0$.

For I_2, by the Hölder inequality we have

$$\begin{aligned} I_2 &\le C_p \left(\int_0^{t_1} \bigl|(t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1}\bigr|^q (t_2-s)^{-\alpha q(1+\gamma)}\,ds \right)^{\frac{1}{q}} \|m\|_{L^p((0,T),\mathbb{R}^+)} \\ &\le C_p \left(\int_0^{t_1} \bigl((t_1-s)^{-q(1+\alpha\gamma)} - (t_2-s)^{-q(1+\alpha\gamma)}\bigr)\,ds \right)^{\frac{1}{q}} \|m\|_{L^p((0,T),\mathbb{R}^+)} \\ &= C_p \left(\frac{t_1^{1-q(1+\alpha\gamma)} - t_2^{1-q(1+\alpha\gamma)}}{1-q(1+\alpha\gamma)} + \frac{(t_2-t_1)^{1-q(1+\alpha\gamma)}}{1-q(1+\alpha\gamma)} \right)^{\frac{1}{q}} \|m\|_{L^p((0,T),\mathbb{R}^+)}. \end{aligned}$$

Moreover, since $0 < q = \frac{p}{p-1} < \frac{1}{1+\alpha\gamma}$, we get $I_2 \to 0$ as $t_1 \to t_2$.

For I_3, by the same reasoning we have

$$\begin{aligned} I_3 &\le \left(\int_{t_1}^{t_2} \bigl((t_2-s)^{\alpha-1}\bigl\|\mathcal{P}_\alpha(t_2-s)\bigr\|\bigr)^q\,ds \right)^{\frac{1}{q}} \|m\|_{L^p((0,T),\mathbb{R}^+)} \\ &\le C_p \left(\frac{(t_2-t_1)^{1-q(1+\alpha\gamma)}}{1-q(1+\alpha\gamma)} \right)^{\frac{1}{q}} \|m\|_{L^p((0,T),\mathbb{R}^+)}. \end{aligned}$$

This gives $I_3 \to 0$ as $t_1 \to t_2$. It follows that $(F_2x)(t)$ is continuous for any $t \in (0,T]$. Hence, F_2 maps the set B into $C([-h,T],X)$.

Step 2. We show that $F_1x + F_2y \in B$ for every pair $x, y \in B$. By the definitions of the operators F_1 and F_2 we have

$$(F_1x)(t) + (F_2y)(t) = \begin{cases} \mathcal{S}_\alpha(t)(\varphi(0) - g(0, x_0)) + g(t, x_t) - \int_0^t (t-s)^{\alpha-1} A\mathcal{P}_\alpha(t-s) g(s, x_s)\, ds \\ \quad + \int_0^t (t-s)^{\alpha-1}\mathcal{P}_\alpha(t-s) f(s, y_s)\, ds, \quad t \in [0, T], \\ \varphi(t), \quad t \in [-h, 0], \end{cases}$$

so we only need to verify that $\max_{t\in[0,T]} \|F_1x + F_2y - \varphi(0)\| \le r$. According to assumptions (H_2)-(H_3) and Lemmas 2.1 and 2.2, we get

$$\begin{aligned} \|F_1x + F_2y - \varphi(0)\| &\le \|\mathcal{S}_\alpha(t)(\varphi(0) - g(0, x_0)) - \varphi(0)\| + \|g(t, x_t)\| \\ &\quad + \left\| \int_0^t (t-s)^{\alpha-1} A\mathcal{P}_\alpha(t-s) g(s, x_s)\, ds \right\| \\ &\quad + \left\| \int_0^t (t-s)^{\alpha-1} \mathcal{P}_\alpha(t-s) f(s, y_s)\, ds \right\| \\ &\le K + r^* M_g |A^{-\beta}| + \frac{\alpha r^* C_0 M_g \Gamma(\beta - \gamma)}{\Gamma(1 - \alpha(\gamma + 1 - \beta))} \cdot \frac{T_0^{-\alpha(\gamma+1-\beta)}}{-\alpha(\gamma + 1 - \beta)} \\ &\quad + C_p \left(\frac{T_0^{1-q(1+\alpha\gamma)}}{1 - q(1 + \alpha\gamma)} \right)^{\frac{1}{q}} \|m\|_{L^p((0,T),\mathbb{R}^+)} \\ &\le r. \end{aligned}$$

Therefore, $F_1x + F_2y \in B$ for every pair $x, y \in B$.

Step 3. The mapping F_1 is contractive. For any $x, y \in B$, by Lemmas 2.1 and 2.2 we have

$$\begin{aligned} &\|(F_1x)(t) - (F_1y)(t)\| \\ &\quad \le \|g(t, x_t) - g(t, y_t)\| + \left\| \int_0^t (t-s)^{\alpha-1} A\mathcal{P}_\alpha(t-s)\big(g(s, x_s) - g(s, y_s)\big)\, ds \right\| \\ &\quad \le \left(L_g |A^{-\beta}| + \frac{\alpha C_0 L_g \Gamma(\beta - \gamma)}{\Gamma(1 - \alpha(\gamma + 1 - \beta))} \cdot \frac{t^{-\alpha(\gamma+1-\beta)}}{-\alpha(\gamma + 1 - \beta)} \right) \|x - y\|. \end{aligned}$$

which, together with (3.6), shows that the mapping F_1 is contractive on B.

Step 4. The operator F_2 is compact. First, we show that F_2 is continuous. Let $\{x^n\} \subseteq B$ with $x^n \to x$ on B. Then, by assumption (H_3) and the fact that $x_t^n \to x_t$ for $t \in [0, T]$ we get

$$f(t, x_t^n) \to f(t, x_t), \quad \text{a.e., } t \in [0, T], \text{ as } n \to \infty.$$

On the other hand, by assumption (H_3) and Proposition 2.5(i) we have

$$\left\| \int_0^t (t-s)^{\alpha-1} \mathcal{P}_\alpha(t-s) f(s, x_s^n)\, ds \right\| \le C_p \frac{t^{-\alpha\gamma}}{-\alpha\gamma} \|m\|_{L^p((0,T),\mathbb{R}^+)}.$$

This implies that $(t-s)^{\alpha-1}\mathcal{P}_\alpha(t-s) f(s, x_s^n) \in L^1((0, t), X)$. Thus, by the Lebesgue dominated convergence theorem we get

$$\left\| \int_0^t (t-s)^{\alpha-1} \mathcal{P}_\alpha(t-s)\big(f(s, x_s^n) - f(s, x_s)\big)\, ds \right\| \to 0 \quad \text{as } n \to \infty.$$

Hence, F_2 is continuous. It remains to show that $F_2(B)$ is relatively compact. According to assumption (H_1) and Theorem 3.5 in [6], the family of functions $\{F_2x : x \in B\}$ is uniformly bounded. From Step 1 we observe that $\{F_2x : x \in B\}$ is a family of equicontinuous functions. So, by the known Ascoli-Arzelà theorem, the family $\{F_2x : x \in B\}$ is relatively compact. Hence, F_2 is compact.

Therefore, by Krasnoselskii's fixed point theorem, we deduce that problem (3.1) has a mild solution on the interval $[-h, T_0]$. The proof is completed. □

Next, we discuss the uniqueness of mild solutions to problem (3.1). For that, we need an additional condition.

(H_4) There exists a constant $L_f > 0$ such that for all $t \in [0, T]$ and $x_t, y_t \in C([-h, 0], X)$, the function f satisfies the Lipschitz condition

$$\bigl\|f(t,x_t) - f(t,y_t)\bigr\| \le L_f|x_t - y_t|_0.$$

The uniqueness result is based on the Banach contraction principle.

Theorem 3.2 *Let $A \in \Theta_\omega^\gamma$ with $-1 < \gamma < 0$ and $0 < \omega < \pi/2$, and $\varphi(t) \in C([-h, 0], X^\beta)$ for $\beta > 1 + \gamma$. Assume that conditions* (H_2)-(H_4) *hold. Then there exists $T_0 \in (0, T]$ such that problem* (3.1) *has a unique mild solution on the interval* $[-h, T_0]$.

Proof For any fixed $r > 0$, we set

$$B = \Bigl\{x \in C\bigl([-h, T], X\bigr) : x(t) = \varphi(t), t \in [-h, 0]; \max_{t\in[0,T]} \bigl\|x(t) - \varphi(0)\bigr\| \le r\Bigr\}.$$

Obviously, B is a closed convex subset of $C([-h, T], X)$. Choose $T_0 \in (0, T]$ such that

$$\begin{aligned} &K + r^* M_g\bigl|A^{-\beta}\bigr| + \frac{\alpha r^* C_0 M_g \Gamma(\beta - \gamma)}{\Gamma(1 - \alpha(\gamma + 1 - \beta))} \cdot \frac{T_0^{-\alpha(\gamma+1-\beta)}}{-\alpha(\gamma + 1 - \beta)} \\ &\quad + C_p\left(\frac{T_0^{1-q(1+\alpha\gamma)}}{1 - q(1 + \alpha\gamma)}\right)^{\frac{1}{q}} \|m\|_{L^p((0,T),\mathbb{R}^+)} \le r \end{aligned}$$

and

$$L_g\bigl|A^{-\beta}\bigr| + \frac{\alpha C_0 L_g \Gamma(\beta - \gamma)}{\Gamma(1 - \alpha(\gamma + 1 - \beta))} \cdot \frac{T_0^{-\alpha(\gamma+1-\beta)}}{-\alpha(\gamma + 1 - \beta)} + \frac{C_p L_f T_0^{-\alpha\gamma}}{-\alpha\gamma} < 1,$$

where r^* and K are the constants defined by (3.7). Now we consider the operator F defined by

$$(Fx)(t) = \begin{cases} \mathcal{S}_\alpha(t)(\varphi(0) - g(0, x_0)) + g(t, x_t) - \int_0^t (t - s)^{\alpha-1} A\mathcal{P}_\alpha(t - s)g(s, x_s)\, ds \\ \quad + \int_0^t (t - s)^{\alpha-1}\mathcal{P}_\alpha(t - s)f(s, x_s)\, ds, \quad t \in [0, T], \\ \varphi(t), \quad t \in [-h, 0]. \end{cases}$$

Similarly to the proof of Theorem 3.1, we can show that F maps the subset B into itself. Moreover, for any $x(t), y(t) \in B$, we have

$$\begin{aligned}
&\big\|(Fx)(t)-(Fy)(t)\big\| \\
&\quad\le \big\|g(t,x_t)-g(t,y_t)\big\| + \left\|\int_0^t (t-s)^{\alpha-1}A\mathcal{P}_\alpha(t-s)\big(g(s,x_s)-g(s,y_s)\big)\,ds\right\| \\
&\qquad + \left\|\int_0^t (t-s)^{\alpha-1}\mathcal{P}_\alpha(t-s)\big(f(s,x_s)-f(s,y_s)\big)\,ds\right\| \\
&\quad\le \left(L_g\big|A^{-\beta}\big| + \frac{\alpha C_0 L_g\Gamma(\beta-\gamma)}{\Gamma(1-\alpha(\gamma+1-\beta))}\cdot\frac{t^{-\alpha(\gamma+1-\beta)}}{-\alpha(\gamma+1-\beta)} + \frac{C_p L_f t^{-\alpha\gamma}}{-\alpha\gamma}\right)\|x-y\|.
\end{aligned}$$

So the mapping F is contractive. It follows from the Banach contractive principle that problem (3.1) has a unique mild solution on the interval $[-h,T_0]$. □

In particular, if $g(t,x_t)\equiv c$ (c is a constant), then we have the following corollary.

Corollary 3.1 *Let $A\in\Theta_\omega^\gamma$ with $-1<\gamma<0$ and $0<\omega<\pi/2$, and $\varphi(t)\in C([-h,0],X^\beta)$ for $\beta>1+\gamma$. Assume that conditions* (H_2)-(H_3) *hold. Then there exists $T_0\in(0,T]$ such that the abstract fractional functional equation*

$$\begin{cases}({}_0^C D_t^\alpha x)(t)+Ax(t)=f(t,x_t), & 0<\alpha<1, t\in[0,T],\\ x(t)=\varphi(t), & t\in[-h,0]\end{cases} \tag{3.8}$$

has a unique mild solution on the interval $[-h,T_0]$.

In particular, when $\beta=1$, then we have $1>1+\gamma$ ($-1<\gamma<0$). So $X^1=D(A)$ is a Banach space with the graph norm $\|x\|_1=\|Ax\|$, where $x\in D(A)$. In this case, condition (H_2) can be written as follows:

(H_2^*) The function $g(t,x_t):[0,T]\times C([-h,0],X)\to D(A)$ is a continuous function with respect to $t\in[0,T]$; there exists a positive constant M_g such that for any $x_t\in C([-h,0],X)$, $Ag(t,x_t)$ is strongly measurable and satisfies the inequality

$$\big\|Ag(t,x_t)\big\|\le M_g\big(1+|x_t|_0\big);$$

and there exist positive constants L_g and θ_1 with $\theta_1>\alpha(1+\gamma)$ such that for any $t,s\in[0,T]$ and $x_t,y_s\in C([-h,0],X)$, $Ag(t,x_t)$ satisfies the Lipschitz condition

$$\big\|Ag(t,x_t)-Ag(s,y_s)\big\|\le L_g\big(|t-s|^{\theta_1}+|x_t-y_s|_0\big).$$

Consequently, we arrive at the following corollary, which is a particular case of Theorem 3.2.

Corollary 3.2 *Let $A\in\Theta_\omega^\gamma$ with $-1<\gamma<0$ and $0<\omega<\pi/2$, and $\varphi(t)\in C([-h,0],X^1)$. Assume that conditions* (H_2^*), (H_3), *and* (H_4) *hold. Then there exists $T_0\in(0,T]$ such that problem* (3.1) *has a unique mild solution on the interval $[-h,T_0]$.*

Now we turn our attention to further conditions on f and g so that the mild solution becomes a strong solution and a classical solution. We first give the definitions of the strong solution and the classical solution to problem (3.1).

Definition 3.2 A function $x(t) : [-h, T] \to X$ is said to be a strong solution to problem (3.1) if

(i) $x(t)$ is continuous on $[-h, T]$, and $({}_0^C D_t^\alpha x)(t) \in L^1((0, T), X)$; and
(ii) $x(t)$ takes values in $D(A)$ and satisfies problem (3.1).

Definition 3.3 A function $x(t) : [-h, T] \to X$ is said to be a classical solution to problem (3.1) if

(i) $x(t)$ is continuous on $[-h, T]$, and $({}_0^C D_t^\alpha x)(t) \in C([0, T], X)$; and
(ii) $x(t)$ takes values in $D(A)$ and satisfies problem (3.1).

In the following, we prove that a mild solution can become a strong solution to problem (3.1) under some assumptions. To do this, we require stronger conditions than conditions (H_3) and (H_4):

(H_3^*) For almost all $t \in [0, T]$, the function $f(t, x_t) : [0, T] \times C([-h, 0], X) \to X$ is continuous; for each $x_t \in C([-h, 0], X)$, $f(t, x_t)$ is strongly measurable; and there exists a function $m(t) \in L^p((0, T), \mathbb{R}^+)$ with $p > \frac{\alpha\gamma+\alpha-1}{\alpha(2\gamma+1)}$ such that $\|f(t, x_t)\| \le m(t)$ for all $t \in [0, T]$ and $x_t \in C([-h, 0], X)$.

(H_4^*) There exist constants $L_f > 0$ and θ_2 with $\theta_2 > \alpha(1 + \gamma)$ such that for all $t, s \in [0, T]$ and $x_t, y_s \in C([-h, 0], X)$, the function f satisfies the Lipschitz condition

$$\bigl\|f(t, x_t) - f(s, y_s)\bigr\| \le L_f\bigl(|t - s|^{\theta_2} + |x_t - y_t|_0\bigr).$$

Theorem 3.3 *Let $A \in \Theta_\omega^\gamma$ with $-1 < \gamma < -\frac{1}{2}$ and $0 < \omega < \pi/2$, and $\varphi(t) \in C([-h, 0], X^1)$. Suppose that conditions* (H_2^*)-(H_4^*) *hold. In addition, suppose that the following conditions are satisfied*:

(H_a) *For almost all $t \in [0, T]$ and $x_t \in C([-h, 0], X)$, ${}_0^C D_t^\alpha g(t, x_t) \in L^1([0, T], X)$.*

(H_b) *In condition* (H_1^*), $0 < L_g < 1$.

(H_c) *For almost all $t \in [0, T]$ and $x_t \in C([-h, 0], X)$, $Ag(t, x_t) \in L^1((0, T), X^1)$ and $A^2 g(t, x_t) \in L^\infty((0, T), X)$.*

(H_d) *For almost all $t \in [0, T]$ and $x_t \in C([-h, 0], X)$, $f(t, x_t) \in L^1((0, T), X^1)$ and $Af(t, x_t) \in L^\infty((0, T), X)$.*

Then the mild solution x is the unique strong solution to problem (3.1), *provided that* $A(\varphi(0) - g(0, x_0)) \in L^\infty((0, T), X)$.

Proof Using the assumption $-1 < \gamma < -\frac{1}{2}$, we have $\frac{\alpha\gamma+\alpha-1}{\alpha(2\gamma+1)} > -\frac{1}{\alpha\gamma}$. Hence, the conclusion of Theorem 3.2 is also true if we replace conditions (H_2)-(H_4) by (H_2^*)-(H_4^*). Now, we will follow the argument of Wang et al. in (see [6], Theorem 4.1) to prove that the mild solution x is a strong solution to problem (3.1).

First, we show that $x(t)$ is Hölder continuous with an exponent ϑ with $\vartheta > \alpha(1 + \gamma)$ on the interval $[-h, T_0]$. For any $t \in [0, T_0]$, taking $\Delta t > 0$ such that $t + \Delta t \le T_0$, we have

$$\begin{aligned}
&\bigl\|x(t + \Delta t) - x(t)\bigr\| \\
&\quad\le \bigl\|\bigl(\mathcal{S}_\alpha(t + \Delta t) - \mathcal{S}_\alpha(t)\bigr)\bigl(\varphi(0) - g(0, x_0)\bigr)\bigr\| + \bigl\|g(t + \Delta t, x_{t+\Delta t}) - g(t, x_t)\bigr\| \\
&\qquad+ \biggl\|\int_0^{t+\Delta t}(t + \Delta t - s)^{\alpha-1} A\mathcal{P}_\alpha(t + \Delta t - s) g(s, x_s)\, ds
\end{aligned}$$

$$\begin{aligned}
&\quad - \int_0^t (t-s)^{\alpha-1} A\mathcal{P}_\alpha(t-s) g(s,x_s)\,ds \Big\| \\
&\quad + \left\| \int_0^{t+\Delta t} (t+\Delta t-s)^{\alpha-1}\mathcal{P}_\alpha(t+\Delta t-s) f(s,x_s)\,ds - \int_0^t (t-s)^{\alpha-1}\mathcal{P}_\alpha(t-s) f(s,x_s)\,ds \right\| \\
&= I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

For I_1, by Proposition 2.5(i), (iii) we get

$$\begin{aligned}
I_1 &= \big\| \big(\mathcal{S}_\alpha(t+\Delta t) - \mathcal{S}_\alpha(t)\big)\big(\varphi(0) - g(0,x_0)\big) \big\| \\
&= \left\| \int_t^{t+\Delta t} -s^{\alpha-1} A\mathcal{P}_\alpha(s)\big(\varphi(0) - g(0,x_0)\big)\,ds \right\| \\
&\le C_p \frac{(t+\Delta t)^{-\alpha\gamma} - t^{-\alpha\gamma}}{-\alpha\gamma} \big\| A\big(\varphi(0) - g(0,x_0)\big) \big\|.
\end{aligned}$$

For I_3, using Lemma 2.1, condition (H_2^*), and Proposition 2.5(i), we get

$$\begin{aligned}
I_3 &= \Big\| \int_0^{t+\Delta t} (t+\Delta t-s)^{\alpha-1} A\mathcal{P}_\alpha(t+\Delta t-s) g(s,x_s)\,ds \\
&\quad - \int_0^t (t-s)^{\alpha-1} A\mathcal{P}_\alpha(t-s) g(s,x_s)\,ds \Big\| \\
&\le \left\| \int_0^t (t-s)^{\alpha-1} A\mathcal{P}_\alpha(t-s)\big(g(s+\Delta t, x_{s+\Delta t}) - g(s,x_s)\big)\,ds \right\| \\
&\quad + \left\| \int_0^{\Delta t} (t+\Delta t-s)^{\alpha-1} A\mathcal{P}_\alpha(t+\Delta t-s) g(s,x_s)\,ds \right\| \\
&\le L_g \int_0^t (t-s)^{\alpha-1} \big\|\mathcal{P}_\alpha(t-s)\big\| \big((\Delta t)^{\theta_1} + |x_{s+\Delta t} - x_s|_0\big)\,ds \\
&\quad + r^* M_g \int_0^{\Delta t} (t+\Delta t-s)^{\alpha-1} \big\|\mathcal{P}_\alpha(t+\Delta t-s)\big\|\,ds \\
&\le C_p L_g \frac{T_0^{-\alpha\gamma}}{-\alpha\gamma}(\Delta t)^{\theta_1} + C_p L_g \int_0^t (t-s)^{-\alpha\gamma-1} |x_{s+\Delta t} - x_s|_0\,ds + r^* M_g \frac{(t+\Delta t)^{-\alpha\gamma} - t^{-\alpha\gamma}}{-\alpha\gamma}.
\end{aligned}$$

Similarly to I_3, we have

$$\begin{aligned}
I_4 &= \left\| \int_0^{t+\Delta t} (t+\Delta t-s)^{\alpha-1}\mathcal{P}_\alpha(t+\Delta t-s) f(s,x_s)\,ds - \int_0^t (t-s)^{\alpha-1}\mathcal{P}_\alpha(t-s) f(s,x_s)\,ds \right\| \\
&\le C_p \frac{(t+\Delta t)^{1-q(1+\alpha\gamma)} - t^{1-q(1+\alpha\gamma)}}{1-q(1+\alpha\gamma)} \|m\|_{L^p((0,T),\mathbb{R}^+)} + \frac{L_f C_p T_0^{-\alpha\gamma}}{-\alpha\gamma}(\Delta t)^{\theta_2} \\
&\quad + L_f C_p \int_0^t (t-s)^{-\alpha\gamma-1} |x_{s+\Delta t} - x_s|_0\,ds.
\end{aligned}$$

As a consequence, we get

$$\begin{aligned}
&\big\| x(t+\Delta t) - x(t) \big\| \\
&\quad \le C_p \frac{(t+\Delta t)^{-\alpha\gamma} - t^{-\alpha\gamma}}{-\alpha\gamma} \big\| A\big(\varphi(0) - g(0,x_0)\big) \big\| + L_g(\Delta t)^{\theta_1} + L_g |x_{t+\Delta t} - x_t|_0
\end{aligned}$$

$$+ \frac{C_p L_g T_0^{-\alpha\gamma}}{-\alpha\gamma}(\Delta t)^{\theta_1} + C_p L_g \int_0^t (t-s)^{-\alpha\gamma-1} |x_{s+\Delta t} - x_s|_0 \, ds$$
$$+ r^* M_g \frac{(t+\Delta t)^{-\alpha\gamma} - t^{-\alpha\gamma}}{-\alpha\gamma} + C_p \frac{(t+\Delta t)^{1-q(1+\alpha\gamma)} - t^{1-q(1+\alpha\gamma)}}{1-q(1+\alpha\gamma)} \|m\|_{L^p((0,T),\mathbb{R}^+)}$$
$$+ \frac{L_f C_p T_0^{-\alpha\gamma}}{-\alpha\gamma}(\Delta t)^{\theta_2} + L_f C_p \int_0^t (t-s)^{-\alpha\gamma-1} |x_{s+\Delta t} - x_s|_0 \, ds.$$

Using the inequality $b^c - a^c \leq (b-a)^c$ $(0 < a < b, 0 < c < 1)$, we have

$$\begin{aligned} &\|x(t+\Delta t) - x(t)\| \\ &\quad\leq \frac{C_p \|A(\varphi(0) - g(0,x_0))\| + r^* M_g}{-\alpha\gamma}(\Delta t)^{-\alpha\gamma} + \left(L_g + \frac{C_p L_g T_0^{-\alpha\gamma}}{-\alpha\gamma}\right)(\Delta t)^{\theta_1} \\ &\qquad + L_g |x_{t+\Delta t} - x_t|_0 + \frac{L_f C_p T_0^{-\alpha\gamma}}{-\alpha\gamma}(\Delta t)^{\theta_2} + \frac{C_p \|m\|_{L^p((0,T),\mathbb{R}^+)}}{1-q(1+\alpha\gamma)}(\Delta t)^{1-q(1+\alpha\gamma)} \\ &\qquad + C_p(L_g + L_f) \int_0^t (t-s)^{-\alpha\gamma-1} |x_{s+\Delta t} - x_s|_0 \, ds. \end{aligned} \tag{3.9}$$

Putting

$$\begin{aligned} &\vartheta = \min\{-\alpha\gamma, \theta_1, \theta_2, 1 - q(1+\alpha\gamma)\} > \alpha(\gamma+1), \\ &M = \frac{C_p \|A(\varphi(0) - g(0,x_0))\| + r^* M_g}{-\alpha\gamma} + L_g + \frac{C_p L_g T_0^{-\alpha\gamma}}{-\alpha\gamma} + \frac{L_f C_p T_0^{-\alpha\gamma}}{-\alpha\gamma} \\ &\qquad + \frac{C_p \|m\|_{L^p((0,T),\mathbb{R}^+)}}{1-q(1+\alpha\gamma)}, \end{aligned}$$

we can rewrite (3.9) in the form

$$\begin{aligned} \|x(t+\Delta t) - x(t)\| \leq{}& M(\Delta t)^{\vartheta} + L_g |x_{t+\Delta t} - x_t|_0 \\ &+ C_p(L_g + L_f) \int_0^t (t-s)^{-\alpha\gamma-1} |x_{s+\Delta t} - x_s|_0 \, ds. \end{aligned}$$

Then it follows from the definition of $|\cdot|_0$ that the inequality

$$|x_{t+\Delta t} - x_t|_0 \leq M(\Delta t)^{\vartheta} + L_g |x_{t+\Delta t} - x_t|_0 + C_p(L_g + L_f) \int_0^t (t-s)^{-\alpha\gamma-1} |x_{s+\Delta t} - x_s|_0 \, ds$$

holds. Then, in view of condition (H_b), we have

$$|x_{t+\Delta t} - x_t|_0 \leq \frac{M}{1-L_g}(\Delta t)^{\vartheta} + \frac{C_p(L_g + L_f)}{1-L_g} \int_0^t (t-s)^{-\alpha\gamma-1} |x_{s+\Delta t} - x_s|_0 \, ds. \tag{3.10}$$

Applying the generalized Gronwall inequality [24] to (3.10), we get the estimate

$$|x_{t+\Delta t} - x_t|_0 \leq Q(\Delta t)^{\vartheta} \tag{3.11}$$

with

$$Q = \frac{M}{1-L_g} E_{-\alpha\gamma}\left(\frac{C_p(L_g + L_f)}{1-L_g}\Gamma(-\alpha\gamma) T_0^{-\alpha\gamma}\right). \tag{3.12}$$

It follows from (3.11) that $x(t)$ is Hölder continuous with an exponent $\vartheta > \alpha(1+\gamma)$ on $[-h, T_0]$.

Next, we show that $x(t)$ satisfies problem (3.1). To do this, let

$$u(t) = \int_0^t (t-s)^{\alpha-1} A\mathcal{P}_\alpha(t-s) g(s, x_s)\, ds. \tag{3.13}$$

By the assumption $A^2 g(t, x_t) \in L^\infty((0,T), X)$ and Proposition 2.5(i), we have

$$\|Au\| \le \int_0^t (t-s)^{\alpha-1} \left\|\mathcal{P}_\alpha(t-s)\right\| ds \left\|A^2 g\right\|_{L^\infty((0,T),X)} \le C_p \frac{t^{-\alpha\gamma}}{-\alpha\gamma} \left\|A^2 g\right\|_{L^\infty((0,T),X)}.$$

This implies that $u(t) \in D(A)$ for all $t \in [0,T]$. Note that $u(0) = 0$. So by Proposition 2.1(ii) and Proposition 2.5(v) we get

$$\begin{aligned}
\left({}_0^C D_t^\alpha u\right)(t) &= \left({}_0 D_t^\alpha u\right)(t) = \frac{d}{dt}\left({}_0 I_t^{1-\alpha} u\right)(t) \\
&= \frac{d}{dt}\left({}_0 I_t^{1-\alpha} t^{\alpha-1} A\mathcal{P}_\alpha(t)(t) * g(t, x_t)\right) \\
&= \frac{d}{dt}\left(A\mathcal{S}_\alpha(t) * g(t, x_t)\right).
\end{aligned}$$

Now we need to calculate the first derivative of $v(t) := A\mathcal{S}_\alpha(t) * g(t, x_t)$. Let $\Delta t > 0$ and $t + \Delta t \le T_0$. Then, by Proposition 2.6(ii) we obtain

$$\begin{aligned}
\frac{v(t+\Delta t) - v(t)}{\Delta t} &= \int_0^t \frac{A\mathcal{S}_\alpha(t+\Delta t - s) - A\mathcal{S}_\alpha(t-s)}{\Delta t} g(s, x_s)\, ds \\
&\quad + \frac{1}{\Delta t}\int_t^{t+\Delta t} A\mathcal{S}_\alpha(t+\Delta t - s) g(s, x_s)\, ds \\
&= -A\int_0^t (t-s)^{\alpha-1} A\mathcal{P}_\alpha(t-s) g(s, x_s)\, ds \\
&\quad + \frac{1}{\Delta t}\int_t^{t+\Delta t} A\mathcal{S}_\alpha(t+\Delta t - s) g(s, x_s)\, ds \\
&= -Au(t) + I,
\end{aligned}$$

where

$$\begin{aligned}
I &= \frac{1}{\Delta t}\int_t^{t+\Delta t} A\mathcal{S}_\alpha(t+\Delta t - s) g(s, x_s)\, ds \\
&= \frac{1}{\Delta t}\int_0^{\Delta t} A\mathcal{S}_\alpha(\tau) g(t+\Delta t - \tau, x_{t+\Delta t-\tau})\, d\tau \\
&= \frac{1}{\Delta t}\int_0^{\Delta t} A\mathcal{S}_\alpha(\tau)\left(g(t+\Delta t - \tau, x_{t+\Delta t-\tau}) - g(t-\tau, x_{t-\tau})\right) d\tau \\
&\quad + \frac{1}{\Delta t}\int_0^{\Delta t} A\mathcal{S}_\alpha(\tau)\left(g(t-\tau, x_{t-\tau}) - g(t, x_t)\right) d\tau + \frac{1}{\Delta t}\int_0^{\Delta t} A\mathcal{S}_\alpha(\tau) g(t, x_t)\, d\tau \\
&= I_1 + I_2 + I_3.
\end{aligned}$$

For I_1, combining Proposition 2.5(i) and relation (3.9), we have

$$\begin{aligned}
&\frac{1}{\Delta t}\left\| \int_0^{\Delta t} A\mathcal{S}_\alpha(\tau)\big(g(t+\Delta t-\tau, x_{t+\Delta t-\tau}) - g(t-\tau, x_{t-\tau})\big)\,d\tau \right\| \\
&\quad\le \frac{1}{\Delta t}\int_0^{\Delta t} \big\|\mathcal{S}_\alpha(\tau)\big\| \big\|Ag(t+\Delta t-\tau, x_{t+\Delta t-\tau}) - Ag(t-\tau, x_{t-\tau})\big\|\,d\tau \\
&\quad\le \frac{C_s L_f}{\Delta t}\int_0^{\Delta t} \tau^{-\alpha(1+\gamma)}\big((\Delta t)^{\theta_1} + |x_{t+\Delta t-\tau} - x_{t-\tau}|_0\big)\,d\tau \\
&\quad\le C_s L_f \frac{(\Delta t)^{\theta_1-\alpha(\gamma+1)}}{1-\alpha(\gamma+1)} + C_s L_f Q\frac{(\Delta t)^{\vartheta-\alpha(\gamma+1)}}{1-\alpha(\gamma+1)}.
\end{aligned}$$

Moreover, since $\vartheta > \alpha(\gamma+1)$ and $1-\alpha(\gamma+1) > 0$, we get $I_1 \to 0$ as $\Delta t \to 0$.

Similarly, we can show that $I_2 \to 0$ as $\Delta t \to 0$. For I_3, by Proposition 2.5(vi) we have $\lim_{\Delta t\to 0} I_3 = Ag(t,x_t)$. Hence, we get $v'(t) = -Au(t) + Ag(t,x_t)$. It follows that

$$\big({}_0^C D_t^\alpha u\big)(t) = -Au(t) + Ag(t,x_t). \tag{3.14}$$

Analogously, taking

$$w(t) = \int_0^t (t-s)^{\alpha-1}\mathcal{P}_\alpha(t-s) f(s,x_s)\,dt, \tag{3.15}$$

we can show that

$$\big({}_0^C D_t^\alpha w\big)(t) = -Aw(t) + f(t,x_t). \tag{3.16}$$

Combining Proposition 2.5(iv), (3.14), and (3.16), we obtain that the mild solution $x(t)$ satisfies problem (3.1).

To complete the proof, it remains to show that $({}_0^C D_t^\alpha x)(t) \in L^1((0,T_0),X)$. In other words, we need to prove that $\int_0^{T_0} |({}_0^C D_t^\alpha x)(t)|\,dt < \infty$. As before, we have

$$\begin{aligned}
&\big\|{}_0^C D_t^\alpha x\big\|_{L^1((0,T_0),X)} \\
&\quad\le \big\|{}_0^C D_t^\alpha g\big\|_{L^1((0,T_0),X)} + \int_0^{T_0} \big\|\mathcal{S}_\alpha(t)\big\|\,dt \big\|A\big(\varphi(0) - g(0,x_0)\big)\big\|_{L^\infty((0,T_0),X)} \\
&\qquad + \int_0^{T_0}\int_0^t (t-s)^{\alpha-1}\big\|\mathcal{P}_\alpha(t-s)\big\|\,ds\,dt \big\|A^2 g\big\|_{L^\infty((0,T_0),X)} + \|Ag\|_{L^1((0,T_0),X^1)} \\
&\qquad + \int_0^{T_0}\int_0^t (t-s)^{\alpha-1}\big\|\mathcal{P}_\alpha(t-s)\big\|\,ds\,dt \|Af\|_{L^\infty((0,T_0),X)} + \|f\|_{L^1((0,T_0),X^1)} \\
&\quad\le \big\|{}_0^C D_t^\alpha g\big\|_{L^1((0,T_0),X)} + C_s\frac{T_0^{1-\alpha(1+\gamma)}}{1-\alpha(1+\gamma)}\big\|A\big(\varphi(0) - g(0,x_0)\big)\big\|_{L^\infty((0,T_0),X)} \\
&\qquad + \frac{C_p T_0^{1-\alpha\gamma}}{-\alpha\gamma(1-\alpha\gamma)}\big(\big\|A^2 g\big\|_{L^\infty((0,T_0),X)} + \|Af\|_{L^\infty((0,T_0),X)}\big) + \|Ag\|_{L^1((0,T_0),X^1)} \\
&\qquad + \|f\|_{L^1((0,T_0),X^1)},
\end{aligned}$$

which shows that $({}_0^C D_t^\alpha x)(t) \in L^1((0,T_0),X)$. Hence, the mild solution $x(t)$ is a strong solution to problem (3.1). The proof is completed. □

Corollary 3.3 *Let $A \in \Theta_\omega^\gamma$ with $-1 < \gamma < -\frac{1}{2}$ and $0 < \omega < \pi/2$, and $\varphi(t) \in C([-h,0],X^1)$. Suppose that conditions* (H_3^*), (H_4^*), *and* (H_d) *hold. Then the mild solution x to equation* (3.8) *is its unique strong solution, provided that $A\varphi(0) \in L^\infty((0,T),X)$.*

Finally, under suitable conditions, we show that the mild solution becomes a classical solution to problem (3.1).

Theorem 3.4 *Let $A \in \Theta_\omega^\gamma$ with $-1 < \gamma < -\frac{1}{2}$ and $0 < \omega < \pi/2$, and $\varphi(t) \in C([-h,0],X^1)$. Suppose that conditions* (H_2^*)-(H_4^*) *hold. In addition, suppose that the following conditions hold*:

(H_a) *For all $t \in [0,T]$ and $x_t \in C([-h,0],X)$, ${}_0^C D_t^\alpha g(t,x_t) \in C([0,T],X)$.*

(H_b) *In condition* (H_1^*), $0 < L_g < 1$.

(H_c) *For almost all $t \in [0,T]$ and $x_t \in C([-h,0],X)$, $Ag(t,x_t) \in L^1((0,T),X^1)$ and $A^2 g(t,x_t) \in L^\infty((0,T),X)$.*

(H_d) *For almost all $t \in [0,T]$ and $x_t \in C([-h,0],X)$, $f(t,x_t) \in L^1((0,T),X^1)$ and $Af(t,x_t) \in L^\infty((0,T),X)$.*

Then the mild solution is a classical solution to problem (3.1), *provided that $A(\varphi(0) - g(0,x_0)) \in D(A^\beta)$ with $\beta > 1+\gamma$.*

Proof To establish the conclusion, we observe from the proof of Theorem 3.3 that it is sufficient to establish that $({}_0^C D_t^\alpha x)(t) \in C([0,T_0],X)$.

We now define $u(t)$ and $w(t)$ as in (3.13) and (3.15). We first prove that $({}_0^C D_t^\alpha u)(t) \in C([0,T_0],X)$. By (3.14) we only need to prove that $-Au(t) + Ag(t,x_t) \in C([0,T_0],X)$. According to the assumption, $Ag(t,x_t)$ is continuous for all $t \in [0,T_0]$ and $x_t \in C([-h,0],X)$. So, it remains to prove that $I(t) = -Au(t)$ is continuous for all $t \in [0,T_0]$. For that, we express $I(t)$ as $I(t) = I_1(t) + I_2(t)$, where

$$\begin{aligned} I_1(t) &= -A\int_0^t (t-s)^{\alpha-1} A\mathcal{P}_\alpha(t-s)\bigl(g(s,x_s) - g(t,x_t)\bigr)\,ds, \\ I_2(t) &= A\int_0^t (t-s)^{\alpha-1} A\mathcal{P}_\alpha(t-s) g(t,x_t)\,ds. \end{aligned}$$

Using Proposition 2.5(iii), we get $I_2(t) = -(\mathcal{S}_\alpha(t) - I)Ag(t,x_t)$. So, by Proposition 2.5(v) and condition (H_c) we have that $I_2(t)$ is continuous for $t \in [0,T_0]$. Next, we prove that $I_1(t)$ is continuous for $t \in [0,T_0]$. Let $\Delta t > 0$ be such that $t + \Delta t \leq T_0$. Then

$$\begin{aligned} &\bigl\|I_1(t+\Delta t) - I_1(t)\bigr\| \\ &\quad\leq \biggl\|A\int_0^t \bigl((t+\Delta t-s)^{\alpha-1} A\mathcal{P}_\alpha(t+\Delta t-s) - (t-s)^{\alpha-1} \\ &\qquad\times A\mathcal{P}_\alpha(t-s)\bigr)\bigl(g(s,x_s) - g(t,x_t)\bigr)\,ds\biggr\| \\ &\qquad+ \biggl\|A\int_0^t (t+\Delta t-s)^{\alpha-1} A\mathcal{P}_\alpha(t+\Delta t-s)\bigl(g(t,x_t) - g(t+\Delta t,x_{t+\Delta t})\bigr)\,ds\biggr\| \\ &\qquad+ \biggl\|A\int_t^{t+\Delta t} (t+\Delta t-s)^{\alpha-1} A\mathcal{P}_\alpha(t+\Delta t-s)\bigl(g(s,x_s) - g(t+\Delta t,x_{t+\Delta t})\bigr)\,ds\biggr\| \\ &\quad= h_1(t) + h_2(t) + h_3(t). \end{aligned}$$

For $h_1(t)$, on the one hand, we have

$$\begin{aligned}&\left\|A\int_0^t (t+\Delta t-s)^{\alpha-1}A\mathcal{P}_\alpha(t+\Delta t-s)\big(g(s,x_s)-g(t,x_t)\big)\,ds\right\|\\&\quad\le L_g\frac{t^{\theta_1-\alpha(1+\gamma)}}{\theta_1-\alpha(1+\gamma)}+L_gQ\frac{t^{\vartheta-\alpha(1+\gamma)}}{\vartheta-\alpha(1+\gamma)}.\end{aligned}$$

Moreover, since $\theta_1>\alpha(1+\gamma)$ and $\vartheta>\alpha(1+\gamma)$, we get $A(t+\Delta t-s)^{\alpha-1}A\mathcal{P}_\alpha(t+\Delta t-s)(g(s,x_s)-g(t,x_t))\in L^1((0,t),X)$. On the other hand, by Proposition 2.5(v) we have

$$\begin{aligned}&\lim_{\Delta t\to 0}A(t+\Delta t-s)^{\alpha-1}A\mathcal{P}_\alpha(t+\Delta t-s)\big(g(s,x_s)-g(t,x_t)\big)\\&\quad=A(t-s)^{\alpha-1}A\mathcal{P}_\alpha(t-s)\big(g(s,x_s)-g(t,x_t)\big),\end{aligned}$$

so that by the Lebesgue dominated convergence theorem we get $h_1(t)\to 0$ as $\Delta t\to 0$.

For $h_2(t)$, we have the estimate

$$\begin{aligned}h_2(t)&\le\left\|A\int_0^t (t+\Delta t-s)^{\alpha-1}A\mathcal{P}_\alpha(t+\Delta t-s)\big(g(t,x_t)-g(t+\Delta t,x_{t+\Delta t})\big)\,ds\right\|\\&\le L_gC_p\int_0^t (t+\Delta t-s)^{-\alpha-\alpha\gamma-1}\big((\Delta t)^{\theta_1}+Q(\Delta t)^{\vartheta}\big)\,ds\\&=L_gC_p\frac{(\Delta t)^{-\alpha(1+\gamma)}-(t+\Delta t)^{-\alpha(1+\gamma)}}{-\alpha(1+\gamma)}\big((\Delta t)^{\theta_1}+Q(\Delta t)^{\vartheta}\big).\end{aligned}$$

Moreover, since $\theta_1>\alpha(1+\gamma)$ and $\vartheta>\alpha(1+\gamma)$, we get $h_2(t)\to 0$ as $\Delta t\to 0$.

For $h_3(t)$, by Proposition 2.5(i) and condition (H_1^*) we have the estimate

$$h_3(t)\le\frac{2(\Delta t)^{-\alpha\gamma}}{-\alpha\gamma}\big\|A^2g\big\|_{L^\infty((0,T_0),X)},$$

which implies that $h_3(t)\to 0$ as $\Delta t\to 0$. Hence, $-Au(t)+Ag(t,x_t)$ is continuous for all $t\in[0,T_0]$.

By the argument used earlier we have that $({}_0^CD_t^\alpha w)(t)$ is continuous for all $t\in[0,T_0]$. Combining Proposition 2.5(v) and the assumption $A(\varphi(0)-g(0,x_0))\in D(A^\beta)$ with $\beta>1+\gamma$, it follows that $({}_0^CD_t^\alpha x)(t)$ is continuous for all $t\in[0,T_0]$, and the proof is completed. □

Corollary 3.4 *Let $A\in\Theta_\omega^\gamma$ with $-1<\gamma<-\frac{1}{2}$ and $0<\omega<\pi/2$, and $\varphi(t)\in C([-h,0],X^1)$. Suppose that conditions (H_3^*), (H_4^*), and (H_d) hold. Then the mild solution x to equation* (3.8) *is its unique classical solution, provided that $A\varphi(0)\in D(A^\beta)$ with $\beta>1+\gamma$.*

4 Application

In this section, we demonstrate the applicability of the obtained results to the following problem:

$$\begin{cases}{}_0^C\partial_t^\alpha\big(u(t,y)-\frac{1}{2}e^{-t}\sin(y(t-h))\big)=\partial_y^2u(t,y)+\int_{t-h}^t\chi(s-t)u(s,y)\,ds,\\\quad t\in[0,T],y\in[0,\pi],\\u(t,0)=u(t,\pi)=0,\quad t\in[0,T],\\u(t,y)=(\varphi(t))(y),\quad t\in[-h,0],y\in[0,\pi],\end{cases}\tag{4.1}$$

in the space of Hölder continuous functions $X := C^l([0,\pi],\mathbb{R})$ $(0 < l < 1)$, where ${}_0^C\partial_t^\alpha$ is the Caputo fractional partial derivative of order $0 < \alpha < 1$ with respect to t, that is,

$$\left({}_0^C\partial_t^\alpha u\right)(t,y) = \frac{1}{\Gamma(1-\alpha)}\left(\frac{\partial}{\partial t}\int_0^t (t-s)^{-\alpha}u(s,y)\,ds - t^{-\alpha}u(0,y)\right).$$

We now introduce the operator

$$\widetilde{A} := -\partial_y^2, \qquad D(\widetilde{A}) = \left\{u \in C^{2+l}\left([0,\pi]\right) : u(t,0) = u(t,\pi) = 0\right\},$$

in the space $C^l([0,\pi],\mathbb{R})$ $(0 < l < 1)$ of Hölder continuous functions. It follows from [4] that there exist $\nu, \epsilon > 0$ such that $\widetilde{A} + \nu \in \Theta_{\frac{\pi}{2}-\epsilon}^{\frac{l}{2}-1}(X)$.

To represent this system in the abstract form (3.1), we introduce the function $f : [0,T] \times C([-h,0],X) \to X$ given by

$$f(t,\Psi)(y) = \int_{-h}^0 \chi(s)\Psi(s,y)\,ds.$$

If $\chi \in L^1([0,T],\mathbb{R})$, then $f \in C([0,T],X)$, and it follows that there exists a function $m(t) \in L^1([0,T],\mathbb{R})$ such that

$$\left\|f(t,\cdot)\right\| \leq m(t).$$

It follows from Theorem 3.2 that there exists T_0 such that equation (4.1) has a unique mild solution on $[-h,T_0]$.

5 Conclusions

The research on almost sectorial operators has been of significant interest during the past years. However, it has been found that there is no published material addressing the existence and uniqueness of solutions for fractional neutral evolution equations with almost sectorial operators. To enrich the literature on the topic, we have investigated the existence and uniqueness of mild solutions to fractional neutral evolution equations with almost sectorial operators in this article. Our study relies on some fixed point theorems. Under some suitable assumptions, we have shown that a mild solution can become a strong solution and a classical solution. As an illustration of our work, we have discussed the existence and uniqueness of a mild solution for a fractional partial differential equation.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors XLD and BA contributed to each part of this work equally and read and approved the final version of the manuscript.

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