Nockowska-Rosiak *Advances in Difference Equations* (2016) 2016:198
DOI 10.1186/s13662-016-0923-2

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Existence of uncountably many bounded positive solutions to higher-order nonlinear neutral delay difference equations

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Abstract

This paper studies higher-order nonlinear neutral delay difference equations of the form

$$\Delta(r_n^{m-1}(\Delta(r_n^{m-2}(\cdots(\Delta(r_n^1(\Delta(x_n+p_nx_{n-\tau}))^{\gamma_1}))^{\gamma_2}\cdots)^{\gamma_{m-2}}))^{\gamma_{m-1}})$$
$$=f(n,x_{n-\tau_1},\ldots,x_{n-\tau_s}).$$

Using Krasnoselskii's fixed point theorem, we obtain the existence of uncountably many bounded positive solutions to the considered problem.

Keywords: nonlinear difference equation; neutral type; Krasnoselskii's fixed point theorem

1 Introduction and preliminaries

In mathematical models in diverse areas such as economy, biology, computer science, difference equations appear in a natural way; see, for example, [1, 2]. In the past thirty years, oscillation, nonoscillation, the asymptotic behavior and existence of bounded solutions to many types of difference equation have been widely examined. For the second order, see, for example, [3–9], and for higher orders, [10–15], and references therein.

Liu *et al.* [16] discussed the existence of uncountably many bounded positive solutions to

$$\Delta\big(r_n\Delta(x_n+b_nx_{n-\tau}-c_n)\big)+f\big(n,x\big(f_1(n)\big),\ldots,x\big(f_k(n)\big)\big)=d_n$$

with respect (b_n). Using techniques of the measures of noncompactness, Galewski *et al.* [4] considered

$$\Delta\big(r_n\big(\Delta(x_n+p_nx_{n-\tau})\big)^{\gamma}\big)+q_nx_n^{\alpha}+a_nf(x_{n+1})=0.$$

Migda and Schmeidel [12] studied the following equation:

$$\Delta\big(r_n^{m-1}\Delta\big(r_n^{m-2}\cdots\Delta\big(r_n^1\Delta(x_n+p_nx_{n-\tau})\big)\cdots\big)\big)=a_nf(x_{n-\sigma})+b_n.$$

They established sufficient conditions under which for every real constant, there exists a solution to the studied problem convergent to this constant.

In this paper, we study higher-order nonlinear neutral delay difference equations of the form

$$\begin{aligned}&\Delta\bigl(r_n^{m-1}\bigl(\Delta\bigl(r_n^{m-2}\bigl(\cdots\bigl(\Delta\bigl(r_n^1\bigl(\Delta(x_n+p_nx_{n-\tau})\bigr)^{\gamma_1}\bigr)\bigr)^{\gamma_2}\cdots\bigr)^{\gamma_{m-2}}\bigr)\bigr)^{\gamma_{m-1}}\bigr)\\&\quad=f(n,x_{n-\tau_1},\dots,x_{n-\tau_s})\end{aligned}\tag{1}$$

under the following general settings:

(H_1) $m\geq 2$, $\gamma_1,\dots,\gamma_{m-1}\leq 1$ are ratios of odd positive integers, $\tau\in\mathbb{N}$, $\tau_1,\dots,\tau_s\in\mathbb{Z}$, $(p_n)\subset\mathbb{R}$, $r^i=(r_n^i)\subset\mathbb{R}\setminus\{0\}$, $i=1,\dots,m-1$, and $f:\mathbb{N}\times\mathbb{R}^s\to\mathbb{R}$.

Additional conditions will be added to obtain the existence of uncountably many positive (nonoscillatory) solutions to equation (1). Krasnoselskii's fixed point theorem will be used to prove our results. To illustrate them, three examples are included.

Throughout this paper, we assume that Δ is the forward difference operator. By a solution to equation (1) we mean a sequence $x:\mathbb{N}\to\mathbb{R}$ that satisfies (1) for every $n\geq k$ for some $k\geq\max\{\tau,\tau_1,\dots,\tau_s\}$.

We consider the Banach space l^∞ of all real bounded sequences $x:\mathbb{N}\to\mathbb{R}$ equipped with the standard supremum norm, that is, for $x=(x_n)\in l^\infty$,

$$\|x\|=\sup_{n\in\mathbb{N}}|x_n|.$$

Definition 1 ([17]) A subset A of l^∞ is said to be uniformly Cauchy if for every $\varepsilon>0$, there exists $n_0\in\mathbb{N}$ such that $|x_i-x_j|<\varepsilon$ for any $i,j\geq n_0$ and $x=(x_n)\in A$.

Theorem 1 ([17]) *A bounded, uniformly Cauchy subset of l^∞ is relatively compact.*

We shall use Krasnoselskii's fixed point theorem in the following form.

Theorem 2 ([18], 11.B, p.501) *Let X be a Banach space, B be a bounded closed convex subset of X, and $S,G:B\to X$ be mappings such that $Sx+Gy\in B$ for any $x,y\in B$. If S is a contraction and G is a compact, then the equation*

$$Sx+Gx=x$$

has a solution in B.

2 Main results

For any nonnegative sequence $y=(y_n)$ and $n\in\mathbb{N}$, we use the notation

$$W_1(n,y)=\sum_{l_1=n}^{\infty}y_{l_1};$$

$$W_2(n,y)=\sum_{l_2=n}^{\infty}\left(\left|\frac{1}{r_{l_2}^{m-1}}\right|\sum_{l_1=l_2}^{\infty}y_{l_1}\right)^{\gamma_{m-1}^{-1}}=\sum_{l_2=n}^{\infty}\left(\frac{W_1(l_2,y)}{|r_{l_2}^{m-1}|}\right)^{\gamma_{m-1}^{-1}};$$

$\dots;$

$$W_k(n,y) = \sum_{l_k=n}^{\infty}\left(\frac{W_{k-1}(l_k,y)}{|r_{l_k}^{m-k+1}|}\right)^{\gamma_{m-k+1}^{-1}}, \quad k = 2,\dots,m.$$

By $[0,M]^s$ we denote the set $[0,M] \times \cdots \times [0,M] \subset \mathbb{R}^s$.

Now we are in position to formulate and prove the main theorem.

Theorem 3 *Suppose that* (H_1) *is satisfied. Assume further that*

(H_2) $\sup_{n\in\mathbb{N}} |p_n| = p^\star < 1/4$;

(H_3) *there exists* $M > 0$ *such that for any* $n \in \mathbb{N}$, *the function* $f(n,\cdot)$ *is a Lipschitz function on* $[0,2M]^s$ *with Lipschitz constant* $P(n,M)$ *satisfying*

$$\sum_{l_m=1}^{\infty}\left|\frac{1}{r_{l_m}^1}\right|^{\gamma_1^{-1}} \sum_{l_{m-1}=l_m}^{\infty}\left|\frac{1}{r_{l_{m-1}}^2}\right|^{\gamma_2^{-1}} \cdots \sum_{l_2=l_3}^{\infty}\left|\frac{1}{r_{l_2}^{m-1}}\right|^{\gamma_{m-1}^{-1}} \sum_{l_1=l_2}^{\infty} P(l_1,M) < \infty;$$

(H_4) $W_m(1,|f(\cdot,0_{\mathbb{R}^s})|) < \infty$.

Then, equation (1) *possesses uncountably many bounded positive solutions lying in* $[M/2,2M]$.

Proof Let $M > 0$ be a constant fulfilling assumption (H_3). It is easy to see that (H_3) implies that

$$\sum_{l_1=1}^{\infty} P(l_1,M) < \infty \tag{2}$$

and

$$\sum_{l_k=1}^{\infty}\left|\frac{1}{r_{l_k}^{m-k+1}}\right|^{\gamma_{m-k+1}^{-1}} \cdots \sum_{l_2=l_3}^{\infty}\left|\frac{1}{r_{l_2}^{m-1}}\right|^{\gamma_{m-1}^{-1}} \sum_{l_1=l_2}^{\infty} P(l_1,M) < \infty, \quad k = 2,\dots,m-1. \tag{3}$$

From (H_4) it is clear that

$$\sum_{n=1}^{\infty}\bigl|f(n,0_{\mathbb{R}^s})\bigr| < \infty, W_k\bigl(1,\bigl|f(\cdot,0_{\mathbb{R}^s})\bigr|\bigr) < \infty, \quad k = 2,\dots,m. \tag{4}$$

Now we claim that (H_3) and (3) imply that

$$W_k\bigl(1,P(\cdot,M)\bigr) < \infty, \quad k = 2,\dots,m. \tag{5}$$

Indeed, from (2) we get that there exists n_1 such that for any $n \ge n_1$, we have $\sum_{l_1=n}^{\infty} P(l_1, M) < 1$; hence, since $\gamma_{m-1} \le 1$, we get that for any $n \ge n_1$,

$$\left(\sum_{l_1=n}^{\infty} P(l_1,M)\right)^{\gamma_{m-1}^{-1}} \le \sum_{l_1=n}^{\infty} P(l_1,M).$$

Thus, for any $n \geq n_1$, we have

$$W_2\big(n, P(\cdot, M)\big) = \sum_{l_2=n}^{\infty} \left| \frac{1}{r_{l_2}^{m-1}} \sum_{l_1=l_2}^{\infty} P(l_1, M) \right|^{\gamma_{m-1}^{-1}} \leq \sum_{l_2=n}^{\infty} \left| \frac{1}{r_{l_2}^{m-1}} \right|^{\gamma_{m-1}^{-1}} \sum_{l_1=l_2}^{\infty} P(l_1, M).$$

To prove that $W_2(1, P(\cdot, M)) < \infty$, we use the classical inequality

$$(a+b)^{\alpha} \leq 2^{\alpha-1}\big(a^{\alpha} + b^{\alpha}\big) \quad \text{for } \alpha \geq 1, a, b > 0, \tag{6}$$

which gives

$$\begin{aligned} W_2\big(1, P(\cdot, M)\big) &\leq W_2\big(n_1, P(\cdot, M)\big) + \sum_{l_2=1}^{n_1-1} \left| \frac{W_1(l_2, P(\cdot, M))}{r_{l_2}^{m-1}} \right|^{\gamma_{m-1}^{-1}} \\ &\leq W_2\big(n_1, P(\cdot, M)\big) \\ &\quad + \sum_{l_2=1}^{n_1-1} \frac{2^{\gamma_{m-1}^{-1}-1}}{|r_{l_2}^{m-1}|^{\gamma_{m-1}^{-1}}} \left(\left(\sum_{l_1=l_2}^{n_1-1} P(l_1, M) \right)^{\gamma_{m-1}^{-1}} + \left(\sum_{l_1=n_1}^{\infty} P(l_1, M) \right)^{\gamma_{m-1}^{-1}} \right) < \infty. \end{aligned}$$

In an analogous way, we prove the remaining conditions in (5). We now claim that

$$\sum_{n=1}^{\infty} \left(\frac{W_k(n, 2M\sqrt{s}P(\cdot, M) + |f(\cdot, 0_{\mathbb{R}^s})|)}{|r_n^{m-k}|} \right)^{\gamma_{m-k}^{-1}} < \infty, \quad k = 1, \ldots, m-1. \tag{7}$$

We give the proof of (7) for the case $k = 2$ and $m = 3$; the other cases are analogous and are left to the reader. Indeed, using (6), we have

$$\begin{aligned} &\sum_{l_3=1}^{\infty} \left(\frac{W_2(l_3, 2M\sqrt{s}P(\cdot, M) + |f(\cdot, 0_{\mathbb{R}^s})|)}{|r_{l_3}^1|} \right)^{\gamma_1^{-1}} \\ &= \sum_{l_3=1}^{\infty} \left(\frac{1}{|r_{l_3}^1|} \sum_{l_2=l_3}^{\infty} \left(\frac{W_1(l_2, 2M\sqrt{s}P(\cdot, M))}{|r_{l_2}^2|} + \frac{W_1(l_2, |f(\cdot, 0_{\mathbb{R}^s})|)}{|r_{l_2}^2|} \right)^{\gamma_2^{-1}} \right)^{\gamma_1^{-1}} \\ &\leq 2^{(\gamma_2^{-1}-1)\gamma_1^{-1}} \sum_{l_3=1}^{\infty} \left(\frac{1}{|r_{l_3}^1|} \left(\sum_{l_2=l_3}^{\infty} \left(\frac{W_1(l_2, 2M\sqrt{s}P(\cdot, M))}{|r_{l_2}^2|} \right)^{\gamma_2^{-1}} \right.\right. \\ &\quad \left.\left. + \sum_{l_2=l_3}^{\infty} \left(\frac{W_1(l_2, |f(\cdot, 0_{\mathbb{R}^s})|)}{|r_{l_2}^2|} \right)^{\gamma_2^{-1}} \right) \right)^{\gamma_1^{-1}} \\ &\leq 2^{(\gamma_2^{-1}-1)\gamma_1^{-1}+\gamma_1^{-1}-1} \cdot \left(\sum_{l_3=1}^{\infty} \left(\frac{W_2(l_3, 2M\sqrt{s}P(\cdot, M))}{|r_{l_3}^1|} \right)^{\gamma_1^{-1}} \right. \\ &\quad \left. + \sum_{l_3=1}^{\infty} \left(\frac{W_2(l_3, |f(\cdot, 0_{\mathbb{R}^s})|)}{|r_{l_3}^1|} \right)^{\gamma_1^{-1}} \right) \\ &= 2^{(\gamma_2^{-1}-1)\gamma_1^{-1}+\gamma_1^{-1}-1} \cdot \big((2M\sqrt{s})^{\gamma_1^{-1}\gamma_2^{-1}} W_3\big(1, P(\cdot, M)\big) + W_3\big(1, \big|f(\cdot, 0_{\mathbb{R}^s})\big|\big) \big) < \infty. \end{aligned}$$

Once the claim proved, observe that we may find $n_0 \geq \max\{\tau, \tau_1, \dots, \tau_s\}$ such that

$$W_m\bigl(n_0, 2M\sqrt{s}P(\cdot, M) + \bigl|f(\cdot, 0_{\mathbb{R}^s})\bigr|\bigr) < 2M\left(\frac{1}{4} - p^\star\right). \tag{8}$$

We consider a subset of l^∞ of the form

$$A_{n_0} = \bigl\{x = (x_n) \in l^\infty : x_n = 3M/2, n < n_0 \wedge |x_n - 3M/2| \leq M/2, n \geq n_0\bigr\}.$$

Observe that A_{n_0} is a nonempty, bounded, convex, and closed subset of l^∞.

Let us denote

$$u_1^x(n) = \sum_{l=n}^{\infty} f(l, x_{l-\tau_1}, \dots, x_{l-\tau_s}), \qquad x = (x_n), x_n \in [0, 2M], n \geq \max\{\tau_1, \dots, \tau_s\}.$$

The following takes care of showing that u_1^x is well defined and bounded above. By (H_3), for any $\mathbf{x} = (x_1, \dots, x_s) \in [0, 2M]^s$ and for any $n \in \mathbb{N}$, we have

$$\bigl|f(n, \mathbf{x})\bigr| \leq P(n, M)\|\mathbf{x}\|_{\mathbb{R}^s} + \bigl|f(n, 0_{\mathbb{R}^s})\bigr| \leq 2M\sqrt{s}P(n, M) + \bigl|f(n, 0_{\mathbb{R}^s})\bigr|, \tag{9}$$

where $\|\cdot\|_{\mathbb{R}^s}$ denotes the Euclidean norm in $\mathbb{R}^s$. Thus, for any $x = (x_n) \in A_{n_0}$ and $n \geq \max\{\tau_1, \dots, \tau_s\}$,

$$\bigl|u_1^x(n)\bigr| \leq W_1\bigl(n, 2M\sqrt{s}P(\cdot, M) + \bigl|f(\cdot, 0_{\mathbb{R}^s})\bigr|\bigr). \tag{10}$$

Denote, for any $x = (x_n) \in A_{n_0}$ and $n \geq \max\{\tau_1, \dots, \tau_s\}$,

$$u_2^x(n) = \sum_{l=n}^{\infty} \left(\frac{u_1^x(l)}{r_l^{m-1}}\right)^{\gamma_{m-1}^{-1}}.$$

Thus, for any $x = (x_n) \in A_{n_0}$ and $n \geq \max\{\tau_1, \dots, \tau_s\}$,

$$\bigl|u_2^x(n)\bigr| \leq W_2\bigl(n, 2M\sqrt{s}P(\cdot, M) + \bigl|f(\cdot, 0_{\mathbb{R}^s})\bigr|\bigr). \tag{11}$$

In an analogous way, for any $x = (x_n) \in A_{n_0}$ and $n \geq \max\{\tau_1, \dots, \tau_s\}$, we denote

$$u_k^x(n) = \sum_{l=n}^{\infty} \left(\frac{u_{k-1}^x(l)}{r_l^{m-k+1}}\right)^{\gamma_{m-k+1}^{-1}}, \quad k = 2, \dots, m.$$

Thus, for any $k = 2, \dots, m$, $x = (x_n) \in A_{n_0}$, and $n \geq \max\{\tau_1, \dots, \tau_s\}$,

$$\bigl|u_k^x(n)\bigr| \leq W_k\bigl(n, 2M\sqrt{s}P(\cdot, M) + \bigl|f(\cdot, 0_{\mathbb{R}^s})\bigr|\bigr). \tag{12}$$

Define two mappings $T_1, T_2 : A_{n_0} \to l^\infty$ as follows:

$$(T_1x)_n = \begin{cases} 0 & \text{for } 1 \leq n < n_0, \\ -p_n x_{n-\tau} & \text{for } n \geq n_0; \end{cases} \tag{13}$$

$$(T_2x)_n = \begin{cases} 3M/2 & \text{for } 1 \le n < n_0, \\ 3M/2 + (-1)^m u_m^x(n) & \text{for } n \ge n_0. \end{cases} \tag{14}$$

Our next goal is to check the assumptions of Theorem 2 (Krasnoselskii's fixed point theorem). Firstly, we show that $T_1x + T_2y \in A_{n_0}$ for $x, y \in A_{n_0}$. Let $x, y \in A_{n_0}$. For $n < n_0$, $(T_1x + T_2y)_n = 3M/2$. For $n \ge n_0$, from assumption (H_2), (8), and (12) we get

$$\left|(T_1x + T_2y)_n - 3M/2\right| \le |p_n x_{n-\tau}| + \left|u_m^x(n)\right| \le p^\star 2M + 2M\bigl(1/4 - p^\star\bigr) = M/2.$$

It is easy to see that

$$\|T_1x - T_1y\| \le p^\star \|x - y\| \quad \text{for } x, y \in A_{n_0},$$

so that T_1 is a contraction.

To prove the continuity of T_2, notice that from (12) we get

$$\left|u_{m-1}^x(n)\right| \le W_{m-1}\bigl(1, 2M\sqrt{s}P(\cdot, M) + \left|f(\cdot, 0_{\mathbb{R}^s})\right|\bigr) =: d_{m-1}$$

for any $x = (x_n) \in A_{n_0}$ and $n \ge \max\{\tau_1, \dots, \tau_s\}$. From the Lipschitz continuity of the function $x \mapsto x^{\gamma_1^{-1}}$ on $[0, d_{m-1}]$ with constant L_{γ_1}, say, we have

$$\left|(T_2x - T_2y)_n\right| \le \sum_{l_m=n}^{\infty} \left|\frac{1}{r_{l_m}^1}\right|^{\gamma_1^{-1}} L_{\gamma_1} \left|u_{m-1}^x(n) - u_{m-1}^y(n)\right|$$

for any $x, y \in A_{n_0}$ and $n \ge n_0$. In an analogous way, by (12), for any $k = 2, \dots, m$, we get intervals $[0, d_k]$ on which the function $x \mapsto x^{\gamma_k^{-1}}$ is Lipschitz continuous, say, with constant $L_{\gamma_k} > 0$. Hence, for any $x, y \in A_{n_0}$ and $n \ge n_0$, we have

$$\begin{aligned} \left|(T_2x - T_2y)_n\right| &\le L_{\gamma_1} \cdot \ldots \cdot L_{\gamma_{m-1}} \cdot \sum_{l_m=n}^{\infty} \left|\frac{1}{r_m^1}\right|^{\gamma_1^{-1}} \cdots \sum_{l_k=l_{k+1}}^{\infty} \left|\frac{1}{r_{l_k}^{m-k+1}}\right|^{\gamma_{m-k+1}^{-1}} \cdots \sum_{l_2=l_3}^{\infty} \left|\frac{1}{r_{l_2}^{m-1}}\right|^{\gamma_{m-1}^{-1}} \\ &\quad \times \sum_{l_1=l_2}^{\infty} \left|f(k, x_{k-\tau_1}, \dots, x_{k-\tau_s}) - f(k, y_{k-\tau_1}, \dots, y_{k-\tau_s})\right| \\ &\le \sqrt{s} \cdot \prod_{j=1}^{m-1} L_{\gamma_j} \cdot \left(\sum_{l_m=n}^{\infty} \left|\frac{1}{r_m^1}\right|^{\gamma_1^{-1}} \cdots \sum_{l_2=l_3}^{\infty} \left|\frac{1}{r_{l_2}^{m-1}}\right|^{\gamma_{m-1}^{-1}} \sum_{l_1=l_2}^{\infty} P(l_1, M) \right) \|x - y\|, \end{aligned}$$

which, combined with (H_3), means that T_2 is continuous on A_{n_0}.

Now we show that $T_2(A_{n_0})$ is uniformly Cauchy. Let $\varepsilon > 0$. From (7) we get the existence of $n_\varepsilon \in \mathbb{N}$ such that $n_\varepsilon \ge n_0$ and

$$2W_m\bigl(n_\varepsilon, 2M\sqrt{s}P(\cdot, M) + \left|f(\cdot, 0_{\mathbb{R}^s})\right|\bigr) < \varepsilon.$$

From (12) we have, for $k, n \ge n_\varepsilon \ge n_0$ and for $x = (x_n) \in A_{n_0}$,

$$\left|(T_2x)_n - (T_2x)_k\right| \le 2\left|u_m^x(n_\varepsilon)\right| \le 2W_m\bigl(n_\varepsilon, 2M\sqrt{s}P(\cdot, M) + \left|f(\cdot, 0_{\mathbb{R}^s})\right|\bigr) < \varepsilon.$$

Since $T_2(A_{n_0})$ is uniformly Cauchy and bounded, by Theorem 1, $T_2(A_{n_0})$ is relatively compact in l^∞, which means that T_2 is a compact operator.

From Theorem 2 we get that there exists a fixed point $x = (x_n)$ of $T_1 + T_2$ on A_{n_0}. Hence,

$$x_n + p_n x_{n-\tau} = (-1)^m u_m^x(n) + 3M/2$$

for $n \geq n_0$. Applying the operator Δ to both sides of the last equation, raising to the power γ_1 (recalling that it is the ratio of odd positive integers), and multiplying by r_n^1, we get

$$r_n^1\bigl(\Delta(x_n + p_n x_{n-\tau})\bigr)^{\gamma_1} = (-1)^{m-1} u_{m-1}^x(n)$$

for $n \geq n_0$. Repeating this procedure $m-2$ times, we get that $x = (x_n)$ is a solution to equation (1) for $n \geq n_0$ with $x_n \in [M, 2M]$.

Now we prove the existence of uncountably many solutions to (1) lying in $[M/2, 2M]$. Let M_1, M_2 be such that $M/2 < M_1 < M_2 < M$. It is easy to see that the assumptions of the theorem are fulfilled for M_1, M_2. So there exist $n_1, n_2 \geq \max\{\tau, \tau_1, \dots, \tau_s\}$ and $x^1 = (x_n^1)$ and $x^2 = (x_n^2)$, each a fixed point of the operator $T_1^i + T_2^i$ in A_{n_i}, respectively, where

$$\bigl(T_1^i x\bigr)_n = \begin{cases} 0 & \text{for } 1 \leq n < n_i, \\ -p_n x_{n-\tau} & \text{for } n \geq n_i; \end{cases}$$

$$\bigl(T_2^i x\bigr)_n = \begin{cases} 3M_i/2 & \text{for } 1 \leq n < n_i, \\ 3M_i/2 + (-1)^m u_m^x(n) & \text{for } n \geq n_i. \end{cases}$$

Thus, x^i are solutions to (1) for $n \geq \max\{n_1, n_2\}$. By (12) there exists $n_3 \in \mathbb{N}$, $n_3 \geq \max\{n_1, n_2\}$, such that

$$\bigl|u_m^{x^1}(n)\bigr| + \bigl|u_m^{x^2}(n)\bigr| \leq 3/4(M_2 - M_1) \quad \text{for } n \geq n_3.$$

From this we get that, for $n \geq n_3$,

$$\bigl|x_n^1 - x_n^2 + p_n\bigl(x_{n-\tau}^1 - x_{n-\tau}^2\bigr)\bigr| \geq 3/2(M_2 - M_1) - \bigl(\bigl|u_m^{x^1}(n)\bigr| + \bigl|u_m^{x^2}(n)\bigr|\bigr) > 0,$$

which means that x^1 and x^2 are different solutions to (1) lying in $[M/2, 2M]$. □

Remark 1 It is obvious that condition (H_4) in Theorem 3 can be replaced by the condition

(H_4') $W_m(1, |f(\cdot, \overline{\mathbf{x}})|) < \infty$ for some $\overline{\mathbf{x}} \in [0, 2M]^s$.

3 Examples

Now, we present examples of equations for which our method can be applied.

Example 1 Let us consider the second-order nonlinear neutral delay difference equation

$$\Delta\bigl(\sqrt{n}\,\Delta(x_n + p_n x_{n-\tau})\bigr) = \frac{x_{n-\tau_1}^2}{4(2n-1)(2n+1)}, \tag{15}$$

where $\tau \in \mathbb{N}$, $\tau_1 \in \mathbb{Z}$, $\gamma_1 = 1$, and (p_n) is any sequence of real numbers such that $\sup_{n\in\mathbb{N}} |p_n| < 1/4$. Moreover, $r_n^1 = \sqrt{n}$, and $f(n,u) = \frac{u^2}{4(2n-1)(2n+1)}$ for $n \in \mathbb{N}$ and $u \in \mathbb{R}$.

Since $f(n,\cdot) \in C^1(\mathbb{R})$ for any $n \in \mathbb{N}$, it follows that, for any $n \in \mathbb{N}$, $f(n,\cdot)$ is a locally Lipschitz function on $\mathbb{R}$. Hence, for any $n \in \mathbb{N}$, $f(n,\cdot)$ is a Lipschitz function on $[0,2M]$ for any $M > 0$. It is easy to calculate that $P(n,M) = \frac{M}{(2n-1)(2n+1)}$ and

$$\sum_{l_2=1}^{\infty} \frac{1}{r_{l_2}^1} \sum_{l_1=l_2}^{\infty} P(l_1,M) = \sum_{l_2=1}^{\infty} \frac{1}{\sqrt{l_2}} \sum_{l_1=l_2}^{\infty} \frac{M}{(2l_1-1)(2l_1+1)} = \sum_{l_2=1}^{\infty} \frac{M}{2\sqrt{l_2}(2l_2-1)} < \infty,$$

so that assumption (H_3) of Theorem 3 is satisfied. To see that assumption (H_4) of Theorem 3 is fulfilled, notice that $f(n,0) = 0$, $n \in \mathbb{N}$. Hence, there exist uncountably many solutions to (15) in any interval $[M/2,2M]$ for any $M > 0$. On the other hand, Theorem 3.1 in [16] is inapplicable because

$$\sum_{l_2=1}^{\infty} \frac{1}{r_{l_2}^1} \sum_{l_1=1}^{l_2-1} P(l_1,M) = \sum_{l_2=1}^{\infty} \frac{1}{\sqrt{l_2}} \sum_{l_1=1}^{l_2-1} \frac{M}{(2l_1-1)(2l_1+1)} = \sum_{l_2=1}^{\infty} \frac{M(l_2-1)}{\sqrt{l_2}(2l_2-1)} = \infty.$$

Example 2 Consider the third-order nonlinear neutral delay difference equation

$$\begin{aligned} &\Delta\left(n\left[\Delta\left(\sqrt{n+2}\left[\Delta\left(x_n + \frac{2+(-1)^n}{16}x_{n-\tau}\right)\right]^{1/3}\right)\right]^{1/5}\right) \\ &\quad = \frac{(-1)^n \sin(x_{n-\tau_1}) - n^2 x_{n-\tau_2}^6}{(n^5+7n^3+1)(x_{n-\tau_1}^4 + x_{n-\tau_2}^2 + 1)}, \end{aligned} \tag{16}$$

where $\tau \in \mathbb{N}$, $\tau_1, \tau_2 \in \mathbb{Z}$, $\gamma_1 = 1/3$, $\gamma_2 = 1/5$. Moreover, $p_n = \frac{2+(-1)^n}{16}$, $r_n^1 = \sqrt{n+2}$, $r_n^2 = n$, and $f(n,u,v) = \frac{(-1)^n \sin(u) - n^2 v^6}{(n^5+7n^3+1)(u^4+v^2+1)}$ for any $n \in \mathbb{N}$ and $u, v \in \mathbb{R}$.

Because $f(n,\cdot) \in C^1(\mathbb{R}^2)$ for any $n \in \mathbb{N}$, $f(n,\cdot)$ is a Lipschitz function on $[0,2M]^2$ for any $M > 0$. It is easy to calculate that there exists $D(M) > 0$ such that $P(n,M) \leq \frac{D(M)n^2}{n^5+7n^3+1}$ for sufficiently large n and

$$\sum_{l_1=1}^{\infty} \frac{D(M)l_1^2}{l_1^5+7l_1^3+1} < \infty, \qquad \sum_{l_2=1}^{\infty} \frac{1}{l_2^5} \sum_{l_1=l_2}^{\infty} \frac{D(M)l_1^2}{l_1^5+7l_1^3+1} < \infty,$$

and

$$\sum_{l_3=1}^{\infty} \frac{1}{\sqrt{l_2+3}^3} \sum_{l_2=l_3}^{\infty} \frac{1}{l_2^5} \sum_{l_1=l_2}^{\infty} \frac{D(M)l_1^2}{l_1^5+7l_1^3+1} < \infty.$$

Moreover, $f(n,0,0) = 0$, $n \in \mathbb{N}$. This means that the assumptions of Theorem 3 are satisfied. Hence, there exist uncountably many solutions to (16) in any interval $[M/2,2M]$ for any $M > 0$.

Example 3 Let us consider a nonlinear neutral delay difference equation of the form

$$\Delta\bigl(\bigl(\Delta\bigl(\cdots\bigl(\Delta\bigl(\Delta(x_n + p_n x_{n-\tau})\bigr)^{\gamma_1}\bigr)^{\gamma_2}\cdots\bigr)^{\gamma_{m-2}}\bigr)^{\gamma_{m-1}}\bigr) = \frac{x_{n-\tau_1}^6}{6^n}, \tag{17}$$

where $m \in \mathbb{N}$, $\tau \in \mathbb{N}$, $\tau_1 \in \mathbb{Z}$, and $\gamma_1, \ldots, \gamma_{m-1}$ are the ratios of odd positive integers. Moreover, $(p_n)_{n\in\mathbb{N}}$ is any sequence of real numbers such that $\sup_{n\in\mathbb{N}} |p_n| < 1/4$, $r_n^1 = \cdots = r_n^{m-1} = 1$, and $f(n,u) = \frac{u^6}{6^n}$ for any $n \in \mathbb{N}$ and $u \in \mathbb{R}$.

In an analogous way to Example 1, we have to check only assumption (H_4) of Theorem 3. We have that $f(n,\cdot) \in C^1(\mathbb{R})$ for any $n \in \mathbb{N}$, and it is easy to calculate that $P(n,M) = \frac{192M^5}{6^n}$ and

$$\sum_{l_m=1}^{\infty} \sum_{l_{m-1}=l_m}^{\infty} \cdots \sum_{l_2=l_3}^{\infty} \sum_{l_1=l_2}^{\infty} \frac{192M^5}{6^{l_1}} = 32M^5 \left(\frac{6}{5}\right)^m < \infty.$$

Hence, there exist uncountably many solutions to (17) in any interval $[M/2, 2M]$ for any $M > 0$.

Competing interests
The author declares that she has no competing interests.

Author's contributions
The author solely contributed in this article.

Acknowledgements
The author wishes to express her thanks to the referee for insightful remarks improving quality of the paper. The research was supported by Lodz University of Technology.

Received: 5 May 2016 Accepted: 20 July 2016 Published online: 27 July 2016

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