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Solvability for a fractional p-Laplacian multipoint boundary value problem at resonance on infinite interval

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Abstract

In this paper, we study the multipoint boundary value problem for a fractional p-Laplacian equation at resonance on infinite interval and establish the existence result of solutions by using extension of Mawhin's continuation theorem. Our paper enriches some known existing articles. In order to illustrate our main result, we give an example.

MSC: 34A08; 34B15

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1 Introduction

Many boundary value problems (BVPs) on infinite interval arise naturally in the study of radially symmetric solutions of nonlinear elliptic equations (see [1]). There are numerous physical models giving us motivations to investigate BVPs on infinite interval, such as the following two important examples.

The first model is for determining the electric potential in an isolated atom derived independently by Thomas and Fermi:

$$\begin{cases} u''(t) = t^{-1/2} u^{3/2}, \\ u(0) = 1, \qquad u(\infty) = 0. \end{cases}$$

Another famous model is the well-known Blasius boundary layer equation that shows flow past a semiinfinite flat plate in hydromechanics:

$$\begin{cases} f'''(\eta) + \frac{1}{2} f(\eta) f''(\eta) = 0, & \eta \in [0, +\infty), \\ f(0) = f'(0) = 0, \qquad f'(\infty) = 1. \end{cases}$$

Although the Blasius equation is simple, it can clearly reveal the essence of the problems, which is convenient for people to carry out theoretic analysis and mathematical research for boundary layer problem. So, BVPs on infinite interval have important significance and have been received much attention (see [1]). There are a large number of papers discussing

the existence of solutions for both integral-order and fractional-order differential BVPs on infinite interval by using the techniques of nonlinear analysis such as variational method (see [2]), fixed-point theorems (see [3–8]), upper and lower solutions method (see [9, 10]), fixed-point index theory (see [11, 12]), coincidence degree theory (see [13–15]), *etc.*

Jiang [13] proved the existence of solutions for the p-Laplacian boundary value problem

$$\begin{cases} (\phi_p(u'(t)))' + f(t, u, u') = 0, & 0 < t < +\infty, \\ u(0) = 0, & \phi_p(u'(\infty)) = \sum_{i=1}^n \alpha_i \phi_p(u'(\xi_i)), \end{cases}$$

where $\phi_p(s) = |s|^{p-2}s$, $\phi_p(0) = 0$, $p > 1$. The main method of the paper was the coincidence degree theory.

Su and Zhang [6] investigated the existence of unbounded solutions of boundary value problem

$$\begin{cases} D_{0+}^\alpha u(t) = f(t, u(t), D_{0+}^{\alpha-1}u(t)), & t \in J := [0, \infty), \\ u(0) = 0, & D_{0+}^{\alpha-1}u(+\infty) = u_\infty, & u_\infty \in \mathbb{R}, \end{cases}$$

where D_{0+}^α and $D_{0+}^{\alpha-1}$ are the standard Riemann-Liouville fractional derivatives of order $1 < \alpha \le 2$. The main result of this paper was obtained by using Schauder's fixed point theorem.

Zhou, Kou, and Xie (see [15]) studied the existence of solutions for the following multi-point boundary value problem:

$$\begin{cases} D_{0+}^\alpha x(t) = f(t, x(t), D_{0+}^{\alpha-1}x(t)), & t \in (0, \infty), \\ x(0) = 0, & \lim_{t \to +\infty} D_{0+}^{\alpha-1}x(t) = \beta x(\eta), \end{cases}$$

where D_{0+}^α is the standard Riemann-Liouville fractional derivative of order $1 < \alpha \le 2$, $\eta > 0$. The analysis of this paper relied on the coincidence degree of Mawhin.

Motivated by the results mentioned, in this paper, we use the extension of Mawhin's continuation theorem (see [16]) to discuss the existence of solutions for the following multipoint boundary value problem of fractional p-Laplacian equation at resonance:

$$\begin{cases} (\phi_p(D_{0+}^\alpha x(t)))' + f(t, x(t), D_{0+}^{\alpha-1}x(t), D_{0+}^\alpha x(t)) = 0, & 0 < t < +\infty, \\ x(0) = x'(0) = 0, & \phi_p(D_{0+}^\alpha x(+\infty)) = \sum_{i=1}^n \alpha_i \phi_p(D_{0+}^\alpha x(\xi_i)), \end{cases} \tag{1.1}$$

where $1 < \alpha \le 2$, D_{0+}^α is the standard Riemann-Liouville fractional derivative, $0 < \xi_1 < \xi_2 < \cdots < \xi_n < +\infty$, $\alpha_i > 0$, $\sum_{i=1}^n \alpha_i = 1$, ϕ_p is reversible, and by ϕ_q we denote the inverse operator of ϕ_p, where $1/p + 1/q = 1$.

As we know, fractional differential equations have been applied in various fields (see [17, 18]). So, it is meaningful to discuss the boundary value problems of fractional differential equations on infinite interval.

Throughout this paper, we suppose that the following hypothesis is satisfied:

(A_1) $f \in C[0, +\infty) \times \mathbb{R}^3 \to \mathbb{R}$ is an L^1-Caratheodory function, that is, f is a Caratheodory function, and for any $r > 0$, there exists a nonnegative function $g_r(t) \in L^1[0, +\infty)$ such

that

$$\left| f(t,u,v,w) \right| \le g_r(t), \quad \text{a.e. } t \in [0,+\infty), u,v,w \in \mathbb{R}, \|u\| \le r, \|v\| \le r, \|w\| \le r.$$

2 Preliminaries

In this section, we introduce some definitions and lemmas.

Definition 2.1 (See [19, 20]) The Rieman-Liouville fractional integral of order $\alpha > 0$ for a function $u : (0,+\infty) \to \mathbb{R}$ is defined as

$$I_{0+}^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s)\,ds,$$

where $\Gamma(\alpha)$ is the gamma function, provided that the right-hand side is pointwise defined on $(0,+\infty)$.

Definition 2.2 (See [19, 20]) The Riemann-Liouville functional derivative of order $\alpha > 0$ for a function $u : (0,+\infty) \to \mathbb{R}$ is defined as

$$D_{0+}^{\alpha} u(t) = \frac{d^n}{dt^n} I_{0+}^{n-\alpha} u(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\alpha-1} u(s)\,ds,$$

where $n = [\alpha] + 1$, provided that the right-hand side is pointwise defined on $(0,+\infty)$.

Lemma 2.1 (See [8, 15]) *Assume that $u \in C[0,\infty) \cap L(0,\infty)$ with a fractional derivative of order $\alpha > 0$ that belongs to $C[0,\infty) \cap L(0,\infty)$. Then*

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}$$

for some $c_i \in R$, $i = 1, 2, \ldots, n$, $n = [\alpha] + 1$.

Definition 2.3 (See [13, 16]) Let X and Y be two Banach spaces with norms $\| \cdot \|_X$ and $\| \cdot \|_Y$, respectively. A continuous operator $M : X \cap \operatorname{dom} M \to Y$ is said to be quasi-linear if

   (a) $\operatorname{Im} M := M(X \cap \operatorname{dom} M)$ is a closed subset of Y, and
   (b) $\operatorname{Ker} M := \{ x \in X \cap \operatorname{dom} M : Mx = 0 \}$ is linearly homeomorphic to $\mathbb{R}^n$, $n < \infty$.

Take $X_1 = \operatorname{Ker} M$ and let X_2 be the complement space of X_1 in X, so that, $X = X_1 \oplus X_2$. On the other hand, suppose that Y_1 is a subspace of Y and Y_2 is the complement space of Y_1 in Y, so that $Y = Y_1 \oplus Y_2$. Let $P : X \to X_1$ and $Q : Y \to Y_1$ be two projectors, and $\Omega \subset X$ be an open bounded set with the origin $\theta \in \Omega$. Throughout we use θ to denote the origin in a linear space (see [13, 16]).

Definition 2.4 (See [13, 16]) Suppose that $N_\lambda : \bar{\Omega} \to Y$, $\lambda \in [0,1]$, is a continuous operator. Denote N_1 by N. Let $\sum_\lambda = \{ x \in \bar{\Omega} : Mx = N_\lambda x \}$. N_λ is said to be M-compact in $\bar{\Omega}$ if there is a vector subspace Y_1 of Y with $\dim Y_1 = \dim X_1$ and a continuous and compact operator $R : \bar{\Omega} \times [0,1] \to X_2$ such that, for $\lambda \in [0,1]$,

(a_1) $(I-Q)N_\lambda(\bar{\Omega}) \subset \operatorname{Im} M \subset (I-Q)Y$;
(a_2) $QN_\lambda x = \theta$, $\lambda \in (0,1) \Leftrightarrow QNx = \theta$;

(a_3) $R(\cdot, 0)$ is the zero operator, and $R(\cdot, \lambda)|_{\sum_\lambda} = (I - P)|_{\sum_\lambda}$;

(a_4) $M[P + R(\cdot, \lambda)] = (I - Q)N_\lambda$.

Lemma 2.2 (Extension of Mawhin's continuation theorem) *Let X and Y be two Banach spaces with the norms $\| \cdot \|_X$ and $\| \cdot \|_Y$ respectively, and $\Omega \subset X$ be an open bounded nonempty set. Suppose that*

$$M : X \cap \operatorname{dom} M \to Y$$

is a quasi-linear operator and

$$N_\lambda : \bar{\Omega} \to Y, \quad \lambda \in [0,1],$$

is M-compact on $\bar{\Omega}$. In addition, let the following conditions hold:

(B_1) $Mx \neq N_\lambda x$, $\forall(x, \lambda) \in \partial\Omega \times (0, 1)$,

(B_2) $QNx \neq 0$, $\forall x \in \operatorname{Ker} M \cap \partial\Omega$,

(B_3) $\deg(JQN, \operatorname{Ker} M \cap \Omega, 0) \neq 0$.

Then the abstract equation $Mx = Nx$ has at least one solution in $\operatorname{dom} M \cap \bar{\Omega}$, where $N = N_1$, $Q : Y \to \operatorname{Im} Q$ is a projector, $J : \operatorname{Im} Q \to \operatorname{Ker} M$ is a homeomorphism with $J(\theta) = \theta$ (see [16]).

3 Main result

In this section, we give a theorem on the existence of solutions for BVP (1.1). Let

$$X = \left\{ x \middle| x, D_{0^+}^\alpha x \in C[0, +\infty), \sup_{t \in [0, +\infty)} \frac{|x(t)|}{1 + t^\alpha} < +\infty, \right.$$

$$\left. \sup_{t \in [0, +\infty)} \frac{|D_{0^+}^{\alpha-1} x(t)|}{1 + t^\alpha} < +\infty, \sup_{t \in [0, +\infty)} \left| D_{0^+}^\alpha x(t) \right| < +\infty \right\},$$

$$Y = L^1[0, +\infty),$$

with norms

$$\|x\|_X = \max\left\{ \|x\|_0, \left\| D_{0^+}^{\alpha-1} x \right\|_0, \left\| D_{0^+}^\alpha x \right\|_\infty \right\}, \qquad \|y\|_Y = \|y\|_1,$$

where $\|y\|_1 = \int_0^{+\infty} |y(t)| \, dt$, $\|D_{0^+}^\alpha x\|_\infty = \sup_{t \in [0, +\infty)} |D_{0^+}^\alpha x(t)|$, $\|x\|_0 = \|\frac{x(t)}{1+t^\alpha}\|_\infty$. Clearly, $(X, \| \cdot \|_X)$ and $(Y, \| \cdot \|_Y)$ are Banach spaces.

Define the operators $M : \operatorname{dom} M \subset X \to Y$ and $N_\lambda : X \to Y$ as follows:

$$Mx = \left(\phi_p(D_{0^+}^\alpha x) \right)', \qquad N_\lambda x(t) = -\lambda f\left(t, x(t), D_{0^+}^{\alpha-1} x(t), D_{0^+}^\alpha x(t)\right), \quad \lambda \in [0,1], x \in X,$$

where

$$\operatorname{dom} M = \left\{ x \in X \middle| \phi_p(D_{0^+}^\alpha x) \in AC[0, +\infty), x(0) = x'(0) = 0, \right.$$

$$\left. \phi_p\left(D_{0^+}^\alpha x(+\infty)\right) = \sum_{i=1}^n \alpha_i \phi_p\left(D_{0^+}^\alpha x(\xi_i)\right) \right\}.$$

Then the BVP (1.1) is equivalent to $Mx = Nx$, $x \in \operatorname{dom} M$.

Lemma 3.1 *For M defined as before, we have*

$$\operatorname{Ker} M = \left\{ x \in \operatorname{dom} M | x(t) = ct^{\alpha}, \forall t \in [0, +\infty), c \in \mathbb{R} \right\}, \tag{3.1}$$

$$\operatorname{Im} M = \left\{ y \in Y \,\Big|\, \sum_{i=1}^{n} \alpha_i \int_{\xi_i}^{+\infty} y(s)\,ds = 0 \right\}, \tag{3.2}$$

and M is a quasi-linear operator.

Proof For $x \in \operatorname{Ker} M$, $Mx = 0$, that is, $(\phi_p(D_{0+}^{\alpha}x))' = 0$, by $\phi_p(D_{0+}^{\alpha}x(+\infty)) = \sum_{i=1}^{n} \alpha_i \times \phi_p(D_{0+}^{\alpha}x(\xi_i))$ we easily get

$$\phi_p\big(D_{0+}^{\alpha}x(t)\big) = \phi_p\big(D_{0+}^{\alpha}x(+\infty)\big) = \sum_{i=1}^{n} \alpha_i \phi_p\big(D_{0+}^{\alpha}x(\xi_i)\big).$$

Based on Lemma 2.1, since $x(0) = x'(0) = 0$, we have

$$x(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \frac{\phi_q(\sum_{i=1}^{n} \alpha_i \phi_p(D_{0+}^{\alpha}x(\xi_i)))}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\,ds$$

$$= \frac{\phi_q(\sum_{i=1}^{n} \alpha_i \phi_p(D_{0+}^{\alpha}x(\xi_i)))}{\Gamma(\alpha+1)} t^{\alpha}.$$

Conversely, if $x = ct^{\alpha}$, then $Mx = 0$ by (3.1). If $y \in \operatorname{Im} M$, then there exists a function $x \in \operatorname{dom} M$ such that $y(t) = (\phi_p(D_{0+}^{\alpha}x(t)))'$. Then

$$\phi_p\big(D_{0+}^{\alpha}x(t)\big) = \phi_p\big(D_{0+}^{\alpha}x(+\infty)\big) - \int_t^{+\infty} y(s)\,ds$$

$$= \sum_{i=1}^{n} \alpha_i \phi_p\big(D_{0+}^{\alpha}x(\xi_i)\big) - \int_t^{+\infty} y(s)\,ds$$

$$= \sum_{i=1}^{n} \alpha_i \left[\phi_p\big(D_{0+}^{\alpha}x(+\infty)\big) - \int_{\xi_i}^{+\infty} y(s)\,ds \right] - \int_t^{+\infty} y(s)\,ds,$$

that is,

$$\sum_{i=1}^{n} \alpha_i \int_{\xi_i}^{+\infty} y(s)\,ds = 0. \tag{3.3}$$

On the other hand, if $y \in Y$ satisfies (3.3), then take

$$x(t) = I_{0+}^{\alpha} \phi_q\left(-\int_t^{+\infty} y(s)\,ds \right).$$

Then $x \in \operatorname{dom} M$ and $Mx = y$. Hence, (3.2) holds. Clearly, $\dim \operatorname{Ker} M = 1 < +\infty$ and $\operatorname{Im} M := M(\operatorname{dom} M \cap X)$ is a closed subset of Y. Therefore, we get that M is a quasi-linear operator. □

Lemma 3.2 (See [1]) *Let $V \subset C_{\infty} = \{ u \in C[0, \infty), \lim_{t \to +\infty} u(t) \text{ exists} \}$. Then V is relatively compact if the following conditions hold:*

(b_1) *all functions from V are uniformly bounded,*

(b_2) *all functions from V are equicontinuous on any compact interval of $[0, +\infty)$,*

(b_3) *all functions from V are equiconvergent at infinity.*

Remark 3.1 By Lemma 3.2, any set $V \subset X$ (defined as before) is relatively compact, we only need to show that the sets

$$V_1 = \left\{ \frac{x(t)}{1 + t^\alpha} \Big| x \in V \right\}, \qquad V_2 = \left\{ \frac{D_{0+}^{\alpha-1} x(t)}{1 + t^\alpha} \Big| x \in V \right\}, \qquad V_3 = \left\{ D_{0+}^\alpha x(t) | x \in V \right\}$$

are uniformly bounded in X, equicontinuous on any compact intervals of $[0, +\infty)$, and equiconvergent at infinity.

Lemma 3.3 *Let $\Omega \subset X$ be nonempty, open, and bounded. Then N_λ is M-compact on $\bar{\Omega}$.*

Proof Define two projectors $P : X \to X_1$ and $Q : Y \to Y_1$ by

$$Px(t) = \frac{D_{0+}^\alpha x(+\infty)}{\Gamma(\alpha + 1)} t^\alpha, \qquad Qy(t) = \frac{\sum_{i=1}^n \alpha_i \int_{\xi_i}^{+\infty} y(s) \, ds}{\sum_{i=1}^n \alpha_i e^{-\xi_i}} e^{-t}, \quad \forall t \in [0, +\infty),$$

where $X_1 = \operatorname{Ker} M$ and $Y_1 = \operatorname{Im} Q$.

First, we show that (a_1) and (a_2) in Definition 2.4 hold. In fact, by the definition of P we get $\operatorname{Im} P = \operatorname{Ker} M$ and $P^2 x(t) = Px(t)$. For $x \in X$, since $x = (x - Px) + Px$ and $\operatorname{Im} P = \operatorname{Ker} M$, we have $(x - Px) \in \operatorname{Ker} P$, $Px \in \operatorname{Ker} M$. We easily see that $\operatorname{Ker} M \cap \operatorname{Ker} P = \{0\}$. So, $X = \operatorname{Ker} M \oplus \operatorname{Ker} P = X_1 \oplus X_2$. Similarly, by the definition of Q, we can obtain

$$Q^2 y = Q(Qy) = \frac{\sum_{i=1}^n \alpha_i \int_{\xi_i}^{+\infty} Qy \, ds}{\sum_{i=1}^n \alpha_i e^{-\xi_i}} e^{-t} = Qy,$$

and $\operatorname{Ker} Q = \operatorname{Im} M$. For $y \in Y$, since $y = (y - Qy) + Qy$ and $\operatorname{Ker} Q = \operatorname{Im} M$, we have $(y - Qy) \in \operatorname{Ker} Q$, $Qy \in \operatorname{Im} M$. Clearly, $\operatorname{Im} Q \cap \operatorname{Im} M = \{0\}$. So, we have $Y = \operatorname{Im} Q \oplus \operatorname{Im} M = Y_1 \oplus Y_2$ and $\dim X_1 = \dim \operatorname{Ker} M = \dim \operatorname{Im} Q = \dim Y_1$, where $X_2 = \operatorname{Ker} P$, $Y_2 = \operatorname{Im} M$. Let $\Omega \subset X$ be open bounded, and let $\theta \in \Omega$. On the one hand, for $x \in \bar{\Omega}$, since $Q(I - Q)$ is a zero operator, we have $Q[(I - Q)N_\lambda x] = 0$; thus, $(I - Q)N_\lambda x \in \operatorname{Ker} Q = \operatorname{Im} M$, that is, $(I - Q)N_\lambda(\bar{\Omega}) \subset \operatorname{Im} M$. On the other hand, for $y \in \operatorname{Im} M$, since $y = (y - Qy) + Qy$ and $\operatorname{Ker} Q = \operatorname{Im} M$, we have $y \in (I - Q)Y$, that is, $\operatorname{Im} M \subset (I - Q)Y$. Clearly, $QN_\lambda x = 0$, $\lambda \in (0, 1) \Leftrightarrow QNx = 0$. So, conditions ($a_1$) and ($a_2$) of Definition 2.4 hold.

Second, we give the definition of operator R and aim to show that R is compact. For notational convenience, let

$$l(t, x, \lambda) = \int_t^{+\infty} (Q - I)N_\lambda x(s) \, ds + \phi_p\left(D_{0+}^\alpha x(+\infty)\right), h(t) = \phi_q\left(l(t, x, \lambda)\right) - D_{0+}^\alpha x(+\infty).$$

Define the operator $R : \bar{\Omega} \times [0, 1] \to X_2$ by

$$R(x, \lambda)(t) = I_{0+}^\alpha h(t).$$

By (A_1) it is easy to know that $R(x, \lambda)(t)$ is continuous on $\bar{\Omega} \times [0, 1]$.

Step 1. We prove that $R(x, \lambda)(\bar{\Omega})$ is both uniformly bounded in X and equicontinuous on any compact interval of $[0, +\infty)$. In fact, since $\Omega \subset X$ is nonempty, open, and bounded, by (A_1) there exist a constant $r > 0$ and a nonnegative function $g_r(t) \in L^1[0, +\infty)$ such that

$$\|x\|_X \le r, \qquad \left| f\left(t, x(t), D_{0+}^{\alpha-1}x(t), D_{0+}^{\alpha}x(t)\right) \right| \le g_r(t), \quad \text{a.e. } t \in [0, +\infty), x \in \bar{\Omega}.$$

Since

$$
\left| \int_s^{+\infty} (Q - I)N_\lambda x \, d\tau \right|
$$

$$
\le \int_s^{+\infty} |QN_\lambda x - N_\lambda x| \, d\tau
$$

$$
\le \int_0^{+\infty} \left| f\left(\tau, x(\tau), D_{0+}^{\alpha-1}x(\tau), D_{0+}^{\alpha}x(\tau)\right) \right.
$$

$$
\left. - \frac{\sum_{i=1}^n \alpha_i \int_{\xi_i}^{+\infty} f(s, x(s), D_{0+}^{\alpha-1}x(s), D_{0+}^{\alpha}x(s)) \, ds}{\sum_{i=1}^n \alpha_i e^{-\xi_i}} e^{-\tau} \right| d\tau
$$

$$
\le \int_0^{+\infty} \left| f\left(\tau, x(\tau), D_{0+}^{\alpha-1}x(\tau), D_{0+}^{\alpha}x(\tau)\right) \right| d\tau
$$

$$
+ \left| \frac{\sum_{i=1}^n \alpha_i \int_{\xi_i}^{+\infty} f(s, x(s), D_{0+}^{\alpha-1}x(s), D_{0+}^{\alpha}x(s)) \, ds}{\sum_{i=1}^n \alpha_i e^{-\xi_i}} \right|
$$

$$
\le \|g_r\|_1 \left(1 + \frac{1}{\sum_{i=1}^n \alpha_i e^{-\xi_i}}\right) =: \tilde{r}, \tag{3.4}
$$

we have

$$
\left| h(s) \right| = \left| \phi_q \left[\int_s^{+\infty} (Q - I)N_\lambda x(\tau) \, d\tau + \phi_p\left(D_{0+}^{\alpha}x(+\infty)\right) \right] - D_{0+}^{\alpha}x(+\infty) \right|
$$

$$
\le \phi_q\left[\tilde{r} + \phi_p(r)\right] + r =: m.
$$

Therefore, for any $x \in \bar{\Omega}$, we have

$$
\left\| R(x, \lambda)(t) \right\|_0 = \sup_{t \in [0, +\infty)} \left| \frac{R(x, \lambda)(t)}{1 + t^\alpha} \right|
$$

$$
= \sup_{t \in [0, +\infty)} \frac{1}{\Gamma(\alpha)} \left| \int_0^t \frac{(t-s)^{\alpha-1}}{1 + t^\alpha} h(s) \, ds \right|
$$

$$
\le \frac{m}{\Gamma(\alpha)} \sup_{t \in [0, +\infty)} \int_0^t \frac{(t-s)^{\alpha-1}}{1 + t^\alpha} \, ds
$$

$$
\le \frac{m}{\Gamma(\alpha+1)} \le m,
$$

$$
\left\| D_{0+}^{\alpha-1} R(x, \lambda)(t) \right\|_0 = \sup_{t \in [0, +\infty)} \left| \frac{D_{0+}^{\alpha-1} R(x, \lambda)(t)}{1 + t^\alpha} \right|
$$

$$
= \sup_{t \in [0, +\infty)} \left| \int_0^t \frac{1}{1 + t^\alpha} h(s) \, ds \right|
$$

$$
\le m \sup_{t \in [0, +\infty)} \int_0^t \frac{1}{1 + t^\alpha} \, ds \le m,
$$

and

$$\left\| D_{0+}^{\alpha} R(x,\lambda)(t) \right\|_{\infty} = \sup_{t \in [0,+\infty)} \left| D_{0+}^{\alpha} R(x,\lambda)(t) \right| = \sup_{t \in [0,+\infty)} \left| h(t) \right| \leq m.$$

That is, $R(x,\lambda)(\bar{\Omega})$ is uniformly bounded in X. Next, we show that $R(x,\lambda)(\bar{\Omega})$ is equicontinuous on any compact interval of $[0,+\infty)$. In fact, for any $K > 0$, $t_1, t_2 \in [0,K]$, $x \in \bar{\Omega}$, $\lambda \in [0,1]$, we have

$$
\begin{aligned}
&\left| \frac{R(x,\lambda)(t_1)}{1+t_1^{\alpha}} - \frac{R(x,\lambda)(t_2)}{1+t_2^{\alpha}} \right| \\
&= \frac{1}{\Gamma(\alpha)} \left[\left| \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha}} h(s)\,ds - \int_0^{t_2} \frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha}} h(s)\,ds \right| \right] \\
&\leq \frac{1}{\Gamma(\alpha)} \left[\left| \int_0^{t_1} \left(\frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha}} - \frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha}} \right) h(s)\,ds \right| + \left| \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha}} h(s)\,ds \right| \right] \\
&\leq \frac{m}{\Gamma(\alpha)} \left[\int_0^{t_1} \left| \frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha}} - \frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha}} \right| ds + \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha}}\,ds \right] \\
&\leq \frac{m}{\Gamma(\alpha)} \left[\int_0^{t_1} \left| \frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha}} - \frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha}} \right| ds + \frac{1}{\alpha} \frac{(t_2-t_1)^{\alpha}}{1+t_2^{\alpha}} \right] \to 0 \quad \text{as } t_1 \to t_2.
\end{aligned}
$$

So, $\{\frac{R(x,\lambda)(t)}{1+t^{\alpha}}, x \in \bar{\Omega}\}$ is equicontinuous on $[0,K]$. Similarly, we obtain that $\{\frac{D_{0+}^{\alpha-1} R(x,\lambda)(t)}{1+t^{\alpha}}, x \in \bar{\Omega}\}$ is equicontinuous on $[0,K]$. In addition, since

$$
\begin{aligned}
\left| D_{0+}^{\alpha} R(x,\lambda)(t_1) - D_{0+}^{\alpha} R(x,\lambda)(t_2) \right| &= \left| h(t_1) - h(t_2) \right| \\
&= \left| \phi_q\big(l(t_1,x,\lambda) \big) - \phi_q\big(l(t_2,x,\lambda) \big) \right|
\end{aligned}
$$

and

$$
\begin{aligned}
\left| l(t,x,\lambda) \right| &= \left| \int_t^{+\infty} (Q-I) N_{\lambda} x(s)\,ds + \phi_p\big(D_{0+}^{\alpha} x(+\infty) \big) \right| \\
&\leq \tilde{r} + \phi_p(r) \quad \big(t \in [0,K], x \in \bar{\Omega} \big),
\end{aligned}
$$

we have

$$
\left| l(t_1,x,\lambda) - l(t_2,x,\lambda) \right| = \left| \int_{t_1}^{t_2} (Q-I) N_{\lambda} x(s)\,ds \right| \leq \int_{t_1}^{t_2} \left(g_r(s) + \frac{\|g_r\|_{L^1}}{\sum_{i=1}^n \alpha_i e^{-\xi_i}} e^{-s} \right) ds.
$$

By the absolute continuity of the integral, $\{l(t,x,\lambda), x \in \bar{\Omega}\}$ is equicontinuous on $[0,K]$, which combined with the uniform continuity of $\phi_q(x)$ on $[-\tilde{r}-\phi_p(r), \tilde{r}+\phi_p(r)]$, gives that $\{D_{0+}^{\alpha} R(x,\lambda)(t), x \in \bar{\Omega}\}$ is equicontinuous on $[0,K]$.

Step 2. We establish the fact that $R(x,\lambda)(\bar{\Omega})$ is equiconvergent at infinity. In fact, for any $x \in \bar{\Omega}$, by (3.4) we have

$$
\lim_{s \to +\infty} \int_s^{+\infty} (Q-I) N_{\lambda} x(\tau)\,d\tau = 0.
$$

Since $\phi_q(x)$ is uniformly continuous on $[-\tilde{r} - \phi_p(r), \tilde{r} + \phi_p(r)]$, for any $\varepsilon > 0$, there exists a constant $L_1 > 0$ such that, for $s \geq L_1$, we have

$$
\left| h(s) \right| = \left| \phi_q \left[\int_s^{+\infty} (Q-I)N_\lambda x(\tau)\, d\tau + \phi_p\big(D_{0+}^\alpha x(+\infty)\big) \right] - \phi_q\big[\phi_p\big(D_{0+}^\alpha x(+\infty)\big) \big] \right|
$$
$$
< \varepsilon, \quad \forall x \in \bar{\Omega}.
$$

Therefore, $|h(s)| < \varepsilon$ for $s \geq L_1$ and $|h(s)| \leq m$ for $s < L_1$. On the other hand, since $\lim_{t \to +\infty} \frac{t^{\alpha-1}}{1+t^\alpha} = 0$ and $\lim_{t \to +\infty} \frac{1}{1+t^\alpha} = 0$ for the above $\varepsilon > 0$, there exists a constant $L > L_1 > 0$ such that, for any $t_1, t_2 \geq L$ and $0 \leq s \leq L_1$, we have

$$
\left| \frac{(t_1-s)^{\alpha-1}}{1+t_1^\alpha} - \frac{(t_2-s)^{\alpha-1}}{1+t_2^\alpha} \right| \leq \frac{t_1^{\alpha-1}}{1+t_1^\alpha} + \frac{t_2^{\alpha-1}}{1+t_2^\alpha} < \varepsilon
$$

and

$$
\left| \frac{1}{1+t_1^\alpha} - \frac{1}{1+t_2^\alpha} \right| \leq \frac{1}{1+t_1^\alpha} + \frac{1}{1+t_2^\alpha} < \varepsilon.
$$

Then, for $t_1, t_2 \geq L$, from the above we obtain

$$
\left| \frac{R(x,\lambda)(t_1)}{1+t_1^\alpha} - \frac{R(x,\lambda)(t_2)}{1+t_2^\alpha} \right|
$$
$$
= \frac{1}{\Gamma(\alpha)} \left| \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{1+t_1^\alpha} h(s)\, ds - \int_0^{t_2} \frac{(t_2-s)^{\alpha-1}}{1+t_2^\alpha} h(s)\, ds \right|
$$
$$
\leq \frac{1}{\Gamma(\alpha)} \int_0^{L_1} \left| \frac{(t_1-s)^{\alpha-1}}{1+t_1^\alpha} - \frac{(t_2-s)^{\alpha-1}}{1+t_2^\alpha} \right| \left| h(s) \right| ds + \frac{1}{\Gamma(\alpha)} \int_{L_1}^{t_1} \frac{(t_1-s)^{\alpha-1}}{1+t_1^\alpha} \left| h(s) \right| ds
$$
$$
+ \frac{1}{\Gamma(\alpha)} \int_{L_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{1+t_2^\alpha} \left| h(s) \right| ds
$$
$$
\leq \frac{m}{\Gamma(\alpha)} L_1 \varepsilon + \frac{2m}{\Gamma(\alpha+1)} \varepsilon.
$$

Similarly, we get

$$
\left| \frac{D_{0+}^{\alpha-1} R(x,\lambda)(t_1)}{1+t_1^\alpha} - \frac{D_{0+}^{\alpha-1} R(x,\lambda)(t_2)}{1+t_2^\alpha} \right| \leq mL_1\varepsilon + 2\varepsilon
$$

and, in addition,

$$
\left| D_{0+}^\alpha R(x,\lambda)(t_1) - D_{0+}^\alpha R(x,\lambda)(t_2) \right| = \left| h(t_1) - h(t_2) \right|
$$
$$
\leq \left| h(t_1) \right| + \left| h(t_2) \right|
$$
$$
< 2\varepsilon.
$$

So, $R(x,\lambda)(\bar{\Omega})$ is equiconvergent at infinity. By Lemma 3.2, $R : \bar{\Omega} \times [0,1] \to X_2$ is completely continuous.

Finally, we prove that the (a_3) and (a_4) in Definition 2.4 hold. Let $x \in \sum_\lambda = \{x \in \bar{\Omega} | Mx = N_\lambda x\}$. Then $(\phi_p(D_{0+}^\alpha x(t)))' = N_\lambda x(t) \in \operatorname{Im} M = \operatorname{Ker} Q$ and

$$
\begin{aligned}
R(x, \lambda)(t) &= I_{0+}^\alpha h(t) \\
&= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[\phi_q \left(\int_s^{+\infty} (Q-I)N_\lambda x(\tau)\, d\tau + \phi_p\big(D_{0+}^\alpha x(+\infty)\big) \right) \right. \\
&\quad \left. - D_{0+}^\alpha x(+\infty) \right] ds \\
&= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \phi_q \left(\int_s^{+\infty} -N_\lambda x(\tau)\, d\tau + \phi_p\big(D_{0+}^\alpha x(+\infty)\big) \right) ds - (Px)(t) \\
&= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \phi_q \left(\int_s^{+\infty} -\big(\phi_p\big(D_{0+}^\alpha x(\tau)\big)\big)'\, d\tau + \phi_p\big(D_{0+}^\alpha x(+\infty)\big) \right) ds \\
&\quad - (Px)(t) \\
&= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} D_{0+}^\alpha x(s)\, ds - (Px)(t),
\end{aligned}
$$

which, combined with boundary conditions, yields that

$$
R(x, \lambda)(t) = x(t) - (Px)(t) = \big[(I-P)x\big](t).
$$

It is clear that $R(x, 0)(t)$ is a zero operator, and for any $x \in \bar{\Omega}$, we have

$$
\begin{aligned}
M\big[Px + R(x, \lambda)\big](t) &= M\left[\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \phi_q \left(\int_s^{+\infty} (Q-I)N_\lambda x(\tau)\, d\tau + \phi_p\big(D_{0+}^\alpha x(+\infty)\big) \right) ds \right] \\
&= \left[\int_t^{+\infty} (Q-I)N_\lambda x(s)\, ds + \phi_p\big(D_{0+}^\alpha x(+\infty)\big) \right]' \\
&= (I-Q)N_\lambda x(t).
\end{aligned}
$$

By the above, N_λ is M-compact on $\bar{\Omega}$. □

Theorem 3.1 *Suppose that* (A_1) *and the following conditions hold*:

(A_2) *there exist nonnegative functions* $a(t), b(t), c(t), d(t) \in Y$ *such that*

$$
\begin{aligned}
\big|f(t, u, v, w)\big| &\le a(t) + b(t) \frac{|u|^{p-1}}{(1+t^\alpha)^{p-1}} + c(t) \frac{|v|^{p-1}}{(1+t^\alpha)^{p-1}} \\
&\quad + d(t)|w|^{p-1}, \quad \forall t \in [0, +\infty), (u, v, w) \in \mathbb{R}^3;
\end{aligned}
$$

(A_3) *there exists a positive constant* B *such that one of the following inequalities hold*:

$$
wf(t, u, v, w) > 0, \quad \forall t \in [0, +\infty), u, v \in \mathbb{R}, |w| > B, \tag{3.5}
$$

$$
wf(t, u, v, w) < 0, \quad \forall t \in [0, +\infty), u, v \in \mathbb{R}, |w| > B. \tag{3.6}
$$

Then BVP (1.1) *has at least one solution in* X, *provided that* $\|b\|_1 + \|c\|_1 + \|d\|_1 < 1$.

Before we prove Theorem 3.1, we show two lemmas.

Lemma 3.4 *Let* $\Omega_1 = \{x \in \operatorname{dom} M \setminus \operatorname{Ker} M | Mx = N_\lambda x, \lambda \in (0,1)\}$. *Suppose that* (A_2) *and* (A_3) *hold. Then* Ω_1 *is bounded in* X.

Proof Let $x \in \Omega_1$. Then $Mx = N_\lambda x$ and thus $QN_\lambda x = 0$, that is,

$$\sum_{i=1}^{n} \alpha_i \int_{\xi_i}^{+\infty} f\left(s, x(s), D_{0+}^{\alpha-1}x(s), D_{0+}^{\alpha}x(s)\right) ds = 0.$$

By Lemma 2.1 and the boundary conditions we have

$$x(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + I_{0+}^{\alpha} D_{0+}^{\alpha} x(t) = I_{0+}^{\alpha} D_{0+}^{\alpha} x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} D_{0+}^{\alpha} x(s) \, ds.$$

Thus,

$$\|x\|_0 \leq \left\| D_{0+}^{\alpha} x \right\|_\infty, \qquad \left\| D_{0+}^{\alpha-1} x \right\|_0 \leq \left\| D_{0+}^{\alpha} x \right\|_\infty.$$

By (A_3) there exists a constant $s_0 \in [0, +\infty)$ such that $|D_{0+}^{\alpha} x(s_0)| \leq B$, which, combined with $Mx = N_\lambda x$ and (A_2), gives

$$
\begin{aligned}
&\left| \phi_p \left(D_{0+}^{\alpha} x(t) \right) \right| \\
&= \left| \phi_p \left(D_{0+}^{\alpha} x(s_0) \right) - \int_{s_0}^{t} \lambda f\left(s, x(s), D_{0+}^{\alpha-1}x(s), D_{0+}^{\alpha}x(s)\right) ds \right| \\
&\leq \left| \phi_p \left(D_{0+}^{\alpha} x(s_0) \right) \right| + \left| \int_{s_0}^{t} \lambda f\left(s, x(s), D_{0+}^{\alpha-1}x(s), D_{0+}^{\alpha}x(s)\right) ds \right| \\
&\leq \phi_p(B) + \int_0^{+\infty} \left| f\left(s, x(s), D_{0+}^{\alpha-1}x(s), D_{0+}^{\alpha}x(s)\right) \right| ds \\
&\leq \phi_p(B) + \int_0^{+\infty} \left[a(s) + b(s)\frac{|x|^{p-1}}{(1+s^\alpha)^{p-1}} + c(s)\frac{|D_{0+}^{\alpha-1}x|^{p-1}}{(1+s^\alpha)^{p-1}} + d(s)\left|D_{0+}^{\alpha}x\right|^{p-1} \right] ds \\
&\leq \phi_p(B) + \|a\|_1 + \|b\|_1 \phi_p\left(\|x\|_0\right) + \|c\|_1 \phi_p\left(\left\|D_{0+}^{\alpha-1}x\right\|_0\right) + \|d\|_1 \phi_p\left(\left\|D_{0+}^{\alpha}x\right\|_\infty\right) \\
&\leq \phi_p(B) + \|a\|_1 + \left[\|b\|_1 + \|c\|_1 + \|d\|_1\right] \phi_p\left(\left\|D_{0+}^{\alpha}x\right\|_\infty\right).
\end{aligned}
$$

Then

$$
\begin{aligned}
\left\| \phi_p \left(D_{0+}^{\alpha} x \right) \right\|_\infty &= \phi_p\left(\left\|D_{0+}^{\alpha}x\right\|_\infty\right) \\
&\leq \phi_p(B) + \|a\|_1 + \left[\|b\|_1 + \|c\|_1 + \|d\|_1\right] \phi_p\left(\left\|D_{0+}^{\alpha}x\right\|_\infty\right).
\end{aligned}
$$

Thus,

$$\phi_p\left(\left\|D_{0+}^{\alpha}x\right\|_\infty\right) \leq \frac{\phi_p(B) + \|a\|_1}{1 - \left[\|b\|_1 + \|c\|_1 + \|d\|_1\right]} := A.$$

That is,

$$\left\|D_{0+}^{\alpha}x\right\|_\infty \leq \phi_q(A).$$

Therefore,

$$\|x\|_X = \max\{\|x\|_0, \|D_{0+}^{\alpha-1}x\|_0, \|D_{0+}^{\alpha}x\|_\infty\} \le \phi_q(A).$$

So, Ω_1 is bounded in X. $\qquad\square$

Lemma 3.5 *Let* $\Omega_2 = \{x \in \operatorname{Ker} M | Nx \in \operatorname{Im} M\}$. *Suppose that* (A_3) *holds. Then* Ω_2 *is bounded in* X.

Proof Let $x \in \Omega_2$, that is, $x = ct^\alpha$, $c \in \mathbb{R}$, $QNx = 0$, so that

$$\sum_{i=1}^{n} \alpha_i \int_{\xi_i}^{+\infty} f\big(t, ct^\alpha, c\Gamma(\alpha+1)t, c\Gamma(\alpha)\big)\, dt = 0.$$

By (A_3) we have $|c\Gamma(\alpha)| \le B$, that is, $|c| \le \frac{B}{\Gamma(\alpha)}$. Therefore,

$$
\begin{aligned}
\|x\|_X &= \max\{\|x\|_0, \|D_{0+}^{\alpha-1}x\|_0, \|D_{0+}^{\alpha}x\|_\infty\} \\
&= \max\left\{\left\|\frac{ct^\alpha}{1+t^\alpha}\right\|_\infty, \left\|\frac{c\Gamma(\alpha+1)t}{1+t^\alpha}\right\|_\infty, |c\Gamma(\alpha)|\right\} \\
&\le \left\|\frac{ct^\alpha}{1+t^\alpha}\right\|_\infty + \left\|\frac{c\Gamma(\alpha+1)t}{1+t^\alpha}\right\|_\infty + |c\Gamma(\alpha)| \le \big(1 + \Gamma(\alpha) + \Gamma(\alpha+1)\big)|c| := C.
\end{aligned}
$$

So, Ω_2 is bounded in X. $\qquad\square$

Proof of Theorem 3.1 Set $\Omega = \{x \in X | \|x\|_X < \max\{B, \phi_q(A), C\} + 1\}$. By Lemma 3.1 and Lemma 3.3 we know that M is quasi-linear and N_λ is M-compact on $\bar{\Omega}$. From Lemma 3.4 and Lemma 3.5 we obtain:

(B_1) $Mx \ne N_\lambda x$, $\forall (x, \lambda) \in \partial\Omega \times (0,1)$,
(B_2) $QNx \ne 0$, $\forall x \in \operatorname{Ker} M \cap \partial\Omega$.

Now we show (B_3) holds. Let $J : \operatorname{Im} Q \to \operatorname{Ker} M$ be the homeomorphism defined by

$$J\big(ce^{-t}\big) = ct^\alpha, \quad c \in \mathbb{R}, t \in [0, +\infty).$$

Without loss of generality, we suppose that (3.6) holds. Define the homotopic mapping

$$H(x, \lambda) = \lambda x - (1-\lambda)JQNx, \quad \forall x \in \bar{\Omega} \cap \operatorname{Ker} M, \lambda \in [0,1].$$

Then $H(x, \lambda) \ne 0$, $x \in \partial\Omega \cap \operatorname{Ker} M$, $\lambda \in [0,1]$. Indeed, for $x \in \partial\Omega \cap \operatorname{Ker} M$, we have $x = ct^\alpha$ and thus

$$H(x, \lambda) = \lambda ct^\alpha - (1-\lambda)\frac{\sum_{i=1}^{n} \alpha_i \int_{\xi_i}^{+\infty} f(s, cs^\alpha, c\Gamma(\alpha+1)s, c\Gamma(\alpha))\, ds}{\sum_{i=1}^{n} \alpha_i e^{-\xi_i}} t^\alpha.$$

Clearly, $H(x, 1) \ne 0$, $x \in \partial\Omega \cap \operatorname{Ker} M$. For $\lambda \in [0,1)$ and $x = ct^\alpha \in \partial\Omega \cap \operatorname{Ker} M$, if $H(x, \lambda) = 0$, then

$$\frac{\sum_{i=1}^{n} \alpha_i \int_{\xi_i}^{+\infty} c\Gamma(\alpha)f(s, cs^\alpha, c\Gamma(\alpha+1)s, c\Gamma(\alpha))\, ds}{\sum_{i=1}^{n} \alpha_i e^{-\xi_i}} = \frac{\lambda}{1-\lambda}c^2\Gamma(\alpha) \ge 0,$$

which contradicts (3.6). If (3.5) holds, then defining the homotopic mapping

$$H(x, \lambda) = \lambda x + (1 - \lambda)JQNx, \quad x \in \bar{\Omega} \cap \operatorname{Ker} M, \lambda \in [0, 1],$$

we also get contradiction in a similar way. Therefore, via the homotopy property of degree, we obtain

$$\deg(JQN, \Omega \cap \operatorname{Ker} M, 0) = \deg\big(H(\cdot, 0), \Omega \cap \operatorname{Ker} M, 0\big)$$
$$= \deg\big(H(\cdot, 1), \Omega \cap \operatorname{Ker} M, 0\big)$$
$$= \deg(I, \Omega \cap \operatorname{Ker} M, 0) \neq 0.$$

Applying Lemma 2.2, we conclude that (1.1) has at least one solution in $\bar{\Omega}$. □

4 Example

Example 4.1 Consider the BVP

$$\begin{cases} (\phi_{3/2}(D_{0+}^{3/2}x(t)))' + \frac{e^{-3t}}{4\sqrt{1+t^{3/2}}} \sin\sqrt{|x(t)|} + \frac{1}{2}e^{-t}\phi_{3/2}(D_{0+}^{3/2}x(t)) + \frac{1}{4}e^{-3t} = 0, \\ 0 < t < +\infty, \\ x(0) = x'(0) = 0, \qquad \phi_{3/2}(D_{0+}^{3/2}x(+\infty)) = \sum_{i=1}^{n}\alpha_i\phi_{3/2}(D_{0+}^{3/2}x(\xi_i)), \end{cases} \qquad (4.1)$$

where $0 < \xi_1 < \xi_2 < \cdots < \xi_n < +\infty$, $\alpha_i > 0$, $\sum_{i=1}^{n}\alpha_i = 1$. Let

$$a(t) = b(t) = \frac{1}{4}e^{-3t}, \qquad c(t) = 0, \quad d(t) = \frac{1}{2}e^{-t}, \qquad B = 4,$$
$$g_r(t) = \left(\frac{1}{4}e^{-3t} + \frac{1}{2}e^{-t}\right)r + \frac{1}{4}e^{-3t}.$$

We easily check (A_1)-(A_3). By Theorem 3.1, problem (4.1) has at least one solution.

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