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On Appell-type Changhee polynomials and numbers

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Abstract

In this paper, we consider the Appell-type Changhee polynomials and derive some properties of these polynomials. Furthermore, we investigate certain identities for these polynomials.

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Keywords: Changhee polynomials; Appell-type Changhee polynomials; degenerate Bernoulli polynomials; beta functions

1 Introduction

Let p be a fixed odd prime number. Throughout this paper, we denote by $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ the ring of p-adic integers, the field of p-adic numbers, and the completion of algebraic closure of $\mathbb{Q}_p$. The p-adic norm $|\cdot|_p$ is normalized as $|p|_p = \frac{1}{p}$. Let $C(\mathbb{Z}_p)$ be the space of continuous functions on $\mathbb{Z}_p$. For $f \in C(\mathbb{Z}_p)$, the fermionic p-adic integral on $\mathbb{Z}_p$ is defined by Kim to be

$$I_{-1}(f) = \int_{\mathbb{Z}_p} f(x)\,d\mu_{-1}(x) = \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)(-1)^x \tag{1}$$

(see [1–19]). For $f_1(x) = f(x+1)$, we have

$$I_{-1}(f_1) + I_{-1}(f) = 2f(0). \tag{2}$$

As is well known, the Changhee polynomials are defined by the generating function

$$\int_{\mathbb{Z}_p} (1+t)^{x+y}\,d\mu_{-1}(y) = \frac{2}{2+t}(1+t)^x = \sum_{n=0}^{\infty} \mathrm{Ch}_n(x)\frac{t^n}{n!}. \tag{3}$$

When $x = 0$, $\mathrm{Ch}_n = \mathrm{Ch}_n(0)$ are called the Changhee numbers (see [17, 18, 20]). The gamma and beta functions are defined by the following definite integrals: for $\alpha > 0$, $\beta > 0$,

$$\Gamma(\alpha) = \int_0^{\infty} e^{-t}t^{\alpha-1}\,dt \tag{4}$$

and

$$B(\alpha,\beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,dt = \int_0^\infty \frac{t^{\alpha-1}}{(1+t)^{\alpha+\beta}}\,dt \tag{5}$$

(see[20, 21]). Thus, by (4) and (5) we have

$$\Gamma(\alpha+1) = \alpha\Gamma(\alpha), \qquad B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}. \tag{6}$$

Stirling numbers of the first kind are defined by

$$\bigl(\log(1+t)\bigr)^n = n!\sum_{m=n}^{\infty} S_1(m,n)\frac{t^m}{m!}, \tag{7}$$

and the Stirling numbers of the second kind are defined by

$$\bigl(e^t-1\bigr)^n = n!\sum_{l=n}^{\infty} S_2(n,l)\frac{t^l}{l!} \quad (n \geq 0). \tag{8}$$

Recently, Lim and Qi [20] have derived integral identities for Appell-type λ-Changhee numbers from the fermionic integral equation. The degenerate Bernoulli polynomials, a degenerate version of the well-known family of polynomials, were introduced by Carlitz, and after that, many researchers have studied the degenerate special polynomials (see [1–3, 20, 22–28]).

The goal of this paper is to consider the Appell-type Changhee polynomials, another version of the Changhee polynomials in (3), and derive some properties of these polynomials. Furthermore, we investigate certain identities for these polynomials.

2 Some identities for Appell-type Changhee polynomials

Now we define the Appell-type Changhee polynomials $\mathrm{Ch}_n^*(x)$ by

$$\frac{2}{2+t}e^{xt} = \sum_{n=0}^{\infty} \mathrm{Ch}_n^*(x)\frac{t^n}{n!}. \tag{9}$$

When $x = 0$, the Changhee numbers $\mathrm{Ch}_n^* = \mathrm{Ch}_n^*(0)$ are equal to the Changhee numbers $\mathrm{Ch}_n = \mathrm{Ch}_n(0)$. From (9) we have

$$\frac{2}{2+t}e^{xt} = \left(\sum_{m=0}^{\infty} \mathrm{Ch}_m^*\frac{t^m}{m!}\right)\left(\sum_{l=0}^{\infty} x^l\frac{t^l}{l!}\right) = \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}\mathrm{Ch}_m^* x^{n-m}\right)\frac{t^n}{n!}. \tag{10}$$

By (10) we have the following theorem.

Theorem 1 *For $n \in \mathbb{N}$, we have*

$$\mathrm{Ch}_n^*(x) = \sum_{m=0}^{n} \binom{n}{m} \mathrm{Ch}_m^* x^{n-m}. \tag{11}$$

By (9), replacing t by $e^t - 1$, we get

$$\frac{2}{2+e^t-1} e^{x(e^t-1)} = \sum_{n=0}^{\infty} \mathrm{Ch}_n^*(x) \frac{(e^t-1)^n}{n!}. \tag{12}$$

Then we have

$$\begin{aligned} \mathrm{RHS} &= \sum_{n=0}^{\infty} \mathrm{Ch}_n^*(x) \frac{(e^t-1)^n}{n!} \\ &= \sum_{n=0}^{\infty} \mathrm{Ch}_n^*(x) \frac{1}{n!} n! \sum_{l=n}^{\infty} S_2(l,n) \frac{t^l}{l!} \\ &= \sum_{l=0}^{\infty} \sum_{n=0}^{l} \mathrm{Ch}_n^*(x) S_2(l,n) \frac{t^l}{l!}, \end{aligned} \tag{13}$$

where $S_2(l,n)$ are the Stirling numbers of the second kind, and

$$\begin{aligned} \mathrm{LHS} &= \frac{2}{1+e^t} e^{x(e^t-1)} \\ &= \sum_{m=0}^{\infty} E_m \frac{t^m}{m!} \sum_{n=0}^{\infty} \mathrm{Bel}_n(x) \frac{t^n}{n!} \\ &= \sum_{l=0}^{\infty} \sum_{n=0}^{l} \binom{l}{n} E_n \mathrm{Bel}_{l-n}(x) \frac{t^l}{l!}. \end{aligned} \tag{14}$$

It is well known that the Bell polynomials are defined by the generating function

$$e^{x(e^t-1)} = \sum_{n=0}^{\infty} \mathrm{Bel}_n(x) \frac{t^n}{n!}$$

(see [8]). By (13) and (14) we have the following theorem.

Theorem 2 *For $l \in \mathbb{N}$, we have*

$$\sum_{n=0}^{l} \mathrm{Ch}_n^*(x) S_2(l,n) = \sum_{n=0}^{l} \binom{l}{n} E_n \mathrm{Bel}_{l-n}(x). \tag{15}$$

By (11) we can derive the following equation:

$$\begin{aligned} \frac{d}{dx} \mathrm{Ch}_n^*(x) &= \sum_{m=0}^{n-1} \binom{n}{m} \mathrm{Ch}_m^* (n-m) x^{n-m-1} \\ &= n \, \mathrm{Ch}_{n-1}^*(x). \end{aligned} \tag{16}$$

From (16) we get

$$
\begin{aligned}
n\int_0^x \mathrm{Ch}_{n-1}^*(s)\,ds &= \int_0^x \frac{d}{ds}\mathrm{Ch}_n^*(s)\,ds \\
&= \mathrm{Ch}_n^*(s)\big|_{s=0}^x \\
&= \mathrm{Ch}_n^*(x) - \mathrm{Ch}_n^*. \qquad (17)
\end{aligned}
$$

By (17) we can derive the following theorem.

Theorem 3 *For $n \in \mathbb{N}$, we have*

$$
\frac{\mathrm{Ch}_{n+1}^*(x) - \mathrm{Ch}_{n+1}^*}{n+1} = \int_0^x \mathrm{Ch}_n^*(s)\,ds. \qquad (18)
$$

By (4) we note that

$$
\begin{aligned}
2 &= \left(\sum_{n=0}^{\infty} \mathrm{Ch}_n^* \frac{t^n}{n!}\right)(2+t) \\
&= \left(\sum_{n=0}^{\infty} 2\,\mathrm{Ch}_n^* \frac{t^n}{n!}\right) + t\sum_{n=0}^{\infty} \mathrm{Ch}_n^* \frac{t^n}{n!} \\
&= \left(\sum_{n=0}^{\infty} 2\,\mathrm{Ch}_n^* \frac{t^n}{n!}\right) + \sum_{n=1}^{\infty} n\,\mathrm{Ch}_{n-1}^* \frac{t^n}{n!} \\
&= 2\,\mathrm{Ch}_0^* + \sum_{n=1}^{\infty}\left(2\,\mathrm{Ch}_n^* + n\,\mathrm{Ch}_{n-1}^*\right)\frac{t^n}{n!}. \qquad (19)
\end{aligned}
$$

By (19) we have the following theorem.

Theorem 4 *For $n \in \mathbb{N}$, we have*

$$
\mathrm{Ch}_0^* = 1, \qquad 2\,\mathrm{Ch}_n^* + n\,\mathrm{Ch}_{n-1}^* = 0 \quad \textit{if } n \geq 1. \qquad (20)
$$

Now we observe that

$$
\begin{aligned}
\sum_{n=0}^{\infty} \mathrm{Ch}_n^*(1-x)\frac{t^n}{n!} &= \frac{2}{2+t}e^{(1-x)t} \\
&= \frac{2}{2+t}e^t e^{-xt} \\
&= \left(\sum_{l=0}^{\infty} \mathrm{Ch}_l^*(1)\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty}(-x)^m \frac{t^m}{m!}\right) \\
&= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}\mathrm{Ch}_{n-m}^*(1)(-x)^m\right)\frac{t^n}{n!}. \qquad (21)
\end{aligned}
$$

From (21) we obtain the following theorem.

Theorem 5 *For $n \in \mathbb{N}$, we have*

$$\mathrm{Ch}_n^*(1-x) = \sum_{m=0}^{n} \binom{n}{m} \mathrm{Ch}_{n-m}^*(1)(-x)^m. \tag{22}$$

By (22) we get

$$\begin{aligned} \int_0^1 \mathrm{Ch}_n^*(1-x)x^n\,dx &= \sum_{m=0}^{n} \binom{n}{m} \mathrm{Ch}_{n-m}^*(1)(-1)^m \int_0^1 x^{n+m}\,dx \\ &= \sum_{m=0}^{n} \binom{n}{m} (-1)^m \frac{\mathrm{Ch}_{n-m}^*(1)}{n+m+1}. \end{aligned} \tag{23}$$

From (16) we note that

$$\begin{aligned} &\int_0^1 y^n \,\mathrm{Ch}_n^*(x+y)\,dy \\ &\quad= \frac{y^{n+1}}{n+1} \mathrm{Ch}_n^*(x+y)\bigg|_{y=0}^{1} - \frac{1}{n+1}\int_0^1 y^{n+1} \frac{d}{dy} \mathrm{Ch}_n^*(x+y)\,dy \\ &\quad= \frac{\mathrm{Ch}_n^*(x+1)}{n+1} - \frac{n}{n+1}\int_0^1 y^{n+1}\,\mathrm{Ch}_{n-1}^*(x+y)\,dy \\ &\quad= \frac{\mathrm{Ch}_n^*(x+1)}{n+1} - \frac{n}{n+1}\left(\frac{\mathrm{Ch}_{n-1}^*(x+y)}{n+2} y^{n+2}\bigg|_{y=0}^{1}\right) \\ &\qquad+ (-1)^2 \frac{n}{n+1}\frac{1}{n+2}(n-1)\int_0^1 y^{n+2}\,\mathrm{Ch}_{n-2}^*(x+y)\,dy \\ &\quad= \frac{\mathrm{Ch}_n^*(x+1)}{n+1} - \frac{n}{n+1}\frac{\mathrm{Ch}_{n-1}^*(x+1)}{n+2} + (-1)^2\frac{n}{n+1}\frac{n-1}{n+2}\int_0^1 y^{n+2}\,\mathrm{Ch}_{n-2}^*(x+y)\,dy \\ &\quad= \frac{\mathrm{Ch}_n^*(x+1)}{n+1} - \frac{n}{n+1}\frac{\mathrm{Ch}_{n-1}^*(x+1)}{n+2} + (-1)^2\frac{n}{n+1}\frac{n-1}{n+2}\frac{\mathrm{Ch}_{n-2}^*(x+1)}{n+3} \\ &\qquad+ (-1)^3\frac{n}{n+1}\frac{n-1}{n+2}\frac{n-2}{n+3}\int_0^1 y^{n+3}\,\mathrm{Ch}_{n-3}^*(x+y)\,dy. \end{aligned} \tag{24}$$

Also, we get

$$\int_0^1 y^{2n-1}\,\mathrm{Ch}_1^*(x+y)\,dy = \frac{\mathrm{Ch}_1^*(x+y)}{2n} y^{2n}\bigg|_{y=0}^{1} - \frac{1}{2n}\int_0^1 y^{2n}\,\mathrm{Ch}_0^*(x+y)\,dy. \tag{25}$$

From (11) we get

$$\mathrm{Ch}_0^*(x) = 1, \tag{26}$$

and hence

$$\begin{aligned} \int_0^1 y^{2n-1}\,\mathrm{Ch}_1^*(x+y)\,dy &= \frac{\mathrm{Ch}_1^*(x)}{2n} - \frac{1}{2n}\int_0^1 y^{2n}\,dy \\ &= \frac{\mathrm{Ch}_1^*(x)}{2n} - \frac{1}{2n(2n+1)}. \end{aligned} \tag{27}$$

By (27), continuing the process in (24), we have

$$\begin{aligned}&\int_0^1 y^n \operatorname{Ch}_n^*(x+y)\,dy\\&\quad= \frac{\operatorname{Ch}_n^*(x+1)}{n+1} + \sum_{m=1}^{n}(-1)^m \operatorname{Ch}_{n-m}^*(x+1)\frac{n(n-1)\cdots(n-m+1)}{(n+1)(n+2)\cdots(n+m+1)}.\end{aligned}\tag{28}$$

We note that

$$\begin{aligned}\operatorname{Ch}_n^*(x+y) &= \operatorname{Ch}_n^*(x+1+y-1)\\&= \sum_{l=1}^{n}\binom{n}{l}\operatorname{Ch}_l^*(x+1)(-1)^{n-l}(1-y)^{n-l}.\end{aligned}\tag{29}$$

By (29) we get

$$\begin{aligned}&\int_0^1 y^n \operatorname{Ch}_n^*(x+y)\,dy\\&\quad= \sum_{l=1}^{n}\binom{n}{l}\operatorname{Ch}_l^*(x+1)(-1)^{n-l}\int_0^1 y^n(1-y)^{n-l}\,dy\\&\quad= \sum_{l=1}^{n}\binom{n}{l}\operatorname{Ch}_l^*(x+1)(-1)^{n-l}B(n+1,n-l+1)\\&\quad= \sum_{l=0}^{n}\binom{n}{l}\operatorname{Ch}_l^*(x+1)(-1)^{n-l}\frac{\Gamma(n+1)\Gamma(n-l+1)}{\Gamma(2n-l+2)}\\&\quad= \sum_{l=0}^{n}(-1)^{n-l}\binom{n}{l}\frac{n!(n-l)!}{(2n-l+1)!}\operatorname{Ch}_l^*(x+1)\\&\quad= \sum_{l=0}^{n}(-1)^{n-l}\frac{n\binom{n}{l}}{(2n-l+1)\binom{2n-l}{n}}\operatorname{Ch}_l^*(x+1).\end{aligned}\tag{30}$$

By (28) and (30) we have the following theorem.

Theorem 6 *For $n \in \mathbb{N}$, we have*

$$\begin{aligned}&\sum_{l=0}^{n}(-1)^{n-l}\frac{n\binom{n}{l}}{(2n-l+1)\binom{2n-l}{n}}\operatorname{Ch}_l^*(x+1)\\&\quad= \frac{\operatorname{Ch}_n^*(x+1)}{n+1} + \sum_{m=1}^{n}(-1)^m \operatorname{Ch}_{n-m}^*(x+1)\frac{n(n-1)\cdots(n-m+1)}{(n+1)(n+2)\cdots(n+m+1)}.\end{aligned}\tag{31}$$

From (16) we note that

$$\begin{aligned}&\int_0^1 y^n \operatorname{Ch}_n^*(x+y)\,dy\\&\quad= \frac{\operatorname{Ch}_{n+1}^*(x+y)}{n+1}y^n\bigg|_{y=0}^{1} - \frac{1}{n+1}n\int_0^1 y^{n-1}\operatorname{Ch}_{n+1}^*(x+y)\,dy\end{aligned}$$

$$
\begin{aligned}
&= \frac{\mathrm{Ch}^*_{n+1}(x+1)}{n+1} - \frac{n}{n+1}\int_0^1 y^{n-1}\,\mathrm{Ch}^*_{n+1}(x+y)\,dy \\
&= \frac{\mathrm{Ch}^*_{n+1}(x+1)}{n+1} - \frac{n}{n+1}\frac{\mathrm{Ch}^*_{n+2}(x+1)}{n+2} + \frac{n(n-1)}{(n+1)(n+2)}\int_0^1 y^{n-2}\,\mathrm{Ch}^*_{n+2}(x+y)\,dy \\
&= \frac{\mathrm{Ch}^*_{n+1}(x+1)}{n+1} - \frac{n}{n+1}\frac{\mathrm{Ch}^*_{n+2}(x+1)}{n+2} + \frac{n(n-1)}{(n+1)(n+2)}\frac{\mathrm{Ch}^*_{n+3}(x+1)}{n+3} \\
&\quad - \frac{n(n-1)(n-2)}{(n+1)(n+2)(n+3)}\int_0^1 y^{n-3}\,\mathrm{Ch}^*_{n+3}(x+y)\,dy.
\end{aligned}
\tag{32}
$$

Also, we have

$$
\begin{aligned}
&\int_0^1 y\,\mathrm{Ch}^*_{2n-1}(x+y)\,dy \\
&= \frac{\mathrm{Ch}^*_{2n}(x+y)}{2n}y\bigg|_{y=0}^{1} - \frac{1}{2n}\int_0^1 1\cdot \mathrm{Ch}^*_{2n}(x+y)\,dy \\
&= \frac{\mathrm{Ch}^*_{2n}(x+1)}{2n} - \frac{1}{2n}\frac{1}{2n+1}\mathrm{Ch}^*_{2n+1}(x+y)\bigg|_{y=0}^{1} \\
&= \frac{\mathrm{Ch}^*_{2n}(x+1)}{2n} - \frac{\mathrm{Ch}^*_{2n+1}(x+1) - \mathrm{Ch}^*_{2n+1}(x)}{2n(2n+1)}.
\end{aligned}
\tag{33}
$$

By (30), continuing the process in (28), we obtain the following theorem.

Theorem 7 *For $n \in \mathbb{N}$, we have*

$$
\begin{aligned}
&\sum_{l=0}^{n}(-1)^{n-l}\frac{n\binom{n}{l}}{(2n-l+1)\binom{2n-l}{n}}\mathrm{Ch}^*_l(x+1) \\
&= \frac{\mathrm{Ch}^*_{n+1}(x+1)}{n+1} + \sum_{m=1}^{n-1}(-1)^m\,\mathrm{Ch}^*_{n+m+1}(x+1)\frac{n(n-1)\cdots(n-m+1)}{(n+1)(n+2)\cdots(n+m+1)} \\
&\quad + (-1)^n\frac{n!}{(2n+1)_{n+1}}\bigl(\mathrm{Ch}^*_{2n+1}(x+1) - \mathrm{Ch}^*_{2n+1}(1)\bigr).
\end{aligned}
\tag{34}
$$

Now, we have

$$
\begin{aligned}
&\int_0^1 \mathrm{Ch}^*_n(x)\,\mathrm{Ch}^*_m(x)\,dx \\
&= \frac{\mathrm{Ch}^*_{n+1}(x)\,\mathrm{Ch}^*_m(x)}{n+1}\bigg|_0^1 - \frac{1}{n+1}m\int_0^1 \mathrm{Ch}^*_{n+1}(x)\,\mathrm{Ch}^*_{m-1}(x)\,dx \\
&= \frac{1}{n+1}\bigl(\mathrm{Ch}^*_{n+1}(1)\,\mathrm{Ch}^*_m(1) - \mathrm{Ch}^*_{n+1}(0)\,\mathrm{Ch}^*_m(0)\bigr) \\
&\quad - \frac{m}{n+1}\int_0^1 \mathrm{Ch}^*_{n+1}(x)\,\mathrm{Ch}^*_{m-1}(x)\,dx \\
&= \frac{\mathrm{Ch}^*_{n+1}(1)\,\mathrm{Ch}^*_m(1) - \mathrm{Ch}^*_{n+1}\,\mathrm{Ch}^*_m}{n+1} - \frac{m}{n+1}\frac{\mathrm{Ch}^*_{n+2}(1)\,\mathrm{Ch}^*_{m-1}(1) - \mathrm{Ch}^*_{n+2}\,\mathrm{Ch}^*_{m-1}}{n+2} \\
&\quad + (-1)^2\frac{m}{n+1}\frac{m-1}{n+2}\int_0^1 \mathrm{Ch}^*_{n+2}(x)\,\mathrm{Ch}^*_{m-2}(x)\,dx
\end{aligned}
\tag{35}
$$

and

$$\begin{aligned}
&\int_0^1 \mathrm{Ch}_{n+m-1}^*(x)\,\mathrm{Ch}_1^*(x)\,dx \\
&\quad= \frac{\mathrm{Ch}_{n+m}^*(1)\,\mathrm{Ch}_1^*(1) - \mathrm{Ch}_{n+m}^*\,\mathrm{Ch}_1^*}{n+m} - \frac{1}{n+m}\int_0^1 \mathrm{Ch}_{n+m}^*(x)\,\mathrm{Ch}_0^*(x)\,dx \\
&\quad= \frac{\mathrm{Ch}_{n+m}^*(1)\,\mathrm{Ch}_1^*(1) - \mathrm{Ch}_{n+m}^*\,\mathrm{Ch}_1^*}{n+m} - \frac{1}{n+m}\frac{\mathrm{Ch}_{n+m+1}^*(1) - \mathrm{Ch}_{n+m+1}^*}{n+m+1}.
\end{aligned} \tag{36}$$

By (30) with $x = 0$ we get

$$\begin{aligned}
&\int_0^1 \mathrm{Ch}_n^*(x)\,\mathrm{Ch}_m^*(x)\,dx \\
&\quad= \sum_{j=0}^{m}\binom{m}{j}\mathrm{Ch}_j^*\int_0^1 x^{m-j}\,\mathrm{Ch}_m^*(x)\,dx \\
&\quad= \sum_{j=0}^{m}\binom{m}{j}\mathrm{Ch}_j^*\sum_{l=0}^{m-j}(-1)^{m-j-l}\frac{(m-j)\binom{m-j}{l}}{(2(m-j)-l+1)\binom{2(m-j)-l}{m-j}}\mathrm{Ch}_l^*(1) \\
&\quad= \sum_{j=0}^{m}\sum_{l=0}^{m-j}\binom{m}{j}(-1)^{m-j-l}\frac{(m-j)\binom{m-j}{l}}{(2(m-j)-l+1)\binom{2(m-j)-l}{m-j}}\mathrm{Ch}_j^*\,\mathrm{Ch}_l^*(1).
\end{aligned} \tag{37}$$

By (37), continuing the process in (35), we obtain the following theorem.

Theorem 8 *For $n \in \mathbb{N}$, we have*

$$\begin{aligned}
&\sum_{j=0}^{m}\sum_{l=0}^{m-j}\binom{m}{j}(-1)^{m-j-l}\frac{(m-j)\binom{m-j}{l}}{(2(m-j)-l+1)\binom{2(m-j)-l}{m-j}}\mathrm{Ch}_j^*\,\mathrm{Ch}_l^*(1) \\
&\quad= \frac{\mathrm{Ch}_{n+1}^*(1)\,\mathrm{Ch}_m^*(1) - \mathrm{Ch}_{n+1}^*\,\mathrm{Ch}_m^*}{n+1} \\
&\qquad+ \sum_{k=1}^{m-1}(-1)^k\frac{m(m-1)\cdots(m-k+1)}{(n+1)(n+2)\cdots(n+k+1)} \\
&\qquad\times \bigl(\mathrm{Ch}_{n+k+1}^*(1)\,\mathrm{Ch}_{m-k}^*(1) - \mathrm{Ch}_{n+k+1}^*\,\mathrm{Ch}_{m-k}^*\bigr) \\
&\qquad+ (-1)^m\frac{m!}{(n+m+1)_{m+1}}\bigl(\mathrm{Ch}_{n+m+1}^*(1) - \mathrm{Ch}_{n+m+1}^*\bigr).
\end{aligned} \tag{38}$$

3 Remarks

In this section, by using the fermionic p-adic integral on $\mathbb{Z}_p$, we derive some identities for Changhee polynomials, Stirling numbers of the first kind, and Euler numbers. By (2) we note that

$$\begin{aligned}
\frac{2}{2+t}e^{xt} &= \int_{\mathbb{Z}_p}(1+t)^y e^{xt}\,d\mu_{-1}(y) \\
&= \int_{\mathbb{Z}_p} e^{y\log(1+t)+xt}\,d\mu_{-1}(y)
\end{aligned} \tag{39}$$

and

$$\begin{aligned} e^{xt}e^{y\log(1+t)} &= \left(\sum_{m=0}^{\infty} x^m \frac{t^m}{m!}\right)\left(\sum_{l=0}^{\infty} \frac{y^l(log(1+t))^l}{l!}\right) \\ &= \left(\sum_{m=0}^{\infty} x^m \frac{t^m}{m!}\right)\left(\sum_{l=0}^{\infty} y^l \sum_{k=l}^{\infty} S_1(k,l)\frac{t^k}{k!}\right) \\ &= \left(\sum_{m=0}^{\infty} x^m \frac{t^m}{m!}\right)\left(\sum_{k=0}^{\infty}\sum_{l=0}^{k} y^l S_1(k,l)\frac{t^k}{k!}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\sum_{l=0}^{k}\binom{n}{k} x^{n-k} y^l S_1(k,l)\right)\frac{t^n}{n!}. \end{aligned} \tag{40}$$

Thus, by (39) and (40) we have

$$\begin{aligned} \sum_{n=0}^{\infty} \mathrm{Ch}_n^*(x)\frac{t^n}{n!} &= \int_{\mathbb{Z}_p} e^{y\log(1+t)}e^{xt}\,d\mu_{-1}(y) \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\sum_{l=0}^{k}\binom{n}{k} x^{n-k}\int_{\mathbb{Z}_p} y^l\,d\mu_{-1}(y) S_1(k,l)\right)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\sum_{l=0}^{k}\binom{n}{k} x^{n-k} E_l S_1(k,l)\right)\frac{t^n}{n!}. \end{aligned} \tag{41}$$

From (41) we have the following theorem.

Theorem 9 *For $n \in \mathbb{N}$, we have*

$$\mathrm{Ch}_n^*(x) = \sum_{k=0}^{n}\sum_{l=0}^{k}\binom{n}{k} x^{n-k} E_l S_1(k,l). \tag{42}$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to this work. All authors read and approved the final manuscript.

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