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Sums of products of two reciprocal Fibonacci numbers

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Abstract

In this paper, we employ elementary methods to investigate the reciprocal sums of the products of two Fibonacci numbers in several ways. First, we consider the sums of the reciprocals of the products of two Fibonacci numbers and establish five interesting families of identities. Then we extend such analysis to the alternating sums and obtain five analogous results.

MSC: 11B39

Keywords: Fibonacci number; reciprocal sum; floor function

1 Introduction

For an integer $n \geq 0$, the *Fibonacci number* F_n is defined by

$$F_n = F_{n-1} + F_{n-2} \quad \text{for } n \geq 2,$$

with $F_0 = 0$ and $F_1 = 1$. There exists a simple and nonobvious formula for the Fibonacci numbers:

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

The Fibonacci numbers play an important role in the theory and applications of mathematics, and its various properties have been investigated by many authors; see [1–4].

In recent years, there has been an increasing interest in studying the reciprocal sums of the Fibonacci numbers. For example, Elsner, Shimomura, and Shiokawa [5–8] investigated algebraic relations for reciprocal sums of the Fibonacci numbers. Ohtsuka and Nakamura [9] studied the partial infinite sums of the reciprocal Fibonacci numbers. They established the following results, where $\lfloor \cdot \rfloor$ denotes the floor function.

Theorem 1.1 *For all* $n \geq 2$,

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_k} \right)^{-1} \right\rfloor = \begin{cases} F_{n-2} & \text{if } n \text{ is even;} \\ F_{n-2} - 1 & \text{if } n \text{ is odd.} \end{cases}$$

Theorem 1.2 *For each $n \geq 1$,*

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_k^2} \right)^{-1} \right\rfloor = \begin{cases} F_n F_{n-1} - 1 & \text{if } n \text{ is even;} \\ F_n F_{n-1} & \text{if } n \text{ is odd.} \end{cases}$$

Recently, Wang and Wen [10] considered the partial finite sums of the reciprocal Fibonacci numbers and strengthened Theorem 1.1 and Theorem 1.2 to the finite-sum case.

Theorem 1.3

(i) *For all $n \geq 4$,*

$$\left\lfloor \left(\sum_{k=n}^{2n} \frac{1}{F_k} \right)^{-1} \right\rfloor = F_{n-2}.$$

(ii) *If $m \geq 3$ and $n \geq 2$, then*

$$\left\lfloor \left(\sum_{k=n}^{mn} \frac{1}{F_k} \right)^{-1} \right\rfloor = \begin{cases} F_{n-2} & \text{if } n \text{ is even;} \\ F_{n-2} - 1 & \text{if } n \text{ is odd.} \end{cases}$$

Theorem 1.4 *For all $m \geq 2$ and $n \geq 1$, we have*

$$\left\lfloor \left(\sum_{k=n}^{mn} \frac{1}{F_k^2} \right)^{-1} \right\rfloor = \begin{cases} F_n F_{n-1} - 1 & \text{if } n \text{ is even;} \\ F_n F_{n-1} & \text{if } n \text{ is odd.} \end{cases}$$

Furthermore, Wang and Zhang [11] studied the reciprocal sums of the Fibonacci numbers with even or odd indexes and obtained the following main results.

Theorem 1.5 *We have*

$$\left\lfloor \left(\sum_{k=n}^{mn} \frac{1}{F_{2k}} \right)^{-1} \right\rfloor = \begin{cases} F_{2n-1} & \text{if } m = 2 \text{ and } n \geq 3; \\ F_{2n-1} - 1 & \text{if } m \geq 3 \text{ and } n \geq 1. \end{cases}$$

Theorem 1.6 *For all $n \geq 1$ and $m \geq 2$, we have*

$$\left\lfloor \left(\sum_{k=n}^{mn} \frac{1}{F_{2k-1}} \right)^{-1} \right\rfloor = F_{2n-2}.$$

Theorem 1.7 *If $n \geq 1$ and $m \geq 2$, then*

$$\left\lfloor \left(\sum_{k=n}^{mn} \frac{1}{F_{2k}^2} \right)^{-1} \right\rfloor = F_{4n-2} - 1.$$

Theorem 1.8 *For all $n \geq 1$ and $m \geq 2$, we have*

$$\left\lfloor \left(\sum_{k=n}^{mn} \frac{1}{F_{2k-1}^2} \right)^{-1} \right\rfloor = F_{4n-4}.$$

More recently, Wang and Zhang [12] proceeded with investigating the reciprocal sums of the Fibonacci numbers according to the subscripts modulo 3 and found many identities. Here are a few examples.

Theorem 1.9
(i) *For all* $n \geq 2$,

$$\left\lfloor \left(\sum_{k=n}^{2n} \frac{1}{F_{3k}} \right)^{-1} \right\rfloor = 2F_{3n-2}.$$

(ii) *If* $m \geq 3$ *and* $n \geq 1$, *then*

$$\left\lfloor \left(\sum_{k=n}^{mn} \frac{1}{F_{3k}} \right)^{-1} \right\rfloor = \begin{cases} 2F_{3n-2} & \text{if } n \text{ is even;} \\ 2F_{3n-2} - 1 & \text{if } n \text{ is odd.} \end{cases}$$

Theorem 1.10 *If* $n \geq 2$ *and* $m \geq 2$, *we have*

$$\left\lfloor \left(\sum_{k=n}^{mn} \frac{1}{F_{3k}^2} \right)^{-1} \right\rfloor = \begin{cases} F_{3n}^2 - F_{3n-3}^2 & \text{if } n \text{ is even;} \\ F_{3n}^2 - F_{3n-3}^2 - 1 & \text{if } n \text{ is odd.} \end{cases}$$

In this article, we focus ourselves on the sums and alternating sums of the products of two reciprocal Fibonacci numbers. By evaluating the integer parts of these sums, we obtain several interesting families of identities concerning the Fibonacci numbers.

2 Main results I: reciprocal sums

We first introduce several well-known results on the Fibonacci numbers, which will be used throughout the article. The detailed proofs can be found in, for example, [4] and [13].

Lemma 2.1 *For any positive integers* m *and* n, *we have*

$$F_m F_n + F_{m+1} F_{n+1} = F_{m+n+1}. \tag{2.1}$$

Lemma 2.2 *If* $n \geq 1$, *then*

$$F_{2n} = F_{n+1}^2 - F_{n-1}^2, \tag{2.2}$$

$$F_{2n+1} = F_{n+1} F_{n+2} - F_{n-1} F_n. \tag{2.3}$$

Lemma 2.3 *Let* a, b, c, d *be positive integers with* $a + b = c + d$ *and* $b \geq \max\{c, d\}$. *Then*

$$F_a F_b - F_c F_d = (-1)^{a+1} F_{b-c} F_{b-d}. \tag{2.4}$$

2.1 Reciprocal sum of $F_k F_{k+1}$

Lemma 2.4 *For all* $n \geq 1$, *we have*

$$F_{2n+1}^2 + 1 > \left(F_{n+1}^2 + 1\right)^2 > F_n F_{n+1}\left(F_{n+1}^2 + 1\right). \tag{2.5}$$

Proof It follows from (2.1) that $F_{2n+1} = F_n^2 + F_{n+1}^2$. Hence,

$$F_{2n+1}^2 + 1 = F_n^4 + 2F_n^2F_{n+1}^2 + F_{n+1}^4 + 1 > F_{n+1}^4 + 2F_{n+1}^2 + 1 = \left(F_{n+1}^2 + 1\right)^2.$$

It is clear that $F_{n+1}^2 \geq F_nF_{n+1}$; therefore, $F_{n+1}^2 + 1 > F_nF_{n+1}$, which yields the second inequality. □

Theorem 2.5 *If $m \geq 2$ and $n \geq 1$, then*

$$\left\lfloor \left(\sum_{k=n}^{mn} \frac{1}{F_kF_{k+1}}\right)^{-1} \right\rfloor = \begin{cases} F_n^2 & \text{if } n \text{ is even;} \\ F_n^2 - 1 & \text{if } n \text{ is odd.} \end{cases}$$

Proof We first consider the case where n is even. By elementary manipulations and setting $a = k-1$, $b = k+1$, and $c = d = k$ in (2.4), we obtain, for $k \geq 1$,

$$\begin{aligned} \frac{1}{F_k^2} - \frac{1}{F_kF_{k+1}} - \frac{1}{F_{k+1}^2} &= \frac{F_{k+1}^2 - F_kF_{k+1} - F_k^2}{F_k^2F_{k+1}^2} \\ &= \frac{F_{k-1}F_{k+1} - F_k^2}{F_k^2F_{k+1}^2} \\ &= \frac{(-1)^k}{F_k^2F_{k+1}^2}. \end{aligned} \tag{2.6}$$

Now we have

$$\sum_{k=n}^{mn} \frac{1}{F_kF_{k+1}} = \frac{1}{F_n^2} - \frac{1}{F_{mn+1}^2} + \sum_{k=n}^{mn} \frac{(-1)^{k-1}}{F_k^2F_{k+1}^2}.$$

Since n is even, it is easy to see that

$$\sum_{k=n}^{mn} \frac{(-1)^{k-1}}{F_k^2F_{k+1}^2} < 0,$$

which implies that

$$\sum_{k=n}^{mn} \frac{1}{F_kF_{k+1}} < \frac{1}{F_n^2}. \tag{2.7}$$

A direct calculation shows that, for $k \geq 1$,

$$\begin{aligned} &\frac{1}{F_k^2 + 1} - \frac{1}{F_kF_{k+1}} - \frac{1}{F_{k+1}^2 + 1} \\ &\quad = \frac{F_kF_{k+1}(F_{k+1}^2 - F_k^2) - (F_k^2 + 1)(F_{k+1}^2 + 1)}{(F_k^2 + 1)F_kF_{k+1}(F_{k+1}^2 + 1)} \\ &\quad = \frac{F_kF_{k+1}(F_{k+1}^2 - F_k^2 - F_kF_{k+1}) - F_k^2 - F_{k+1}^2 - 1}{(F_k^2 + 1)F_kF_{k+1}(F_{k+1}^2 + 1)} \\ &\quad = \frac{F_kF_{k+1}(F_{k-1}F_{k+1} - F_k^2) - F_k^2 - F_{k+1}^2 - 1}{(F_k^2 + 1)F_kF_{k+1}(F_{k+1}^2 + 1)} \end{aligned}$$

$$
\begin{aligned}
&= \frac{(-1)^k F_k F_{k+1} - F_k^2 - F_{k+1}^2 - 1}{(F_k^2+1)F_k F_{k+1}(F_{k+1}^2+1)} \\
&< 0.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\sum_{k=n}^{mn} \frac{1}{F_k F_{k+1}} &= \frac{1}{F_n^2+1} - \frac{1}{F_{mn+1}^2+1} + \sum_{k=n}^{mn} \frac{(-1)^{k-1} F_k F_{k+1} + F_k^2 + F_{k+1}^2 + 1}{(F_k^2+1)F_k F_{k+1}(F_{k+1}^2+1)} \\
&> \frac{1}{F_n^2+1} - \frac{1}{F_{2n+1}^2+1} + \frac{F_n^2 + F_{n+1}^2 + 1 - F_n F_{n+1}}{(F_n^2+1)F_n F_{n+1}(F_{n+1}^2+1)} \\
&> \frac{1}{F_n^2+1} + \frac{1}{F_n F_{n+1}(F_{n+1}^2+1)} - \frac{1}{F_{2n+1}^2+1} \\
&> \frac{1}{F_n^2+1},
\end{aligned} \tag{2.8}
$$

where the last inequality follows from (2.5).

Combining (2.7) and (2.8), we have

$$
\frac{1}{F_n^2+1} < \sum_{k=n}^{mn} \frac{1}{F_k F_{k+1}} < \frac{1}{F_n^2},
$$

which means that the statement is true when n is even.

We now concentrate ourselves on the case where n is odd. It is obviously true for $n = 1$. Now we assume that $n \geq 3$. A similar calculation shows that, for $k \geq 3$,

$$
\frac{1}{F_k^2-1} - \frac{1}{F_k F_{k+1}} - \frac{1}{F_{k+1}^2-1} = \frac{(-1)^k F_k F_{k+1} + F_k^2 + F_{k+1}^2 + 1}{(F_k^2-1)F_k F_{k+1}(F_{k+1}^2-1)} > 0,
$$

from which we get

$$
\sum_{k=n}^{mn} \frac{1}{F_k F_{k+1}} < \frac{1}{F_n^2-1} - \frac{1}{F_{mn+1}^2-1} < \frac{1}{F_n^2-1}. \tag{2.9}
$$

It follows from (2.1) that

$$
F_{2n+1} = F_{n-1}F_{n+1} + F_n F_{n+2} = F_n^2 + F_{n+1}^2,
$$

which implies that $F_{2n+1} \geq F_n F_{n+2}$, and $F_{2n+2} > F_{2n+1} > F_{n+1}^2$. Therefore,

$$
F_{2n+1}^2 F_{2n+2} > F_n^2 F_{n+1}^2 F_{n+2}^2. \tag{2.10}
$$

Invoking (2.6), (2.2), and the fact n is odd, we have

$$
\begin{aligned}
\sum_{k=n}^{mn} \frac{1}{F_k F_{k+1}} &= \frac{1}{F_n^2} - \frac{1}{F_{mn+1}^2} + \sum_{k=n}^{mn} \frac{(-1)^{k-1}}{F_k^2 F_{k+1}^2} \\
&> \frac{1}{F_n^2} - \frac{1}{F_{2n+1}^2} + \frac{1}{F_n^2 F_{n+1}^2} - \frac{1}{F_{n+1}^2 F_{n+2}^2}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{F_n^2} + \frac{F_{n+2}^2 - F_n^2}{F_n^2 F_{n+1}^2 F_{n+2}^2} - \frac{1}{F_{2n+1}^2} \\
&= \frac{1}{F_n^2} + \frac{F_{2n+2}}{F_n^2 F_{n+1}^2 F_{n+2}^2} - \frac{F_{2n+2}}{F_{2n+1}^2 F_{2n+2}} \\
&> \frac{1}{F_n^2},
\end{aligned} \tag{2.11}
$$

where the last inequality follows from (2.10).

Combining (2.9) and (2.11) yields that

$$\frac{1}{F_n^2} < \sum_{k=n}^{mn} \frac{1}{F_k F_{k+1}} < \frac{1}{F_n^2 - 1},$$

from which the desired result follows immediately. □

2.2 Reciprocal sum of $F_{2k-1}F_{2k}$

Lemma 2.6 *For all $n \geq 1$, we have*

$$F_{8n+1} > F_{2n-1}F_{2n}(F_{4n+1} + 1).$$

Proof Applying (2.1) repeatedly, we have

$$F_{8n+1} = F_{4n+1}^2 + F_{4n}^2 > F_{4n+1}^2 + F_{4n+1} = F_{4n+1}(F_{4n+1} + 1) > F_{2n-1}F_{2n}(F_{4n+1} + 1),$$

which completes the proof. □

Theorem 2.7 *For all $n \geq 2$ and $m \geq 2$,*

$$\left\lfloor \left(\sum_{k=n}^{mn} \frac{1}{F_{2k-1}F_{2k}} \right)^{-1} \right\rfloor = F_{4n-3}.$$

Proof It follows from (2.3) that

$$F_{4n+1} = F_{2n+1}F_{2n+2} - F_{2n-1}F_{2n}, \tag{2.12}$$

$$F_{4n-3} = F_{2n-1}F_{2n} - F_{2n-3}F_{2n-2}. \tag{2.13}$$

Employing (2.4), we can easily get that

$$F_{2n-2}F_{2n+1} = F_{2n-1}F_{2n} - 1, \tag{2.14}$$

$$F_{2n-3}F_{2n+2} = F_{2n-1}F_{2n} + 2. \tag{2.15}$$

Applying (2.12), (2.13), (2.14), and (2.15), it is easy to see that, for all $k \geq 2$,

$$
\begin{aligned}
\frac{1}{F_{4k-3}} - \frac{1}{F_{2k-1}F_{2k}} - \frac{1}{F_{4k+1}} &= \frac{F_{2k-1}F_{2k}(F_{4k+1} - F_{4k-3}) - F_{4k-3}F_{4k+1}}{F_{4k-3}F_{2k-1}F_{2k}F_{4k+1}} \\
&= \frac{F_{2k-3}F_{2k-2}F_{2k+1}F_{2k+2} - F_{2k-1}^2 F_{2k}^2}{F_{4k-3}F_{2k-1}F_{2k}F_{4k+1}}
\end{aligned}
$$

$$= \frac{F_{2k-1}F_{2k} - 2}{F_{4k-3}F_{2k-1}F_{2k}F_{4k+1}}$$
$$> 0,$$

which implies that

$$\sum_{k=n}^{mn} \frac{1}{F_{2k-1}F_{2k}} < \sum_{k=n}^{mn} \left(\frac{1}{F_{4k-3}} - \frac{1}{F_{4k+1}} \right) = \frac{1}{F_{4n-3}} - \frac{1}{F_{4mn+1}} < \frac{1}{F_{4n-3}}. \tag{2.16}$$

It follows from (2.1) that

$$F_{4n+1} > F_{4n} = F_{2n-1}F_{2n} + F_{2n}F_{2n+1} > F_{2n-1}F_{2n} + 1.$$

Therefore, by (2.13) we obtain

$$F_{4n+1} - F_{2n-3}F_{2n-2} > F_{2n-1}F_{2n} - F_{2n-3}F_{2n-2} + 1 = F_{4n-3} + 1. \tag{2.17}$$

Elementary manipulations and (2.17) yield, for $k \geq 2$,

$$\begin{aligned} \frac{1}{F_{4k-3}+1} - \frac{1}{F_{2k-1}F_{2k}} - \frac{1}{F_{4k+1}+1} &= \frac{F_{2k-3}F_{2k-2} - F_{4k+1} - 3}{(F_{4k-3}+1)F_{2k-1}F_{2k}(F_{4k+1}+1)} \\ &< \frac{-1}{F_{2k-1}F_{2k}(F_{4k+1}+1)}. \end{aligned}$$

Now we can deduce that

$$\begin{aligned} \sum_{k=n}^{mn} \frac{1}{F_{2k-1}F_{2k}} &> \frac{1}{F_{4n-3}+1} - \frac{1}{F_{4mn+1}+1} + \sum_{k=n}^{mn} \frac{1}{F_{2k-1}F_{2k}(F_{4k+1}+1)} \\ &> \frac{1}{F_{4n-3}+1} + \frac{1}{F_{2n-1}F_{2n}(F_{4n+1}+1)} - \frac{1}{F_{4mn+1}+1} \\ &> \frac{1}{F_{4n-3}+1} + \frac{1}{F_{2n-1}F_{2n}(F_{4n+1}+1)} - \frac{1}{F_{8n+1}+1} \\ &> \frac{1}{F_{4n-3}+1}, \end{aligned} \tag{2.18}$$

where the last inequality follows from Lemma 2.6.

Combining (2.16) and (2.18), we deduce

$$\frac{1}{F_{4n-3}+1} < \sum_{k=n}^{mn} \frac{1}{F_{2k-1}F_{2k}} < \frac{1}{F_{4n-3}},$$

which yields the desired identity. □

Similarly, we can prove the following result, whose proof is left as an exercise to the readers.

Theorem 2.8 *For all $n \geq 2$ and $m \geq 2$,*

$$\left\lfloor \left(\sum_{k=n}^{mn} \frac{1}{F_{2k}F_{2k+1}} \right)^{-1} \right\rfloor = F_{4n-1} - 1.$$

2.3 Reciprocal sum of $F_{2k-1}F_{2k+1}$

Lemma 2.9 *For all $n \geq 2$, we have*

$$F_{n+2}^2 - F_{n-2}^2 > 4F_{n-1}F_{n+1}.$$

Proof It is easy to check that

$$\begin{aligned} F_{n+2}^2 - F_{n-2}^2 &= (F_{n+2} - F_{n-2})(F_{n+2} + F_{n-2}) \\ &> (F_{n+1} + F_{n-1})(F_{n+1} + F_{n-1}) \\ &= F_{n+1}^2 + F_{n-1}^2 + 2F_{n-1}F_{n+1} \\ &= (F_{n+1} - F_{n-1})^2 + 4F_{n-1}F_{n+1} \\ &> 4F_{n-1}F_{n+1}. \end{aligned}$$

□

Lemma 2.10 *If $n \geq 1$, then*

$$F_{8n+2} > (F_{4n-2} + 1)(F_{4n+2} + 1).$$

Proof It follows from (2.2) that

$$F_{8n+2} = F_{4n+2}^2 - F_{4n}^2 = (F_{4n+2} - F_{4n})(F_{4n+2} + F_{4n}) = F_{4n+1}(F_{4n+2} + F_{4n}).$$

It is obvious that $F_{4n+1} > F_{4n-2} + 1$ and $F_{4n} > 1$, which completes the proof. □

Theorem 2.11 *For all $n \geq 1$ and $m \geq 2$,*

$$\left\lfloor \left(\sum_{k=n}^{mn} \frac{1}{F_{2k-1}F_{2k+1}} \right)^{-1} \right\rfloor = F_{4n-2}.$$

Proof Employing (2.2), we can readily see that

$$\begin{aligned} F_{4n+2} &= F_{2n+2}^2 - F_{2n}^2 = F_{2n+1}(F_{2n+2} + F_{2n}) = F_{2n+1}^2 + 2F_{2n}F_{2n+1}, \\ F_{4n-2} &= F_{2n}^2 - F_{2n-2}^2 = F_{2n-1}(F_{2n} + F_{2n-2}) = 2F_{2n-1}F_{2n} - F_{2n-1}^2. \end{aligned}$$

Applying (2.4), we can establish the following identities:

$$\begin{aligned} F_{2n+1}^2 &= F_{2n}F_{2n+2} + 1, \\ F_{2n-1}^2 &= F_{2n-2}F_{2n} + 1, \\ F_{2n}^2 &= F_{2n-2}F_{2n+2} + 1. \end{aligned}$$

With the help of these identities, we now arrive at

$$\begin{aligned} F_{4n+2} - F_{4n-2} &= F_{2n+1}^2 + F_{2n-1}^2 + 2F_{2n}F_{2n+1} - 2F_{2n-1}F_{2n} \\ &= F_{2n+1}^2 + F_{2n-1}^2 + 2F_{2n}^2 \end{aligned}$$

$$\begin{aligned}
&= (F_{2n}F_{2n+2}+1)+(F_{2n-2}F_{2n}+1)+(F_{2n-2}F_{2n+2}+1)+F_{2n}^2 \\
&= (F_{2n+2}+F_{2n})(F_{2n}+F_{2n-2})+3 \\
&= \frac{F_{4n+2}F_{4n-2}}{F_{2n-1}F_{2n+1}}+3. \qquad (2.19)
\end{aligned}$$

Elementary manipulations and (2.19) yield that, for $k \geq 1$,

$$\begin{aligned}
\frac{1}{F_{4k-2}}-\frac{1}{F_{2k-1}F_{2k+1}}-\frac{1}{F_{4k+2}} &= \frac{F_{2k-1}F_{2k+1}(F_{4k+2}-F_{4k-2})-F_{4k-2}F_{4k+2}}{F_{4k-2}F_{2k-1}F_{2k+1}F_{4k+2}} \\
&= \frac{3F_{2k-1}F_{2k+1}}{F_{4k-2}F_{2k-1}F_{2k+1}F_{4k+2}} \\
&> 0, \qquad (2.20)
\end{aligned}$$

which implies that

$$\sum_{k=n}^{mn}\frac{1}{F_{2k-1}F_{2k+1}} < \sum_{k=n}^{mn}\left(\frac{1}{F_{4k-2}}-\frac{1}{F_{4k+2}}\right) = \frac{1}{F_{4n-2}}-\frac{1}{F_{4mn+2}} < \frac{1}{F_{4n-2}}. \qquad (2.21)$$

Invoking (2.20) and Lemma 2.9, we have

$$\begin{aligned}
&\frac{1}{F_{4k-2}+1}-\frac{1}{F_{2k-1}F_{2k+1}}-\frac{1}{F_{4k+2}+1} \\
&\quad= \frac{F_{2k-1}F_{2k+1}(F_{4k+2}-F_{4k-2})-F_{4k-2}F_{4k+2}}{(F_{4k-2}+1)F_{2k-1}F_{2k+1}(F_{4k+2}+1)} \\
&\qquad - \frac{F_{4k-2}+F_{4k+2}+1}{(F_{4k-2}+1)F_{2k-1}F_{2k+1}(F_{4k+2}+1)} \\
&\quad= \frac{3F_{2k-1}F_{2k+1}-(F_{2k+2}^2-F_{2k-2}^2)-1}{(F_{4k-2}+1)F_{2k-1}F_{2k+1}(F_{4k+2}+1)} \\
&\quad< \frac{3F_{2k-1}F_{2k+1}-4F_{2k-1}F_{2k+1}-1}{(F_{4k-2}+1)F_{2k-1}F_{2k+1}(F_{4k+2}+1)} \\
&\quad< -\frac{1}{(F_{4k-2}+1)(F_{4k+2}+1)},
\end{aligned}$$

from which we deduce that

$$\begin{aligned}
\sum_{k=n}^{mn}\frac{1}{F_{2k-1}F_{2k+1}} &> \sum_{k=n}^{mn}\left(\frac{1}{F_{4k-2}+1}-\frac{1}{F_{4k+2}+1}\right)+\sum_{k=n}^{mn}\frac{1}{(F_{4k-2}+1)(F_{4k+2}+1)} \\
&= \frac{1}{F_{4n-2}+1}-\frac{1}{F_{4mn+2}+1}+\sum_{k=n}^{mn}\frac{1}{(F_{4k-2}+1)(F_{4k+2}+1)} \\
&> \frac{1}{F_{4n-2}+1}+\frac{1}{(F_{4n-2}+1)(F_{4n+2}+1)}-\frac{1}{F_{8n+2}+1} \\
&> \frac{1}{F_{4n-2}+1}, \qquad (2.22)
\end{aligned}$$

where the last inequality follows from Lemma 2.10.

Combining (2.21) and (2.22), we obtain

$$\frac{1}{F_{4n-2}+1} < \sum_{k=n}^{mn} \frac{1}{F_{2k-1}F_{2k+1}} < \frac{1}{F_{4n-2}},$$

from which the desired result follows. □

Similarly, we can obtain the following result, whose proof is omitted here.

Theorem 2.12 *For all $n \geq 1$ and $m \geq 2$,*

$$\left\lfloor \left(\sum_{k=n}^{mn} \frac{1}{F_{2k}F_{2k+2}} \right)^{-1} \right\rfloor = F_{4n} - 1.$$

3 Main results II: alternating reciprocal sums

In this section, we extend the analysis of the sums of the products of two reciprocal Fibonacci numbers to alternating sums.

3.1 Alternating reciprocal sum of F_kF_{k+1}

Lemma 3.1 *For $n \geq 1$, we have*

$$\frac{F_{n+1}}{F_n} - \frac{1}{F_{2n}} = \frac{F_{2n+1} + (-1)^n - 1}{F_{2n}}, \tag{3.1}$$

$$\frac{F_{n+1}}{F_n} + \frac{1}{F_{2n}} = \frac{F_{2n+1} + (-1)^n + 1}{F_{2n}}. \tag{3.2}$$

Proof Applying (2.1) repeatedly and (2.4), we derive that

$$\begin{aligned}
\frac{F_{n+1}}{F_n} - \frac{1}{F_{2n}} &= \frac{F_{n+1}F_{2n} - F_n}{F_nF_{2n}} \\
&= \frac{F_{n+1}(F_{n-1}F_n + F_nF_{n+1}) - F_n}{F_nF_{2n}} \\
&= \frac{F_{n+1}F_{n-1} + F_{n+1}^2 - 1}{F_{2n}} \\
&= \frac{(F_{n+1}F_{n-1} - F_n^2) + (F_n^2 + F_{n+1}^2) - 1}{F_{2n}} \\
&= \frac{F_{2n+1} + (-1)^n - 1}{F_{2n}}.
\end{aligned}$$

Then (3.2) immediately follows from (3.1). □

Lemma 3.2 *If $n \geq 2$, then*

$$\frac{1}{F_{2n}-1} - \frac{F_{n+1}}{F_nF_{2n+1}} > 0, \tag{3.3}$$

$$\frac{1}{F_{2n}+1} - \frac{F_{n+1}}{F_nF_{2n+1}} < 0. \tag{3.4}$$

Proof It follows from (2.4) that

$$\frac{1}{F_{2n}-1}-\frac{F_{n+1}}{F_nF_{2n+1}}=\frac{F_nF_{2n+1}-F_{n+1}F_{2n}+F_{n+1}}{(F_{2n}-1)F_nF_{2n+1}}=\frac{(-1)^{n+1}F_n+F_{n+1}}{(F_{2n}-1)F_nF_{2n+1}}>0.$$

Similarly, we can prove (3.4). □

Lemma 3.3 *For $n\geq 1$ and $m\geq 2$, we have*

$$\frac{F_{2n+1}}{F_{2n}}-\frac{F_{mn+2}}{F_{mn+1}}>0, \tag{3.5}$$

$$\frac{F_{2n+2}}{F_{2n+1}}-\frac{F_{mn+2}}{F_{mn+1}}\leq 0. \tag{3.6}$$

Proof With the help of (2.4), we see that

$$\frac{F_{2n+1}}{F_{2n}}-\frac{F_{mn+2}}{F_{mn+1}}=\frac{F_{2n+1}F_{mn+1}-F_{2n}F_{mn+2}}{F_{2n}F_{mn+1}}=(-1)^{2n+2}\frac{F_{(m-2)n+1}}{F_{2n}F_{mn+1}}>0.$$

A similar analysis yields (3.6). □

Theorem 3.4 *If $m\geq 2$ and $n\geq 1$, then*

$$\left\lfloor\left(\sum_{k=n}^{mn}\frac{(-1)^k}{F_kF_{k+1}}\right)^{-1}\right\rfloor=\begin{cases}F_{2n}-1 & \text{if } n \text{ is even};\\ -F_{2n}-1 & \text{if } n \text{ is odd}.\end{cases}$$

Proof Employing (2.4), we derive that

$$\frac{(-1)^k}{F_kF_{k+1}}=\frac{F_{k+1}^2-F_kF_{k+2}}{F_kF_{k+1}}=\frac{F_{k+1}}{F_k}-\frac{F_{k+2}}{F_{k+1}},$$

which implies that

$$\sum_{k=n}^{mn}\frac{(-1)^k}{F_kF_{k+1}}=\frac{F_{n+1}}{F_n}-\frac{F_{mn+2}}{F_{mn+1}}. \tag{3.7}$$

Furthermore, it follows from (3.6) and (2.4) that

$$\frac{F_{n+1}}{F_n}-\frac{F_{mn+2}}{F_{mn+1}}\leq\frac{F_{n+1}}{F_n}-\frac{F_{2n+2}}{F_{2n+1}}=\frac{F_{n+1}F_{2n+1}-F_nF_{2n+2}}{F_nF_{2n+1}}=(-1)^n\frac{F_{n+1}}{F_nF_{2n+1}}. \tag{3.8}$$

We first assume that n is even. Then combining (3.8) and (3.3), we obtain

$$\frac{F_{n+1}}{F_n}-\frac{F_{mn+2}}{F_{mn+1}}\leq\frac{F_{n+1}}{F_nF_{2n+1}}<\frac{1}{F_{2n}-1}. \tag{3.9}$$

On the other hand, it follows from (3.1) and (3.5) that

$$\frac{F_{n+1}}{F_n}-\frac{1}{F_{2n}}-\frac{F_{mn+2}}{F_{mn+1}}>0,$$

which means that

$$\frac{F_{n+1}}{F_n} - \frac{F_{mn+2}}{F_{mn+1}} > \frac{1}{F_{2n}}. \tag{3.10}$$

Now combining (3.7), (3.9), and (3.10), we deduce that

$$\frac{1}{F_{2n}} < \sum_{k=n}^{mn} \frac{(-1)^k}{F_k F_{k+1}} < \frac{1}{F_{2n} - 1},$$

which shows that the statement is true when n is even.

We now consider the case where n is odd. It is clearly true for $n = 1$, so we assume that $n \geq 3$. Applying (3.8) and (3.4), we can see that

$$\frac{F_{n+1}}{F_n} - \frac{F_{mn+2}}{F_{mn+1}} \leq -\frac{F_{n+1}}{F_n F_{2n+1}} < -\frac{1}{F_{2n} + 1}. \tag{3.11}$$

It follows from (3.2) and (3.5) that

$$\frac{F_{n+1}}{F_n} + \frac{1}{F_{2n}} - \frac{F_{mn+2}}{F_{mn+1}} = \frac{F_{2n+1}}{F_{2n}} - \frac{F_{mn+2}}{F_{mn+1}} > 0,$$

which means that

$$\frac{F_{n+1}}{F_n} - \frac{F_{mn+2}}{F_{mn+1}} > -\frac{1}{F_{2n}}. \tag{3.12}$$

Combining (3.7), (3.11), and (3.12), we get that

$$-\frac{1}{F_{2n}} < \sum_{k=n}^{mn} \frac{(-1)^k}{F_k F_{k+1}} < -\frac{1}{F_{2n} + 1},$$

which yields the desired identity. □

3.2 Alternating reciprocal sums of $F_{2k-1}F_{2k}$

For $n \geq 2$, we define

$$\begin{aligned}
f(n) &= \frac{1}{3F_{2n-2}F_{2n-1}} - \frac{(-1)^n}{F_{2n-1}F_{2n}} - \frac{1}{3F_{2n}F_{2n+1}}, \\
g(n) &= \frac{1}{3F_{2n-2}F_{2n-1} + 1} - \frac{(-1)^n}{F_{2n-1}F_{2n}} - \frac{1}{3F_{2n}F_{2n+1} + 1}, \\
s(n) &= \frac{-1}{3F_{2n-2}F_{2n-1} + 1} - \frac{(-1)^n}{F_{2n-1}F_{2n}} + \frac{1}{3F_{2n}F_{2n+1} + 1}, \\
t(n) &= \frac{-1}{3F_{2n-2}F_{2n-1}} - \frac{(-1)^n}{F_{2n-1}F_{2n}} + \frac{1}{3F_{2n}F_{2n+1}}.
\end{aligned}$$

It is not hard to check that $f(n)$, $g(n)$, $s(n)$, and $t(n)$ are all negative if n is even and positive otherwise.

Lemma 3.5 *For $n \geq 2$, we have*

$$f(n) + f(n+1) > 0. \tag{3.13}$$

Proof The statement is clearly true when n is odd, so we assume that n is even in the rest of the proof. Applying (2.3), we derive that

$$\begin{aligned}
f(n) + f(n+1) &= \frac{1}{3F_{2n-2}F_{2n-1}} - \frac{1}{F_{2n-1}F_{2n}} + \frac{1}{F_{2n+1}F_{2n+2}} - \frac{1}{3F_{2n+2}F_{2n+3}} \\
&= \frac{1}{3}\left(\frac{1}{F_{2n-2}F_{2n-1}} - \frac{1}{F_{2n+2}F_{2n+3}}\right) - \left(\frac{1}{F_{2n-1}F_{2n}} - \frac{1}{F_{2n+1}F_{2n+2}}\right) \\
&= \frac{1}{3}\frac{F_{2n+2}F_{2n+3} - F_{2n-2}F_{2n-1}}{F_{2n-2}F_{2n-1}F_{2n+2}F_{2n+3}} - \frac{F_{2n+1}F_{2n+2} - F_{2n-1}F_{2n}}{F_{2n-1}F_{2n}F_{2n+1}F_{2n+2}} \\
&= \frac{1}{3}\frac{F_{2n+2}F_{2n+3} - F_{2n}F_{2n+1} + F_{2n}F_{2n+1} - F_{2n-2}F_{2n-1}}{F_{2n-2}F_{2n-1}F_{2n+2}F_{2n+3}} \\
&\quad - \frac{F_{2n+1}F_{2n+2} - F_{2n-1}F_{2n}}{F_{2n-1}F_{2n}F_{2n+1}F_{2n+2}} \\
&= \frac{1}{3}\frac{F_{4n+3} + F_{4n-1}}{F_{2n-2}F_{2n-1}F_{2n+2}F_{2n+3}} - \frac{F_{4n+1}}{F_{2n-1}F_{2n}F_{2n+1}F_{2n+2}} \\
&= \frac{1}{3}\frac{3F_{4n+1}}{F_{2n-2}F_{2n-1}F_{2n+2}F_{2n+3}} - \frac{F_{4n+1}}{F_{2n-1}F_{2n}F_{2n+1}F_{2n+2}} \\
&= \frac{F_{4n+1}}{F_{2n-1}F_{2n+2}}\left(\frac{1}{F_{2n-2}F_{2n+3}} - \frac{1}{F_{2n}F_{2n+1}}\right) \\
&= \frac{F_{4n+1}}{F_{2n-1}F_{2n+2}} \cdot \frac{F_{2n}F_{2n+1} - F_{2n-2}F_{2n+3}}{F_{2n-2}F_{2n}F_{2n+1}F_{2n+3}} \\
&= \frac{F_{4n+1}}{F_{2n-1}F_{2n+2}} \cdot \frac{2}{F_{2n-2}F_{2n}F_{2n+1}F_{2n+3}} \\
&> 0,
\end{aligned}$$

where the last equality follows from (2.4). □

Remark From the proof of Lemma 3.5 we can easily derive that if n is odd, then

$$\begin{aligned}
f(n) + f(n+1) &= \frac{F_{4n+1}}{F_{2n-1}F_{2n+2}}\left(\frac{1}{F_{2n-2}F_{2n+3}} + \frac{1}{F_{2n}F_{2n+1}}\right) \\
&> \frac{F_{4n+1}}{F_{2n-1}F_{2n+2}} \cdot \frac{2}{F_{2n-2}F_{2n}F_{2n+1}F_{2n+3}}.
\end{aligned}$$

Therefore, whether n is even or odd, we always have

$$f(n) + f(n+1) \geq \frac{F_{4n+1}}{F_{2n-1}F_{2n+2}} \cdot \frac{2}{F_{2n-2}F_{2n}F_{2n+1}F_{2n+3}} > \frac{2}{F_{2n-2}F_{2n}F_{2n+1}F_{2n+3}}. \tag{3.14}$$

Lemma 3.6 *If $n \geq 2$ and $m \geq 2$, we have*

$$f(n) + f(n+1) + f(mn) > 0.$$

Proof If mn is odd, then the result follows from (3.13) and the fact $f(mn) > 0$. Now we consider the case where mn is even. It is straightforward to check that

$$\begin{aligned}
f(mn) &= \frac{1}{3F_{2mn-2}F_{2mn-1}} - \frac{1}{F_{2mn-1}F_{2mn}} - \frac{1}{3F_{2mn}F_{2mn+1}} \\
&= \frac{F_{2mn}F_{2mn+1} - F_{2mn-2}F_{2mn-1} - 3F_{2mn-2}F_{2mn+1}}{3F_{2mn-2}F_{2mn-1}F_{2mn}F_{2mn+1}} \\
&= \frac{F_{2mn}(F_{2mn} + F_{2mn-1}) - (F_{2mn} - F_{2mn-1})F_{2mn-1} - 3F_{2mn-2}F_{2mn+1}}{3F_{2mn-2}F_{2mn-1}F_{2mn}F_{2mn+1}} \\
&= \frac{F_{2mn}^2 + F_{2mn-1}^2 - 3(F_{2mn} - F_{2mn-1})(F_{2mn} + F_{2mn-1})}{3F_{2mn-2}F_{2mn-1}F_{2mn}F_{2mn+1}} \\
&= \frac{4F_{2mn-1}^2 - 2F_{2mn}^2}{3F_{2mn-2}F_{2mn-1}F_{2mn}F_{2mn+1}}.
\end{aligned}$$

Invoking (2.4), we get

$$\begin{aligned}
F_{2mn}^2 - 2F_{2mn-1}^2 &= (F_{2mn-1} + F_{2mn-2})^2 - 2F_{2mn-1}^2 \\
&= 2F_{2mn-2}F_{2mn-1} + F_{2mn-2}^2 - F_{2mn-1}^2 \\
&= 2F_{2mn-2}F_{2mn-1} - F_{2mn-3}F_{2mn} \\
&= F_{2mn-2}F_{2mn-1} - (F_{2mn-3}F_{2mn} - F_{2mn-2}F_{2mn-1}) \\
&= F_{2mn-2}F_{2mn-1} - (-1)^{2mn-2} \\
&= F_{2mn-2}F_{2mn-1} - 1.
\end{aligned}$$

Therefore, we have

$$f(mn) = \frac{-2(F_{2mn-2}F_{2mn-1} - 1)}{3F_{2mn-2}F_{2mn-1}F_{2mn}F_{2mn+1}} > \frac{-2}{3F_{2mn}F_{2mn+1}}. \tag{3.15}$$

Combining (3.14) and (3.15), we obtain

$$\begin{aligned}
f(n) + f(n+1) + f(mn) &> \frac{2}{F_{2n-2}F_{2n}F_{2n+1}F_{2n+3}} - \frac{2}{3F_{2mn}F_{2mn+1}} \\
&\geq \frac{2}{F_{2n}F_{2n+1}F_{2n-2}F_{2n+3}} - \frac{2}{3F_{4n}F_{4n+1}}.
\end{aligned}$$

Since $F_{4n+1} > F_{4n}$ and

$$F_{4n} = F_{2n-1}F_{2n} + F_{2n}F_{2n+1} = F_{2n-3}F_{2n+2} + F_{2n-2}F_{2n+3},$$

we determine that

$$f(n) + f(n+1) + f(mn) > 0.$$

The proof is completed. □

Lemma 3.7 *If $n \geq 2$ is even, we have*

$$g(n) + g(n+1) < 0.$$

Proof From the proof of Lemma 3.5 we know that

$$\begin{aligned} g(n) + g(n+1) &= \frac{1}{3F_{2n-2}F_{2n-1}+1} - \frac{1}{F_{2n-1}F_{2n}} + \frac{1}{F_{2n+1}F_{2n+2}} - \frac{1}{3F_{2n+2}F_{2n+3}+1} \\ &= \frac{3(F_{2n+2}F_{2n+3} - F_{2n-2}F_{2n-1})}{(3F_{2n-2}F_{2n-1}+1)(3F_{2n+2}F_{2n+3}+1)} - \left(\frac{1}{F_{2n-1}F_{2n}} - \frac{1}{F_{2n+1}F_{2n+2}}\right) \\ &= \frac{9F_{4n+1}}{(3F_{2n-2}F_{2n-1}+1)(3F_{2n+2}F_{2n+3}+1)} - \frac{9F_{4n+1}}{9F_{2n-1}F_{2n}F_{2n+1}F_{2n+2}}. \end{aligned}$$

For $n \geq 2$, we have

$$\begin{aligned} &(3F_{2n-2}F_{2n-1}+1)(3F_{2n+2}F_{2n+3}+1) \\ &\quad > 9F_{2n-2}F_{2n-1}F_{2n+2}F_{2n+3} + 3F_{2n+2}F_{2n+3} \\ &\quad = 9F_{2n-1}F_{2n+2}(F_{2n-2}F_{2n+3} - F_{2n}F_{2n+1}) \\ &\qquad + 3F_{2n+2}F_{2n+3} + 9F_{2n-1}F_{2n}F_{2n+1}F_{2n+2} \\ &\quad = 9F_{2n-1}F_{2n}F_{2n+1}F_{2n+2} + 3F_{2n+2}F_{2n+3} - 18F_{2n-1}F_{2n+2} \\ &\quad = 9F_{2n-1}F_{2n}F_{2n+1}F_{2n+2} + 3F_{2n+2}(F_{2n+3} - 6F_{2n-1}) \\ &\quad = 9F_{2n-1}F_{2n}F_{2n+1}F_{2n+2} + 3F_{2n+2}(3F_{2n-2} - F_{2n-1}) \\ &\quad > 9F_{2n-1}F_{2n}F_{2n+1}F_{2n+2}, \end{aligned}$$

which implies

$$g(n) + g(n+1) < 0.$$

This completes the proof. □

Lemma 3.8 *If $n > 0$ is even, then*

$$g(n) + \frac{1}{3F_{2n}F_{2n+1}+1} < 0.$$

Proof The result follows from the definition of $g(n)$ and the fact $3F_n > F_{n+2}$. □

To introduce the property of $s(n)$, we need two preliminary results.

Lemma 3.9 *If $n \geq 5$, then*

$$F_{2n}F_{2n+1} > 3F_{n-1}F_nF_{n+1}F_{n+2}.$$

Proof It is easy to see that $2F_n > F_{n+1}$ for $n \geq 3$. We claim that $5F_n \geq 3F_{n+1}$ if $n \geq 3$. First, the claim holds for $n = 3$ and $n = 4$. Now we assume that $n \geq 5$. It is straightforward to verify that

$$5F_n = 3F_n + 2F_n = 3F_n + 2F_{n-1} + 2F_{n-2} > 3F_n + 2F_{n-1} + F_{n-1} = 3F_{n+1}.$$

Since $5F_n \geq 3F_{n+1}$ and $2F_{n+1} > F_{n+2}$, we have $5F_n > F_{n+3}$ for $n \geq 3$.

It follows from (2.1) that $F_{2n} = F_{n-1}F_n + F_nF_{n+1}$ and

$$F_{2n+1} = F_{n-2}F_{n+2} + F_{n-1}F_{n+3} = F_{n-2}F_{n+2} + F_{n-1}F_{n+2} + F_{n-1}F_{n+1},$$

from which we derive that

$$\begin{aligned}
\frac{F_{2n}F_{2n+1}}{F_{n-1}F_nF_{n+1}F_{n+2}} &= \frac{F_{2n}}{F_nF_{n+1}} \cdot \frac{F_{2n+1}}{F_{n-1}F_{n+2}} \\
&= \left(1 + \frac{F_{n-1}}{F_{n+1}}\right)\left(1 + \frac{F_{n-2}}{F_{n-1}} + \frac{F_{n+1}}{F_{n+2}}\right) \\
&= 1 + \frac{F_{n-2}}{F_{n-1}} + \frac{F_{n+1}}{F_{n+2}} + \frac{F_{n-1}}{F_{n+1}} + \frac{F_{n-2}}{F_{n+1}} + \frac{F_{n-1}}{F_{n+2}} \\
&= 1 + \frac{F_{n-2}}{F_{n-1}} + \frac{F_{n+1}}{F_{n+2}} + \frac{F_n}{F_{n+1}} + \frac{F_{n-1}}{F_{n+2}} \\
&> 1 + \frac{3}{5} + \frac{3}{5} + \frac{3}{5} + \frac{1}{5} \\
&= 3,
\end{aligned}$$

which completes the proof. □

Lemma 3.10 *For $n \geq 5$, we have*

$$F_{2n+1}(F_{n+3} - 6F_{n-1}) > (F_{n-2}F_{n-1} + 1)F_{n+3}.$$

Proof It is easy to see that $F_{n-2}^2F_{n+2} > F_{n+3}$ for $n \geq 5$, and thus we have

$$\begin{aligned}
F_{2n+1}F_{n-2} &= (F_{n-2}F_{n+2} + F_{n-1}F_{n+3})F_{n-2} \\
&= F_{n-2}^2F_{n+2} + F_{n-2}F_{n-1}F_{n+3} \\
&> F_{n+3} + F_{n-2}F_{n-1}F_{n+3} \\
&= (F_{n-2}F_{n-1} + 1)F_{n+3}.
\end{aligned}$$

It is straightforward to check that $F_{n+3} = 5F_{n-1} + 3F_{n-2}$, from which we get

$$F_{n+3} - 6F_{n-1} = 3F_{n-2} - F_{n-1} > F_{n-2}.$$

Combining the last two inequalities yields the desired result. □

Lemma 3.11 *If $n \geq 3$ is odd, then*

$$s(n) + s(n+1) - \frac{1}{3F_{4n}F_{4n+1} + 1} > 0.$$

Proof Since n is odd, we have

$$\begin{aligned}
s(n)+s(n+1) &= \frac{-1}{3F_{2n-2}F_{2n-1}+1}+\frac{1}{F_{2n-1}F_{2n}}-\frac{1}{F_{2n+1}F_{2n+2}}+\frac{1}{3F_{2n+2}F_{2n+3}+1}\\
&= \left(\frac{1}{F_{2n-1}F_{2n}}-\frac{1}{F_{2n+1}F_{2n+2}}\right)-\left(\frac{1}{3F_{2n-2}F_{2n-1}+1}-\frac{1}{3F_{2n+2}F_{2n+3}+1}\right)\\
&= \frac{F_{4n+1}}{F_{2n-1}F_{2n}F_{2n+1}F_{2n+2}}-\frac{9F_{4n+1}}{(3F_{2n-2}F_{2n-1}+1)(3F_{2n+2}F_{2n+3}+1)}\\
&> \frac{F_{4n+1}}{F_{2n-1}F_{2n}F_{2n+1}F_{2n+2}}-\frac{3F_{4n+1}}{(3F_{2n-2}F_{2n-1}+1)F_{2n+2}F_{2n+3}}\\
&= \frac{1}{F_{2n-1}F_{2n}F_{2n+1}F_{2n+2}}\cdot\frac{F_{4n+1}(F_{2n+3}-6F_{2n-1})}{(3F_{2n-2}F_{2n-1}+1)F_{2n+3}}\\
&> \frac{1}{F_{2n-1}F_{2n}F_{2n+1}F_{2n+2}}\cdot\frac{F_{4n+1}(F_{2n+3}-6F_{2n-1})}{(3F_{2n-2}F_{2n-1}+3)F_{2n+3}}\\
&> \frac{1}{3F_{2n-1}F_{2n}F_{2n+1}F_{2n+2}},
\end{aligned}$$

where the last inequality follows from Lemma 3.10.

Applying Lemma 3.9, we have

$$3F_{4n}F_{4n+1}+1>3F_{4n}F_{4n+1}>9F_{2n-1}F_{2n}F_{2n+1}F_{2n+2},$$

which implies that

$$\begin{aligned}
s(n)+s(n+1)-\frac{1}{3F_{4n}F_{4n+1}+1} &> s(n)+s(n+1)-\frac{1}{9F_{2n-1}F_{2n}F_{2n+1}F_{2n+2}}\\
&> \frac{1}{3F_{2n-1}F_{2n}F_{2n+1}F_{2n+2}}-\frac{1}{9F_{2n-1}F_{2n}F_{2n+1}F_{2n+2}}\\
&> 0,
\end{aligned}$$

which completes the proof. □

Applying a similar analysis of $f(n)$, we can obtain the following properties of $t(n)$, whose proofs are omitted here.

Lemma 3.12 *For* $n \geq 2$,

$$t(n)+t(n+1)<0.$$

Lemma 3.13 *If* $n \geq 2$ *and* $m \geq 2$, *we have*

$$t(n)+t(n+1)+t(mn)<0.$$

Theorem 3.14 *If* $m \geq 2$ *and* $n \geq 3$, *then*

$$\left\lfloor \left(\sum_{k=n}^{mn}\frac{(-1)^k}{F_{2k-1}F_{2k}}\right)^{-1}\right\rfloor = \begin{cases} 3F_{2n-2}F_{2n-1} & \text{if } n \text{ is even};\\ -3F_{2n-2}F_{2n-1}-1 & \text{if } n \text{ is odd}.\end{cases}$$

Proof We first consider the case where n is even. With the help of $f(n)$, we have

$$\sum_{k=n}^{mn} \frac{(-1)^k}{F_{2k-1}F_{2k}} = \frac{1}{3F_{2n-2}F_{2n-1}} - \frac{1}{3F_{2mn}F_{2mn+1}} - \sum_{k=n}^{mn} f(k).$$

Lemma 3.5 implies that

$$\sum_{k=n+2}^{mn-1} f(k) > 0.$$

Furthermore, applying Lemma 3.6, we get

$$\sum_{k=n}^{mn} f(k) = f(n) + f(n+1) + f(mn) + \sum_{k=n+2}^{mn-1} f(k) > 0.$$

Hence, we obtain

$$\sum_{k=n}^{mn} \frac{(-1)^k}{F_{2k-1}F_{2k}} < \frac{1}{3F_{2n-2}F_{2n-1}}. \tag{3.16}$$

It follows from Lemma 3.7 and Lemma 3.8 that

$$\begin{aligned}
\sum_{k=n}^{mn} \frac{(-1)^k}{F_{2k-1}F_{2k}} &= \frac{1}{3F_{2n-2}F_{2n-1}+1} - \frac{1}{3F_{2mn}F_{2mn+1}+1} - \sum_{k=n}^{mn} g(k) \\
&= \frac{1}{3F_{2n-2}F_{2n-1}+1} - \sum_{k=n}^{mn-1} g(k) - \left(g(mn) + \frac{1}{3F_{2mn}F_{2mn+1}+1} \right) \\
&> \frac{1}{3F_{2n-2}F_{2n-1}+1}.
\end{aligned} \tag{3.17}$$

Combining (3.16) and (3.17) yields

$$\frac{1}{3F_{2n-2}F_{2n-1}+1} < \sum_{k=n}^{mn} \frac{(-1)^k}{F_{2k-1}F_{2k}} < \frac{1}{3F_{2n-2}F_{2n-1}},$$

which shows that the statement is true when n is even.

Next, we turn to the case where n is odd. It follows from Lemma 3.11 that

$$\begin{aligned}
\sum_{k=n}^{mn} \frac{(-1)^k}{F_{2k-1}F_{2k}} &= \frac{-1}{3F_{2n-2}F_{2n-1}+1} + \frac{1}{3F_{2mn}F_{2mn+1}+1} - \sum_{k=n}^{mn} s(k) \\
&= \frac{-1}{3F_{2n-2}F_{2n-1}+1} - \left(s(n) + s(n+1) - \frac{1}{3F_{2mn}F_{2mn+1}+1} \right) - \sum_{k=n+2}^{mn} s(k) \\
&< \frac{-1}{3F_{2n-2}F_{2n-1}+1} - \left(s(n) + s(n+1) - \frac{1}{3F_{4n}F_{4n+1}+1} \right) - \sum_{k=n+2}^{mn} s(k) \\
&< \frac{-1}{3F_{2n-2}F_{2n-1}+1}.
\end{aligned} \tag{3.18}$$

If mn is even, then it follows from Lemma 3.12 that $\sum_{k=n}^{mn} t(k) < 0$, and hence

$$\begin{aligned}\sum_{k=n}^{mn} \frac{(-1)^k}{F_{2k-1}F_{2k}} &= \frac{-1}{3F_{2n-2}F_{2n-1}} + \frac{1}{3F_{2mn}F_{2mn+1}} - \sum_{k=n}^{mn} t(k) \\ &> \frac{-1}{3F_{2n-2}F_{2n-1}}.\end{aligned}$$

If mn is odd, then it follows from Lemma 3.13 that

$$\begin{aligned}\sum_{k=n}^{mn} \frac{(-1)^k}{F_{2k-1}F_{2k}} &= \frac{-1}{3F_{2n-2}F_{2n-1}} + \frac{1}{3F_{2mn}F_{2mn+1}} - \sum_{k=n}^{mn} t(k) \\ &= \frac{-1}{3F_{2n-2}F_{2n-1}} + \frac{1}{3F_{2mn}F_{2mn+1}} - \sum_{k=n+2}^{mn-1} t(k) \\ &\quad - \big(t(n) + t(n+1) + t(mn)\big) \\ &> \frac{-1}{3F_{2n-2}F_{2n-1}}.\end{aligned}$$

Therefore, if n is odd, then we always have

$$\sum_{k=n}^{mn} \frac{(-1)^k}{F_{2k-1}F_{2k}} > \frac{-1}{3F_{2n-2}F_{2n-1}}. \tag{3.19}$$

It follows from (3.18) and (3.19) that

$$\frac{-1}{3F_{2n-2}F_{2n-1}} < \sum_{k=n}^{mn} \frac{(-1)^k}{F_{2k-1}F_{2k}} < \frac{-1}{3F_{2n-2}F_{2n-1}+1},$$

which shows that the assertion for odd n also holds. □

Similarly, we can consider the alternating reciprocal sums of $F_{2k}F_{2k+1}$ and obtain the following result, whose proof is similar to that of Theorem 3.14 and is omitted here.

Theorem 3.15 *If $m \geq 2$ and $n \geq 2$, then*

$$\left\lfloor \left(\sum_{k=n}^{mn} \frac{(-1)^k}{F_{2k}F_{2k+1}} \right)^{-1} \right\rfloor = \begin{cases} 3F_{2n-1}F_{2n} - 1 & \text{if } n \text{ is even;} \\ -3F_{2n-1}F_{2n}, & \text{if } n \text{ is odd.} \end{cases}$$

3.3 Alternating sums of $F_{2k-1}F_{2k+1}$

We first introduce the following notation:

$$\begin{aligned}\alpha(n) &= \frac{1}{3F_{2n-1}^2} - \frac{(-1)^n}{F_{2n-1}F_{2n+1}} - \frac{1}{3F_{2n+1}^2}, \\ \beta(n) &= \frac{1}{3F_{2n-1}^2 - 1} - \frac{(-1)^n}{F_{2n-1}F_{2n+1}} - \frac{1}{3F_{2n+1}^2 - 1}, \\ \gamma(n) &= \frac{-1}{3F_{2n-1}^2 - 1} - \frac{(-1)^n}{F_{2n-1}F_{2n+1}} + \frac{1}{3F_{2n+1}^2 - 1},\end{aligned}$$

$$\delta(n) = \frac{-1}{3F_{2n-1}^2} - \frac{(-1)^n}{F_{2n-1}F_{2n+1}} + \frac{1}{3F_{2n+1}^2}.$$

It is not hard to check that $\alpha(n)$, $\beta(n)$, $\gamma(n)$, and $\delta(n)$ are all negative if n is even and positive otherwise.

Lemma 3.16 *If $n > 0$ is even, then*

$$\alpha(n) + \alpha(n+1) < 0.$$

Proof Since n is even, we have

$$\begin{aligned}
\alpha(n) + \alpha(n+1) &= \frac{1}{3F_{2n-1}^2} - \frac{1}{F_{2n-1}F_{2n+1}} + \frac{1}{F_{2n+1}F_{2n+3}} - \frac{1}{3F_{2n+3}^2} \\
&= \frac{1}{3}\left(\frac{1}{F_{2n-1}^2} - \frac{1}{F_{2n+3}^2}\right) - \frac{1}{F_{2n+1}}\left(\frac{1}{F_{2n-1}} - \frac{1}{F_{2n+3}}\right) \\
&= \frac{1}{3}\left(\frac{1}{F_{2n-1}} - \frac{1}{F_{2n+3}}\right)\left(\frac{1}{F_{2n-1}} + \frac{1}{F_{2n+3}} - \frac{3}{F_{2n+1}}\right) \\
&= \frac{1}{3}\left(\frac{1}{F_{2n-1}} - \frac{1}{F_{2n+3}}\right)\left(\frac{1}{F_{2n+3}} - \frac{F_{2n-3}}{F_{2n-1}F_{2n+1}}\right) \\
&= \frac{1}{3}\left(\frac{1}{F_{2n-1}} - \frac{1}{F_{2n+3}}\right)\left(\frac{F_{2n-1}F_{2n+1} - F_{2n-3}F_{2n+3}}{F_{2n-1}F_{2n+1}F_{2n+3}}\right) \\
&= -\left(\frac{1}{F_{2n-1}} - \frac{1}{F_{2n+3}}\right)\frac{1}{F_{2n-1}F_{2n+1}F_{2n+3}} \\
&< 0,
\end{aligned}$$

where the last equality follows from (2.4). □

Lemma 3.17 *For $n > 0$,*

$$6F_{2n-1}F_{2n+1} > \left(3F_{n-1}^2 - 1\right)F_{n+3}^2.$$

Proof It is easy to see that the result holds when $n < 5$. Next we show that, for $n \geq 5$,

$$2F_{2n-1}F_{2n+1} > F_{2n}F_{2n+1} > F_{n-1}^2F_{n+3}^2,$$

from which the desired result follows.

The first inequality is obvious. It follows from (2.1) that

$$\begin{aligned}
F_{2n} &= F_{n-2}F_{n+3} + F_{n-3}F_{n+2}, \\
F_{2n+1} &= F_{n-1}F_{n+3} + F_{n-2}F_{n+2},
\end{aligned}$$

which implies that

$$\begin{aligned}
F_{2n}F_{2n+1} &= F_{n-2}F_{n-1}F_{n+3}^2 + F_{n-2}^2F_{n+2}F_{n+3} + F_{n-3}F_{n-1}F_{n+2}F_{n+3} + F_{n-3}F_{n-2}F_{n+2}^2 \\
&= F_{n-2}F_{n-1}F_{n+3}^2 + \left(F_{n-3}F_{n-1} - (-1)^n\right)F_{n+2}F_{n+3} + F_{n-3}F_{n-1}F_{n+2}F_{n+3}
\end{aligned}$$

$$\begin{aligned}
&\quad + F_{n-3}F_{n-2}F_{n+2}^2 \\
&= F_{n-2}F_{n-1}F_{n+3}^2 + 2F_{n-3}F_{n-1}F_{n+2}F_{n+3} + F_{n-3}F_{n-2}F_{n+2}^2 - (-1)^n F_{n+2}F_{n+3} \\
&> F_{n-2}F_{n-1}F_{n+3}^2 + F_{n-3}F_{n-1}F_{n+3}^2 + F_{n-3}F_{n-2}F_{n+2}^2 - (-1)^n F_{n+2}F_{n+3} \\
&= F_{n-1}^2 F_{n+3}^2 + F_{n-3}F_{n-2}F_{n+2}^2 - (-1)^n F_{n+2}F_{n+3} \\
&> F_{n-1}^2 F_{n+3}^2,
\end{aligned}$$

where the last inequality follows from the fact that, for $n \geq 5$,

$$F_{n-3}F_{n-2}F_{n+2} \geq 2F_{n+2} > F_{n+3}.$$

This completes the proof. □

Lemma 3.18 *For $n \geq 1$,*

$$\beta(n) + \beta(n+1) > 0.$$

Proof It is obviously true when n is odd, so we assume that n is even. Now we have

$$\begin{aligned}
\beta(n) + \beta(n+1) &= \left(\frac{1}{3F_{2n-1}^2 - 1} - \frac{1}{3F_{2n+3}^2 - 1}\right) - \left(\frac{1}{F_{2n-1}F_{2n+1}} - \frac{1}{F_{2n+1}F_{2n+3}}\right) \\
&= \frac{3(F_{2n+3}^2 - F_{2n-1}^2)}{(3F_{2n-1}^2 - 1)(3F_{2n+3}^2 - 1)} - \frac{F_{2n+3} - F_{2n-1}}{F_{2n-1}F_{2n+1}F_{2n+3}} \\
&= \frac{9F_{2n+1}(F_{2n+3} - F_{2n-1})}{(3F_{2n-1}^2 - 1)(3F_{2n+3}^2 - 1)} - \frac{F_{2n+3} - F_{2n-1}}{F_{2n-1}F_{2n+1}F_{2n+3}}.
\end{aligned}$$

Since

$$\begin{aligned}
&\left(3F_{2n-1}^2 - 1\right)\left(3F_{2n+3}^2 - 1\right) \\
&\quad = 9F_{2n-1}^2 F_{2n+3}^2 - 3F_{2n-1}^2 - 3F_{2n+3}^2 + 1 \\
&\quad = 9F_{2n-1}F_{2n+3}\left(F_{2n+1}^2 + 1\right) - 3F_{2n-1}^2 - 3F_{2n+3}^2 + 1 \\
&\quad = 9F_{2n-1}F_{2n+1}^2 F_{2n+3} + 9F_{2n-1}F_{2n+3} - 3F_{2n-1}^2 - 3F_{2n+3}^2 + 1 \\
&\quad < 9F_{2n-1}F_{2n+1}^2 F_{2n+3} + 9F_{2n-1}F_{2n+3} - 3F_{2n+3}^2 \\
&\quad = 9F_{2n-1}F_{2n+1}^2 F_{2n+3} + 3F_{2n+3}(3F_{2n-1} - F_{2n+3}) \\
&\quad = 9F_{2n-1}F_{2n+1}^2 F_{2n+3} - 3F_{2n+3}(2F_{2n-1} + 3F_{2n-2}) \\
&\quad < 9F_{2n-1}F_{2n+1}^2 F_{2n+3},
\end{aligned}$$

we have

$$\beta(n) + \beta(n+1) > 0,$$

which completes the proof. □

Remark From the proof of Lemma 3.18 we can derive that if n is odd, then

$$\begin{aligned}
\beta(n)+\beta(n+1) &= \left(\frac{1}{3F_{2n-1}^2-1}-\frac{1}{3F_{2n+3}^2-1}\right)+\left(\frac{1}{F_{2n-1}F_{2n+1}}-\frac{1}{F_{2n+1}F_{2n+3}}\right) \\
&> \left(\frac{1}{3F_{2n-1}^2-1}-\frac{1}{3F_{2n+3}^2-1}\right)-\left(\frac{1}{F_{2n-1}F_{2n+1}}-\frac{1}{F_{2n+1}F_{2n+3}}\right) \\
&= \frac{9F_{2n+1}(F_{2n+3}-F_{2n-1})}{(3F_{2n-1}^2-1)(3F_{2n+3}^2-1)}-\frac{F_{2n+3}-F_{2n-1}}{F_{2n-1}F_{2n+1}F_{2n+3}} \\
&> \frac{9F_{2n+1}^2}{(3F_{2n-1}^2-1)(3F_{2n+3}^2-1)}-\frac{F_{2n+1}}{F_{2n-1}F_{2n+1}F_{2n+3}} \\
&= \frac{9F_{2n+1}^2}{(3F_{2n-1}^2-1)(3F_{2n+3}^2-1)}-\frac{1}{F_{2n-1}F_{2n+3}} \\
&> \frac{3F_{2n+1}^2}{(3F_{2n-1}^2-1)F_{2n+3}^2}-\frac{1}{F_{2n-1}F_{2n+3}} \\
&= \frac{3F_{2n-1}(F_{2n+1}^2-F_{2n-1}F_{2n+3})+F_{2n+3}}{(3F_{2n-1}^2-1)F_{2n-1}F_{2n+3}^2} \\
&= \frac{F_{2n+3}-3F_{2n-1}}{(3F_{2n-1}^2-1)F_{2n-1}F_{2n+3}^2} \\
&= \frac{2F_{2n}+F_{2n-2}}{(3F_{2n-1}^2-1)F_{2n-1}F_{2n+3}^2}.
\end{aligned}$$

Thus, we have that, for all $n>0$,

$$\beta(n)+\beta(n+1) > \frac{2F_{2n}+F_{2n-2}}{(3F_{2n-1}^2-1)F_{2n-1}F_{2n+3}^2} > \frac{2}{(3F_{2n-1}^2-1)F_{2n+3}^2}. \tag{3.20}$$

Lemma 3.19 *If $n\ge 1$ and $m\ge 2$, we have*

$$\beta(n)+\beta(n+1)+\beta(mn)>0.$$

Proof If mn is odd, then the result follows from the facts $\beta(mn)>0$ and $\beta(n)+\beta(n+1)>0$. Next, we focus ourselves on the case where mn is even. It is easy to see that

$$\begin{aligned}
\beta(mn) &= \frac{1}{3F_{2mn-1}^2-1}-\frac{1}{F_{2mn-1}F_{2mn+1}}-\frac{1}{3F_{2mn+1}^2-1} \\
&= \frac{3F_{2mn+1}^2-3F_{2mn-1}^2}{(3F_{2mn-1}^2-1)(3F_{2mn+1}^2-1)}-\frac{1}{F_{2mn-1}F_{2mn+1}} \\
&> \frac{3F_{2mn+1}^2-3F_{2mn-1}^2}{3F_{2mn-1}^2\cdot 3F_{2mn+1}^2}-\frac{1}{F_{2mn-1}F_{2mn+1}} \\
&= \frac{F_{2mn+1}^2-F_{2mn-1}^2}{3F_{2mn-1}^2F_{2mn+1}^2}-\frac{1}{F_{2mn-1}F_{2mn+1}}.
\end{aligned}$$

Furthermore, since

$$\begin{aligned}
F_{2mn+1}^2-F_{2mn-1}^2 &= F_{2mn+1}(F_{2mn-2}+2F_{2mn-1})-F_{2mn-1}^2 \\
&= 2F_{2mn-1}F_{2mn+1}+F_{2mn-2}(F_{2mn-1}+F_{2mn})-F_{2mn-1}^2
\end{aligned}$$

$$
\begin{aligned}
&= 2F_{2mn-1}F_{2mn+1} + F_{2mn-2}F_{2mn-1} + \left(F_{2mn-2}F_{2mn} - F_{2mn-1}^2\right) \\
&= 2F_{2mn-1}F_{2mn+1} + F_{2mn-2}F_{2mn-1} + (-1)^{2mn-1} \\
&> 2F_{2mn-1}F_{2mn+1},
\end{aligned}
$$

we have

$$
\beta(mn) > \frac{2F_{2mn-1}F_{2mn+1}}{3F_{2mn-1}^2F_{2mn+1}^2} - \frac{1}{F_{2mn-1}F_{2mn+1}} = -\frac{1}{3F_{2mn-1}F_{2mn+1}}.
$$

From (3.20) we see that

$$
\begin{aligned}
\beta(n) + \beta(n+1) + \beta(mn) &> \frac{2}{(3F_{2n-1}^2 - 1)F_{2n+3}^2} - \frac{1}{3F_{2mn-1}F_{2mn+1}} \\
&\geq \frac{2}{(3F_{2n-1}^2 - 1)F_{2n+3}^2} - \frac{1}{3F_{4n-1}F_{4n+1}} \\
&> 0,
\end{aligned}
$$

where the last inequality follows from Lemma 3.17. □

Applying a similar analysis of $\beta(n)$, we can obtain the following properties of $\gamma(n)$, and the details are left as an exercise.

Lemma 3.20 *For $n \geq 1$, we have*

$$
\gamma(n) + \gamma(n+1) < 0.
$$

Lemma 3.21 *If $n \geq 1$ and $m \geq 2$, then*

$$
\gamma(n) + \gamma(n+1) + \gamma(mn) < 0.
$$

Lemma 3.22 *If $n \geq 1$ is odd, then we have*

$$
\delta(n) + \delta(n+1) - \frac{1}{3F_{4n+1}^2} > 0.
$$

Proof Since n is odd, we have

$$
\delta(n) + \delta(n+1) = -\frac{1}{3F_{2n-1}^2} + \frac{1}{F_{2n-1}F_{2n+1}} - \frac{1}{F_{2n+1}F_{2n+3}} + \frac{1}{3F_{2n+3}^2}.
$$

Applying the argument in the proof of Lemma 3.16, we obtain

$$
\delta(n) + \delta(n+1) = \frac{F_{2n+3} - F_{2n-1}}{F_{2n-1}^2F_{2n+1}F_{2n+3}^2} > \frac{1}{F_{2n-1}^2F_{2n+3}^2}.
$$

Since $F_{4n+1} = F_{2n-2}F_{2n+2} + F_{2n-1}F_{2n+3}$, we have $F_{4n+1} \geq F_{2n-1}F_{2n+3}$. Thus,

$$
3F_{4n+1}^2 > F_{2n-1}^2F_{2n+3}^2.
$$

Combining the last two inequalities yields the desired result. □

Theorem 3.23 *If $m \geq 2$ and $n \geq 2$, then*

$$\left\lfloor \left(\sum_{k=n}^{mn} \frac{(-1)^k}{F_{2k-1}F_{2k+1}} \right)^{-1} \right\rfloor = \begin{cases} 3F_{2n-1}^2 - 1 & \text{if } n \text{ is even;} \\ -3F_{2n-1}^2 & \text{if } n \text{ is odd.} \end{cases}$$

Proof We first consider the case where n is even. Now we have

$$\alpha(mn) + \frac{1}{3F_{2mn+1}^2} = \frac{1}{3F_{2mn-1}^2} - \frac{1}{F_{2mn-1}F_{2mn+1}} < 0,$$

where the inequality follows from the fact $3F_{n-1} > F_{n+1}$.

Combining Lemma 3.16 and the last inequality, we derive that

$$\begin{aligned} \sum_{k=n}^{mn} \frac{(-1)^k}{F_{2k-1}F_{2k+1}} &= \frac{1}{3F_{2n-1}^2} - \frac{1}{3F_{2mn+1}^2} - \sum_{k=n}^{mn} \alpha(k) \\ &= \frac{1}{3F_{2n-1}^2} - \sum_{k=n}^{mn-1} \alpha(k) - \left(\alpha(mn) + \frac{1}{3F_{2mn+1}^2} \right) \\ &> \frac{1}{3F_{2n-1}^2}. \end{aligned}$$

With the help of $\beta(n)$, Lemma 3.18, and Lemma 3.19, we get

$$\begin{aligned} \sum_{k=n}^{mn} \frac{(-1)^k}{F_{2k-1}F_{2k+1}} &= \frac{1}{3F_{2n-1}^2 - 1} - \frac{1}{3F_{2mn+1}^2 - 1} - \sum_{k=n}^{mn} \beta(k) \\ &< \frac{1}{3F_{2n-1}^2 - 1} - \big(\beta(n) + \beta(n+1) + \beta(mn) \big) - \sum_{k=n+2}^{mn-1} \beta(k) \\ &< \frac{1}{3F_{2n-1}^2 - 1}. \end{aligned}$$

Therefore, we arrive at

$$\frac{1}{3F_{2n-1}^2} < \sum_{k=n}^{mn} \frac{(-1)^k}{F_{2k-1}F_{2k+1}} < \frac{1}{3F_{2n-1}^2 - 1},$$

which shows that the statement is true when n is even.

We now turn to the case where n is odd. We have

$$\sum_{k=n}^{mn} \frac{(-1)^k}{F_{2k-1}F_{2k+1}} = \frac{-1}{3F_{2n-1}^2 - 1} + \frac{1}{3F_{2mn+1}^2 - 1} - \sum_{k=n}^{mn} \gamma(k).$$

If mn is even, we easily see that

$$\sum_{k=n}^{mn} \gamma(k) < 0$$

by Lemma 3.20. Therefore,

$$\sum_{k=n}^{mn} \frac{(-1)^k}{F_{2k-1}F_{2k+1}} > \frac{-1}{3F_{2n-1}^2 - 1}.$$

If mn is odd, then employing Lemma 3.20 and Lemma 3.21, we deduce

$$\begin{aligned}\sum_{k=n}^{mn} \frac{(-1)^k}{F_{2k-1}F_{2k+1}} &= \frac{-1}{3F_{2n-1}^2 - 1} + \frac{1}{3F_{2mn+1}^2 - 1} - \sum_{k=n+2}^{mn-1} \gamma(k) \\ &\quad - \bigl(\gamma(n) + \gamma(n+1) + \gamma(mn)\bigr) \\ &> \frac{-1}{3F_{2n-1}^2 - 1}.\end{aligned}$$

Thus, we always have

$$\sum_{k=n}^{mn} \frac{(-1)^k}{F_{2k-1}F_{2k+1}} > \frac{-1}{3F_{2n-1}^2 - 1},$$

provided that n is odd.

Since n is odd, it follows from Lemma 3.22 that

$$\sum_{k=n+2}^{mn} \delta(k) > 0.$$

Furthermore, applying the last inequality and Lemma 3.22, we derive that

$$\begin{aligned}\sum_{k=n}^{mn} \frac{(-1)^k}{F_{2k-1}F_{2k+1}} &= \frac{-1}{3F_{2n-1}^2} + \frac{1}{3F_{2mn+1}^2} - \sum_{k=n}^{mn} \delta(k) \\ &= \frac{-1}{3F_{2n-1}^2} - \sum_{k=n+2}^{mn} \delta(k) - \left(\delta(n) + \delta(n+1) - \frac{1}{3F_{2mn+1}^2}\right) \\ &< \frac{-1}{3F_{2n-1}^2} - \left(\delta(n) + \delta(n+1) - \frac{1}{3F_{4n+1}^2}\right) \\ &< \frac{-1}{3F_{2n-1}^2}.\end{aligned}$$

Therefore, when n is odd, we have

$$\frac{-1}{3F_{2n-1}^2 - 1} < \sum_{k=n}^{mn} \frac{(-1)^k}{F_{2k-1}F_{2k+1}} < \frac{-1}{3F_{2n-1}^2},$$

which yields the desired identity. □

Similarly, we can prove the following result, whose proof is omitted here.

Theorem 3.24 *If $m \geq 2$ and $n \geq 2$, then*

$$\left\lfloor \left(\sum_{k=n}^{mn} \frac{(-1)^k}{F_{2k}F_{2k+2}}\right)^{-1} \right\rfloor = \begin{cases} 3F_{2n}^2 & \text{if } n \text{ is even;} \\ -3F_{2n}^2 - 1 & \text{if } n \text{ is odd.} \end{cases}$$

4 Conclusions

In this paper, we investigate the sums and alternating sums of the products of two reciprocal Fibonacci numbers in various ways. The results are new and interesting. In particular, the techniques for dealing with alternating sums can be applied to study other types of alternating sums, which will be presented in a future paper.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors contributed equally to deriving all the results of this article and read and approved the final manuscript.

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