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# Differential equations arising from the generating function of general modified degenerate Euler numbers

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**Abstract**
In this paper, we introduce the general modified degenerate Euler numbers and study ordinary differential equations arising from the generating function of these numbers. In addition, we give some new explicit identities for the general modified degenerate Euler numbers arising from our differential equations.

**MSC:** 05A19; 11B37; 11B83; 34A34

**Keywords:** general modified degenerate Euler numbers; differential equations

## 1 Introduction

As is known, the Euler numbers are defined by the generating function

$$\frac{2}{e^t+1}=\sum_{n=0}^{\infty}E_n\frac{t^n}{n!} \quad \text{(see [1–9])}. \tag{1.1}$$

Carlitz [2] considered the degenerate Euler numbers defined by the generating function

$$\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}=\sum_{n=0}^{\infty}\mathcal{E}_{n,\lambda}\frac{t^n}{n!}. \tag{1.2}$$

In [7], the modified degenerate Euler numbers, which are slightly different from Carlitz's degenerate Euler numbers, are defined by

$$\frac{2}{(1+\lambda)^{\frac{t}{\lambda}}+1}=\sum_{n=0}^{\infty}\tilde{\mathcal{E}}_{n,\lambda}\frac{t^n}{n!}. \tag{1.3}$$

Note that $\lim_{\lambda\to 0}\tilde{\mathcal{E}}_{n,\lambda}=\lim_{\lambda\to 0}\mathcal{E}_{n,\lambda}=E_n$ ($n\geq 0$). Recently, Kim and Kim [6] studied non-linear differential equations given by

$$\left(\frac{d}{dt}\right)^N\left(\frac{1}{(1+\lambda t)^{\frac{1}{\lambda}}+1}\right)=\frac{(-1)^N}{(1+\lambda t)^N}\sum_{i=1}^{N+1}a_i(N,\lambda)F^i, \tag{1.4}$$

where $F=\frac{1}{(1+\lambda t)^{\frac{1}{\lambda}}+1}$.

Let $\alpha$, $a$, $b$ be nonzero real numbers. Then we consider the general modified degenerate Euler numbers as follows:

$$\frac{2}{\alpha(1+\lambda)^{\frac{at}{\lambda}}+b}=\sum_{n=0}^{\infty}\tilde{\mathcal{E}}_{n,\lambda}(\alpha\mid a,b)\frac{t^n}{n!}. \tag{1.5}$$

From (1.5) we note that

$$\begin{aligned}\lim_{\lambda\to 0}\frac{2}{\alpha(1+\lambda)^{\frac{at}{\lambda}}+b}&=\frac{2}{\alpha e^{at}+b}\\&=\frac{1}{b}\frac{2}{\frac{\alpha}{b}e^{at}+1}\\&=\frac{1}{b}\sum_{n=0}^{\infty}E_{n,\frac{\alpha}{b}}a^n\frac{t^n}{n!},\end{aligned} \tag{1.6}$$

where $E_{n,q}$ ($n\geq 0$) are the Apostol-Euler numbers given by the generating function

$$\frac{2}{qe^t+1}=\sum_{n=0}^{\infty}E_{n,q}\frac{t^n}{n!}\quad \text{(see [1, 3])}. \tag{1.7}$$

Thus, by (1.5) and (1.6) we get

$$\frac{a^n}{b}E_{n,\frac{\alpha}{b}}=\lim_{\lambda\to 0}\tilde{\mathcal{E}}_{n,\lambda}(\alpha\mid a,b)\quad (n\geq 0).$$

Bayad and Kim [1] studied the following nonlinear differential equations related to Apostol-Euler numbers:

$$F_q^N=\frac{1}{(N-1)!}\sum_{k=0}^{N}a_k(N)F_q^{(k-1)}\quad (N\in\mathbb{N}), \tag{1.8}$$

where $F_q^{(k)}=(\frac{d}{dt})^kF_q(t)$, $F_q(t)=\frac{1}{qe^t+1}$.

In this paper, we study the ordinary differential equations associated with the generating function of general modified degenerate Euler numbers. In addition, we give some new and explicit formulas and identities for those numbers arising from our differential equations.

## 2 Generalized modified degenerate Euler numbers

For nonzero real numbers $\alpha$, $a$, $b$, let

$$F=F(t)=\frac{1}{\alpha(1+\lambda)^{\frac{at}{\lambda}}+b}. \tag{2.1}$$

Then by (2.1) we get

$$\begin{aligned}F^{(1)}&=\frac{dF}{dt}(t)\\&=\frac{(-1)\frac{a}{\lambda}\log(1+\lambda)}{(\alpha(1+\lambda)^{\frac{at}{\lambda}}+b)^2}\bigl(\alpha(1+\lambda)^{\frac{\alpha t}{\lambda}}\bigr)\end{aligned}$$

$$= \frac{(-1)\frac{a}{\lambda}\log(1+\lambda)}{(\alpha(1+\lambda)^{\frac{at}{\lambda}}+b)^2}\left\{\alpha(1+\lambda)^{\frac{at}{\lambda}}+b-b\right\}$$
$$= (-1)\frac{a}{\lambda}\log(1+\lambda)\left(F-bF^2\right). \tag{2.2}$$

Thus, from (2.2) we have

$$F^{(1)} = \frac{a}{\lambda}\log(1+\lambda)\left(bF^2-F\right). \tag{2.3}$$

From (2.3) we derive the following equation:

$$\begin{aligned}
F^{(2)} &= \frac{d}{dt}F^{(1)} \\
&= \frac{a}{\lambda}\log(1+\lambda)\left\{2bFF^{(1)}-F^{(1)}\right\} \\
&= \frac{a}{\lambda}\log(1+\lambda)(2bF-1)F^{(1)} \\
&= \left(\frac{a}{\lambda}\log(1+\lambda)\right)^2(2bF-1)\left(bF^2-F\right) \\
&= \left(\frac{a}{\lambda}\log(1+\lambda)\right)^2\left(2b^2F^3-3bF^2+F\right).
\end{aligned} \tag{2.4}$$

Continuing this process, we set

$$\begin{aligned}
F^{(N)} &= \left(\frac{d}{dt}\right)^N F(t) \\
&= \left(\frac{a}{\lambda}\log(1+\lambda)\right)^N \sum_{k=1}^{N+1} a_k(N)b^{k-1}F^k.
\end{aligned} \tag{2.5}$$

By taking the derivative of (2.5) with respect to $t$ we have

$$\begin{aligned}
F^{(N+1)} &= \frac{dF^{(N)}}{dt} \\
&= \left(\frac{a}{\lambda}\log(1+\lambda)\right)^N \sum_{k=1}^{N+1} a_k(N)b^{k-1}kF^{k-1}F^{(1)} \\
&= \left(\frac{a}{\lambda}\log(1+\lambda)\right)^{N+1} \sum_{k=1}^{N+1}\left(ka_k(N)b^kF^{k+1}-a_k(N)b^{k-1}kF^k\right) \\
&= \left(\frac{a}{\lambda}\log(1+\lambda)\right)^{N+1}\left\{\sum_{k=2}^{N+2}(k-1)a_{k-1}(N)b^{k-1}F^k-\sum_{k=1}^{N+1}ka_k(N)b^{k-1}F^k\right\}.
\end{aligned} \tag{2.6}$$

Replacing $N$ by $N+1$ in (2.5), we get

$$F^{(N+1)} = \left(\frac{a}{\lambda}\log(1+\lambda)\right)^{N+1} \sum_{k=1}^{N+2} a_k(N+1)b^{k-1}F^k. \tag{2.7}$$

Comparing the coefficients on both sides of (2.6) and (2.7), we obtain

$$a_1(N+1) = -a_1(N). \tag{2.8}$$

Thus, by (2.8) we get

$$a_1(N+1) = -a_1(N) = (-1)^2 a_1(N-1) = \cdots = (-1)^N a_1(1). \tag{2.9}$$

From (2.5) we have

$$\frac{a}{\lambda}\log(1+\lambda)\left\{bF^2 - F\right\} = \frac{a}{\lambda}\log(1+\lambda)\left\{a_1(1)F + a_2(1)bF^2\right\}. \tag{2.10}$$

By (2.10) we get

$$a_1(1) = -1 \quad \text{and} \quad a_2(1) = 1. \tag{2.11}$$

Thus, from (2.9) and (2.11) we have

$$a_1(N+1) = (-1)^N a_1(1) = (-1)^{N+1}. \tag{2.12}$$

By (2.6) and (2.7) we see that

$$\begin{aligned} a_{N+2}(N+1) &= (N+1)a_{N+1}(N) \\ &= (N+1)Na_N(N-1) \\ &= (N+1)N(N-1)a_{N-1}(N-2) \\ &\ \ \vdots \\ &= (N+1)N(N-1)\cdots 2a_2(1) \\ &= (N+1)!. \end{aligned} \tag{2.13}$$

Thus, by (2.13) we have

$$a_{N+2}(N+1) = (N+1)!. \tag{2.14}$$

For $2 \leq k \leq N+1$, by comparing the coefficients on both sides of (2.6) and (2.7) we have

$$a_k(N+1) = (k-1)a_{k-1}(N) - ka_k(N). \tag{2.15}$$

Let $k = 2$ in (2.15). Then we have

$$\begin{aligned} a_2(N+1) &= a_1(N) - 2a_2(N) \\ &= a_1(N) - 2\big(a_1(N-1) - 2a_2(N-1)\big) \\ &= a_1(N) - 2a_1(N-1) + (-1)^2 2^2 a_2(N-1) \\ &= a_1(N) - 2a_1(N-1) + (-1)^2 2^2 \left\{a_1(N-2) - 2a_2(N-2)\right\} \\ &= a_1(N) - 2a_1(N-1) + (-1)^2 2^2 a_1(N-2) + (-1)^3 2^3 a_2(N-2) \\ &\ \ \vdots \end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=0}^{N-1} (-1)^k a_1(N-k)2^k + (-1)^N 2^N a_2(1) \\
&= \sum_{k=0}^{N} (-1)^k a_1(N-k)2^k. \qquad (2.16)
\end{aligned}
$$

For $k = 3$ in (2.15), we have

$$
\begin{aligned}
a_3(N+1) &= 2a_2(N) - 3a_3(N) \\
&= 2a_2(N) - 3\{2a_2(N-1) - 3a_3(N-1)\} \\
&= 2a_2(N) - 3 \cdot 2a_2(N-1) + (-1)^2 3^2 a_3(N-1) \\
&= 2a_2(N) - 3 \cdot 2a_2(N-1) + (-1)^2 3^2 \{2a_2(N-2) - 3a_3(N-2)\} \\
&= 2a_2(N) - 3 \cdot 2a_2(N-1) + (-1)^2 3^2 2a_2(N-2) + (-1)^3 3^3 a_3(N-2) \\
&\ \ \vdots \\
&= 2\sum_{k=0}^{N-2} a_2(N-k)(-1)^k 3^k + (-1)^{N-1} 3^{N-1} a_3(2) \\
&= 2\sum_{k=0}^{N-1} a_2(N-k)(-1)^k 3^k. \qquad (2.17)
\end{aligned}
$$

Continuing this process, we deduce

$$
a_j(N+1) = (j-1) \sum_{k=0}^{N-j+2} a_{j-1}(N-k)(-1)^k j^k, \qquad (2.18)
$$

where $2 \le j \le N+1$.

Now we give an explicit expression for $a_j(N+1)$ in (2.18). From (2.12) and (2.16) we can derive the following equation:

$$
\begin{aligned}
a_2(N+1) &= \sum_{k=0}^{N} (-1)^k a_1(N-k)2^k \\
&= \sum_{k=0}^{N} (-1)^k (-1)^{N-k} 2^k = (-1)^N \sum_{k=0}^{N} 2^k. \qquad (2.19)
\end{aligned}
$$

By (2.17) we get

$$
\begin{aligned}
a_3(N+1) &= 2\sum_{k_2=0}^{N-1} a_2(N-k_2)(-1)^{k_2} 3^{k_2} \\
&= 2\sum_{k_2=0}^{N-1} (-1)^{N-k_2-1} \sum_{k_1=0}^{N-k_2-1} 2^{k_1} (-1)^{k_2} 3^{k_2} \\
&= 2(-1)^{N-1} \sum_{k_2=0}^{N-1} \sum_{k_1=0}^{N-1-k_2} 2^{k_1} 3^{k_2}. \qquad (2.20)
\end{aligned}
$$

Continuing this process, we deduce that, for $2 \leq j \leq N+1$,

$$\begin{aligned} &a_j(N+1) \\ &\quad = (j-1)!(-1)^{N-j+2} \\ &\qquad \times \sum_{k_{j-1}=0}^{N-j+2} \sum_{k_{j-2}=0}^{N-j+2-k_{j-1}} \sum_{k_{j-3}=0}^{N-j+2-k_{j-1}-k_{j-2}} \cdots \sum_{k_1=0}^{N-j+2-k_{j-1}-\cdots-k_2} j^{k_{j-1}}(j-1)^{k_{j-2}} \cdots 3^{k_2} 2^{k_1}. \end{aligned} \tag{2.21}$$

Therefore, by (2.5) and (2.21) we obtain the following theorem.

**Theorem 1** *Let $\alpha$, $a$, $b$ be nonzero real numbers. The family of nonlinear differential equations*

$$F^{(N)} = \left(\frac{a}{\lambda}\log(1+\lambda)\right)^N \sum_{k=1}^{N+1} a_k(N) b^{k-1} F^k$$

*has a solution $F = F(t) = \frac{1}{\alpha(1+\lambda)^{\frac{at}{\lambda}}+b}$, where $a_1(N) = (-1)^N$, and*

$$\begin{aligned} a_j(N) &= (j-1)!(-1)^{N-j+1} \\ &\quad \times \sum_{k_{j-1}=0}^{N-j+1} \sum_{k_{j-2}=0}^{N-j+1-k_{j-1}} \cdots \sum_{k_1=0}^{N-j+1-k_{j-1}-\cdots-k_2} j^{k_{j-1}}(j-1)^{k_{j-2}} \cdots 3^{k_2} 2^{k_1} \end{aligned}$$

*for $2 \leq j \leq N+1$.*

Now we define the general modified degenerate Euler numbers given by the generating function

$$\frac{2}{\alpha(1+\lambda)^{\frac{at}{\lambda}}+b} = \sum_{n=0}^{\infty} \tilde{\mathcal{E}}_{n,\lambda}(\alpha;a,b)\frac{t^n}{n!}. \tag{2.22}$$

Note that $\tilde{\mathcal{E}}_{n,\lambda}(1;1,1)$ are the modified degenerate Euler numbers given by

$$\frac{2}{(1+\lambda)^{\frac{t}{\lambda}}+1} = \sum_{n=0}^{\infty} \tilde{\mathcal{E}}_{n,\lambda}\frac{t^n}{n!}.$$

Now we observe that

$$\begin{aligned} F^{(N)} &= \frac{1}{2}\left(\frac{d}{dt}\right)^N \left(\frac{2}{\alpha(1+\lambda)^{\frac{at}{\lambda}}+b}\right) \\ &= \frac{1}{2}\sum_{n=0}^{\infty} \frac{\tilde{\mathcal{E}}_{n,\lambda}(\alpha;a,b)}{n!}\left(\frac{d}{dt}\right)^N t^n \\ &= \frac{1}{2}\sum_{n=0}^{\infty} \tilde{\mathcal{E}}_{n+N,\lambda}(\alpha;a,b)\frac{t^n}{n!}. \end{aligned} \tag{2.23}$$

For $r \in \mathbb{N}$, the higher-order general modified degenerate Euler numbers are defined by the generating function

$$\left(\frac{2}{\alpha(1+\lambda)^{\frac{at}{\lambda}}+b}\right)^r = \sum_{n=0}^{\infty} \tilde{\mathcal{E}}_{n,\lambda}^{(r)}(\alpha;a,b)\frac{t^n}{n!}. \tag{2.24}$$

Therefore, by Theorem 1, (2.23), and (2.21) we obtain the following theorem.

**Theorem 2** *Let $\alpha$, $a$, $b$ be nonzero real numbers. For $n \geq 0$, we have*

$$\tilde{\mathcal{E}}_{n+N}(\alpha;a,b) = \left(\frac{a}{\lambda}\log(1+\lambda)\right)^N \sum_{k=1}^{N+1} a_k(N) b^{k-1} 2^{1-k} \tilde{\mathcal{E}}_{n,\lambda}^{(k)}(\alpha;a,b),$$

*where $a_1(N) = (-1)^N$, and, for $2 \leq j \leq N+1$,*

$$\begin{aligned} a_j(N) &= (j-1)!(-1)^{N-j+1} \\ &\quad \times \sum_{k_{j-1}=0}^{N-j+1} \sum_{k_{j-2}=0}^{N-j+1-k_{j-1}} \cdots \sum_{k_1=0}^{N-j+1-k_{j-1}-\cdots-k_2} j^{k_{j-1}}(j-1)^{k_{j-2}} \cdots 3^{k_2} 2^{k_1}. \end{aligned}$$

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
All authors contributed equally to this work. All authors read and approved the final manuscript.

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