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Birkhoff's individual ergodic theorem and maximal ergodic theorem for fuzzy dynamical systems

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Abstract

In our previous paper (Tirpáková and Markechová in Adv. Differ. Equ. 2015:171, 2015), we presented fuzzy analogies of Mesiar's ergodic theorems. Our aim in this contribution is to prove analogues of Birkhoff's individual ergodic theorem and the maximal ergodic theorem for the case of fuzzy dynamical systems.

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1 Introduction

Ergodic theory is currently rapidly and massively developing area of theoretical and applied mathematical research. Ergodic theory theorems are studied in many structures, especially, in structures created on the basis of fuzzy approach. Some ergodic theorems valid in the classical ergodic theory [2] have been proven, among others, for D-posets of fuzzy sets [3], for MV-algebras of fuzzy sets [4, 5], and recently also for families of IF-events [6]. In our previous paper [1], we proved fuzzy versions of Mesiar's ergodic theorems for the case of fuzzy dynamical systems defined and studied in [7]. This way, we contributed to the extension of our study of fuzzy dynamical systems. By a fuzzy dynamical system we mean a system (Ω, M, m, U), where (Ω, M, m) is a fuzzy probability space defined by Piasecki [8], and $U : M \to M$ is an m-invariant σ-homomorphism. This structure can serve as an alternative mathematical model of ergodic theory for the case where the observed events are described vaguely. Fuzzy dynamical systems include the classical dynamical systems; on the other hand, they enable one to study more general situations. The aim of this paper is to generalize some other assertions valid in the classical ergodic theory to the case of fuzzy dynamical systems. In Section 3, after the introductory section (Section 2), we prove fuzzy analogies of Birkhoff's individual ergodic theorem and the maximal ergodic theorem. The basic idea of our proofs is based on a factorization of the fuzzy σ-algebra M and on properties of the σ-homomorphism U. Note that other approaches to a fuzzy generalization of the notion of a dynamical system are presented in [9–12]. The authors of these papers used some other connectives to define the fuzzy set operations.

2 Basic definitions and facts

Let us recall some definitions and basic facts.

Definition 2.1 [8] A fuzzy probability space is a triplet (Ω, M, m) where Ω is a nonempty set, M is a fuzzy σ-algebra of fuzzy subsets of Ω (*i.e.*, $M \subset [0,1]^{\Omega}$ such that (i) $1_\Omega \in M$, $(1/2)_\Omega \notin M$; (ii) if $f_n \in M$, $n = 1, 2, \dots$, then $\bigvee_{n=1}^{\infty} f_n \in M$; (iii) if $f \in M$, then $f' := 1_\Omega - f \in M$), and the mapping $m : M \to \langle 0, \infty)$ satisfies the following conditions:

(iv) $m(f \vee f') = 1$ for every $f \in M$;

(v) if $\{f_n\}_{n=1}^{\infty}$ is a sequence of pairwise weakly separated fuzzy subsets from M (*i.e.*, $f_i \leq f_j'$ (point wisely) whenever $i \neq j$), then $m(\bigvee_{n=1}^{\infty} f_n) = \sum_{n=1}^{\infty} m(f_n)$.

The symbols $\bigvee_{n=1}^{\infty} f_n := \sup_n f_n$ and $\bigwedge_{n=1}^{\infty} f_n =: \inf_n f_n$ denote the fuzzy union and fuzzy intersection of a sequence $\{f_n\}_{n=1}^{\infty} \subset M$, respectively, in the sense of Zadeh [13]. A couple (Ω, M), where Ω is a nonempty set, and M is a fuzzy σ-algebra of fuzzy subsets of Ω, is called a fuzzy measurable space. The presented σ-additive fuzzy measure m satisfies all properties analogous to the properties of classical probability measure in the crisp case. The described structure (Ω, M, m) can serve as a mathematical model of random experiments, the results of which are vaguely defined events, the so-called fuzzy events. A probability interpretation of the above notions is as follows: a set Ω is the set of elementary events; a fuzzy set from the system M is a fuzzy event; the value $m(f)$ is the probability of a fuzzy event f; a fuzzy event f' is the opposite event to a fuzzy event f; and weakly separated fuzzy events are interpreted as mutually exclusive events.

Definition 2.2 [7] By a fuzzy dynamical system we mean a quadruplet (Ω, M, m, U), where (Ω, M, m) is a fuzzy probability space, and $U : M \to M$ is an m-invariant σ-homomorphism, that is, $U(f') = (U(f))'$, $U(\bigvee_{n=1}^{\infty} f_n) = \bigvee_{n=1}^{\infty} U(f_n)$, and $m(U(f)) = m(f)$ for every $f \in M$ and any sequence $\{f_n\}_{n=1}^{\infty} \subset M$.

We present some examples of the above notions.

The trivial case of a fuzzy dynamical system is a quadruplet (Ω, M, m, I), where (Ω, M, m) is any fuzzy probability space, and $I : M \to M$ is the identity mapping.

Consider any fuzzy probability space (Ω, M, m). Let $T : \Omega \to \Omega$ be a measure m preserving transformation (*i.e.*, $f \in M$ implies $f \circ T \in M$ and $m(f \circ T) = m(f)$). Then the mapping $U : M \to M$ defined by

$$U(f) = f \circ T \quad \text{for every } f \in M \tag{2.1}$$

is an m-invariant σ-homomorphism, and the quadruplet (Ω, M, m, U) is a fuzzy dynamical system.

Example 2.1 Let (Ω, S, P, T) be a classical dynamical system. Put $M = \{\chi_A; A \in S\}$, where χ_A is the indicator of a set $A \in S$, and define the mapping $m : M \to \langle 0, 1\rangle$ by $m(\chi_A) = P(A)$. Then the triplet (Ω, M, m) is a fuzzy probability space, and the system (Ω, M, m, U), where the mapping $U : M \to M$ is defined by (2.1), is a fuzzy dynamical system. By this procedure a classical model can be imbedded into a fuzzy one.

We can also consider the following extension of a fuzzy dynamical system from the previous example.

Example 2.2 Let (Ω, S, P, T) be a classical dynamical system. Denote by M the system of fuzzy subsets f of Ω such that f is an S-measurable mapping and $P(f \in (1/4; 3/4)) = 0$. If we define the mapping $m : M \to \langle 0,1\rangle$ by the equality $m(f) = P\ (f > (1/2)_\Omega)$, then the triplet (Ω, M, m) is a fuzzy probability space, and the system (Ω, M, m, U), where the mapping $U : M \to M$ is defined by (2.1), is a fuzzy dynamical system.

Example 2.3 Let $\Omega = \langle 0,1\rangle$, $f : \Omega \to \Omega$, $f(x) = x$ for every $x \in \Omega$. Put $M = \{f, f', f \vee f', f \wedge f', 0_\Omega, 1_\Omega\}$ and define the mapping $m : M \to \langle 0,1\rangle$ by the equalities $m(1_\Omega) = m(f \vee f') = 1$, $m(0_\Omega) = m(f \wedge f') = 0$, and $m(f) = m(f') = 1/2$. Then the triplet (Ω, M, m) is a fuzzy probability space. Moreover, if we define the mapping $U : M \to M$ by the equalities $U(f \vee f') = f \vee f'$, $U(1_\Omega) = 1_\Omega$, $U(0_\Omega) = 0_\Omega$, $U(f \wedge f') = f \wedge f'$, $U(f) = f'$, $U(f') = f$, then (Ω, M, m, U) is a fuzzy dynamical system.

An analog of a random variable in terms of the classical probability theory is an F-observable.

Definition 2.3 [14] An F-observable on a fuzzy measurable space (Ω, M) is a mapping $x : B(\Re) \to M$ such that

(i) $x(E^C) = 1_\Omega - x(E)$ for every $E \in B(\Re)$;

(ii) $x(\bigcup_{n=1}^\infty E_n) = \bigvee_{n=1}^\infty x(E_n)$ for any sequence $\{E_n\}_{n=1}^\infty \subset B(\Re)$,

where $B(\Re)$ is the family of all Borel subsets of the real line $\Re$, and E^C denotes the complement of a set $E \subset \Re$.

Example 2.4 Let (Ω, S, P) be a classical probability space, and $\xi : \Omega \to \Re$ be a random variable in the sense of classical probability theory. Then the mapping x defined by $x(E) = \chi_{\xi^{-1}(E)}$, $E \in B(\Re)$, is an F-observable on the fuzzy measurable space (Ω, M) from Example 2.1.

Let x be an F-observable on a fuzzy measurable space (Ω, M). Then the range of F-observable x, that is, the set $R(x) := \{x(E); E \in B(\Re)\}$ is a Boolean σ-algebra of (Ω, M) with minimal and maximal elements $x(\varnothing)$ and $x(\Re)$, respectively. If $U : M \to M$ is a σ-homomorphism, then it is easy to verify that the mapping $U \circ x : B(\Re) \to M$, $U \circ x : E \to U(x(E))$, $E \in B(\Re)$, is an F-observable on (Ω, M), too.

Let any fuzzy probability space (Ω, M, m) be given. If x is an F-observable on (Ω, M), then the mapping $m_x : E \mapsto m(x(E))$, $E \in B(\Re)$, is a probability measure on $B(\Re)$. The value $m(x(E))$ is interpreted as the probability that an F-observable x has a value in $E \in B(\Re)$. By an integral of x with respect to m we mean the expression

$$m(x) = \int x\, dm := \int_\Re t\, dm_x(t)$$

(if the integral on the right-hand side exists and is finite). The integral $m(x)$ is interpreted as the mean value of x [15]. Further, for a Borel-measurable function $\psi : \Re \to \Re$, we put

$$\int \psi(x)\, dm := \int_\Re \psi(t)\, dm_x(t),$$

where the F-observable $\psi(x) : B(\Re) \to M$ is defined by $\psi(x)(E) = x(\psi^{-1}(E))$, $E \in B(\Re)$ (under the conventional assumptions on the integrability).

The integral of an F-observable x on (Ω, M) over a fuzzy set $f \in M$ is defined (see [4]) via

$$\upsilon(f) = \int_f x\,dm := \int x \cdot x_f\,dm,$$

where x_f is the question observable of a fuzzy set f, that is, the mapping defined, for any $E \in B(\Re)$, by

$$x_f(E) = \begin{cases} f \vee f' & \text{if } 0,1 \in E; \\ f' & \text{if } 0 \in E, 1 \notin E; \\ f & \text{if } 0 \notin E, 1 \in E; \\ f \wedge f' & \text{if } 0,1 \notin E \end{cases}$$

(if the integral on the right-hand side exists and is finite).

Definition 2.4 [16] Let (Ω, M, m) be a given fuzzy probability space, and $x, x_1, x_2, \ldots$ be F-observables on (Ω, M). We say that the sequence $\{x_n\}_{n=1}^{\infty}$ converges to x almost everywhere in m (and we write $x_n \to x$ a.e. in m) if, for every $\varepsilon > 0$,

$$m\left(\bigvee_{k=1}^{\infty} \bigwedge_{n=k}^{\infty} (x_n - x)\big(\langle -\varepsilon, \varepsilon\rangle\big)\right) = 1.$$

3 Main results

In this section, we present Birkhoff's individual ergodic theorem and the maximal ergodic theorem for fuzzy dynamical systems. In the proofs, we will use the factorization of a fuzzy σ-algebra M described further and the properties of a σ-homomorphism U. The presented results can be obtained also using the factorization over the σ-ideal of weakly empty sets. Details on this approach can be found in [17].

Let any fuzzy probability space (Ω, M, m) be given. In the set M, we define the relation of equivalence $\sim$ in the following way: for every $f, g \in M$, $f \sim g$ if and only if $m(f \Delta g) = 0$, where $f \Delta g = (f \wedge g') \vee (f' \wedge g)$ is the symmetric difference of fuzzy sets f and g. Put $[f] = \{g \in M; m(f \Delta g) = 0\}$ for $f \in M$. It is easy to verify that if $f_1, f_2 \in [f]$, then $m(f_1) = m(f_2)$. In the system $[M] = \{[f]; f \in M\}$, we can define the relation $\le$ as follows: for every $[f], [g] \in [M]$, $[f] \le [g]$ if and only if $m(f \wedge g') = 0$. The couple $([M], \le)$ is a partially ordered set with a minimal element $[0_\Omega]$ and a maximal element $[1_\Omega]$; moreover, $[M]$ is a Boolean σ-algebra, where $[\bigvee_{n=1}^{\infty} f_n]$ is the least upper bound of a sequence $\{[f_n]\}_{n=1}^{\infty} \subset [M]$, that is, $\bigvee_{n=1}^{\infty} [f_n] = [\bigvee_{n=1}^{\infty} f_n]$. Further, for all $f, g \in M$, $[f] \wedge [g] = [f \wedge g]$. For every $[f] \in [M]$, $[f] \wedge [f'] = [f \wedge f'] = [0_\Omega]$ and $[f] \vee [f'] = [f \vee f'] = [1_\Omega]$; hence, we have $[f]' = [f']$ for every $f \in M$. If we define the mapping $\mu : [M] \to \langle 0, 1\rangle$ by the equality $\mu([f]) := m(f)$ for $[f] \in [M]$, then μ is a probability measure on the Boolean σ-algebra $[M]$, that is, $\mu([1_\Omega]) = 1$, $\mu \ge 0$, and $[f_i] \wedge [f_j] = [0_\Omega]$ $(i \ne j)$ implies $\mu(\bigvee_{n=1}^{\infty} [f_n]) = \sum_{n=1}^{\infty} \mu([f_n])$.

Let (Ω, M, m, U) be any fuzzy dynamical system. Then the mapping $\bar{U} : [M] \to [M]$ defined by $\bar{U}([f]) = [U(f)]$, $[f] \in [M]$, is a σ-homomorphism of the Boolean σ-algebra $[M]$, that is, for every $[f] \in [M]$, $\bar{U}([f]') = (\bar{U}([f]))'$, and for every sequence $\{[f_n]\}_{n=1}^{\infty} \subset [M]$, $\bar{U}(\bigvee_{n=1}^{\infty} [f_n]) = \bigvee_{n=1}^{\infty} \bar{U}([f_n])$; moreover, $\bar{U}$ is invariant in μ, that is, $\mu(\bar{U}[f]) = \mu([f])$ for every $[f] \in [M]$.

Let x be an F-observable on a fuzzy measurable space (Ω, M). Let us define the mapping $h : M \to [M]$ via

$$h(f) = [f], \quad f \in M. \tag{3.1}$$

Then it is easy to see that h is a σ-homomorphism from M onto $[M]$ and $\bar{x} := h \circ x$ is an observable on the Boolean σ-algebra $[M]$.

In the following, we will need the notion of an ergodic fuzzy dynamical system. We introduce this notion analogously as in the classical ergodic theory [2].

Definition 3.1 A fuzzy dynamical system (Ω, M, m, U) is said to be ergodic if a σ-homomorphism U of (Ω, M) is ergodic in m, that is, for every $f \in M$, the statement $m(f \wedge U(f')) = 0 = m(U(f) \wedge f')$ implies $m(f) \in \{0, 1\}$.

The following theorem is a fuzzy analogue of Birkhoff's individual ergodic theorem. It should be noted that the first authors interested in the ergodic theory on fuzzy measurable spaces were Harman and Riečan [18]. They proved Birkhoff's individual ergodic theorem for the compatible case. Theorem 3.1 presents a more general case.

Theorem 3.1 *Let* (Ω, M, m, U) *be an ergodic fuzzy dynamical system, and* x *be an* F-*observable on* (Ω, M). *Suppose* $m(x) = 0$. *Then*

$$\frac{1}{n} \sum_{i=1}^{n} U^i \circ x \to o \quad \textit{almost everywhere in } m, \tag{3.2}$$

where o *is the question observable of the empty fuzzy set* 0_Ω.

Proof At the beginning of the proof, we use arguments similar to those in the proof of Theorem 3.1 in [1]. Let $\mathcal{A}$ be the minimal Boolean sub-σ-algebra of $[M]$ containing all ranges of $\bar{U}^n \circ \bar{x}$, $n = 1, 2, \ldots$. Then $\bar{U}([f]) \in \mathcal{A}$ for any $[f] \in \mathcal{A}$ and the Boolean σ-algebra $\mathcal{A}$ has a countable generator. Therefore, in view of Varadarajan [19], there exists an observable $z : B(\Re) \to [M]$ such that $\{z(E) : E \in B(\Re)\} = \mathcal{A}$. Moreover, there is a sequence of real-valued Borel functions $\{\psi_n\}_{n=0}^{\infty}$ such that

$$(\bar{U}^n \circ \bar{x})(E) = z(\psi_n^{-1}(E)), \quad E \in B(\Re), n = 0, 1, 2, \ldots.$$

Since $\bar{U}$ is z-measurable, by Dvurečenskij and Riečan [14], there exists a Borel-measurable transformation $T : \Re \to \Re$ such that

$$\bar{U}(z(E)) = z(T^{-1}(E)) \quad \text{for every } E \in B(\Re).$$

Therefore, for every $E \in B(\Re)$, we have

$$\bar{U}^n(z(E)) = z(T^{-n}(E)),$$

and consequently,

$$(\bar{U}^n \circ \bar{x})(E) = \bar{U}^n(\bar{x}(E)) = \bar{U}^n(z(\psi_0^{-1}(E))) = z(T^{-n}(\psi_0^{-1}(E))) = z((\psi_0 \circ T^n)^{-1}(E)).$$

Hence, we may assume without loss of generality that $\psi_n = \psi \circ T^n$, $n = 1, 2, \ldots$, for some Borel function ψ. By Definition 2.4 $x_n \to o$ almost everywhere in m if, for every $\varepsilon > 0$, $m(\bigvee_{k=1}^{\infty} \bigwedge_{n=k}^{\infty} x_n(\langle -\varepsilon, \varepsilon\rangle)) = 1$. But

$$m\left(\bigvee_{k=1}^{\infty} \bigwedge_{n=k}^{\infty} x_n\big(\langle -\varepsilon, \varepsilon\rangle\big)\right) = 1 \quad \text{if and only if} \quad \mu\left(\bigvee_{k=1}^{\infty} \bigwedge_{n=k}^{\infty} \bar{x}_n\big(\langle -\varepsilon, \varepsilon\rangle\big)\right) = 1.$$

Therefore,

$$\frac{1}{n} \sum_{i=1}^{n} U^i \circ x \to o \quad \text{a.e. in } m \quad \text{if and only if} \quad \frac{1}{n} \sum_{i=1}^{n} \bar{U}^i \circ \bar{x} \to \bar{o} \quad \text{a.e. in } \mu,$$

where, for every $E \in B(\Re)$, $\bar{o}(E) = [0_\Omega]$ if $0 \notin E$ and $\bar{o}(E) = [1_\Omega]$ if $0 \in E$. The previous convergence is true if and only if

$$\frac{1}{n} \sum_{i=1}^{n} \big(\psi \circ T^i\big)(z) \to \bar{o} \quad \text{almost everywhere in } \mu,$$

which is possible if and only if

$$\frac{1}{n} \sum_{i=1}^{n} \psi\big(T^i(t)\big) \to 0 \quad \text{almost everywhere in } \mu_z,$$

where $\mu_z : B(\Re) \to \langle 0, 1\rangle$ is the probability measure on $B(\Re)$ defined by $\mu_z(E) = \mu(z(E))$, $E \in B(\Re)$. On the other hand,

$$0 = m(x) = \int_{\Re} t \, dm_x(t) = \int_{\Re} t \, d\mu_{\bar{x}}(t) = \int_{\Re} \psi(t) \, d\mu_z(t),$$

where $\mu_{\bar{x}} : B(\Re) \to \langle 0, 1\rangle$ is the probability measure on $B(\Re)$ defined by $\mu_{\bar{x}}(E) = \mu(\bar{x}(E))$, $E \in B(\Re)$. Taking into account the classical dynamical system $(\Re, B(\Re), \mu_z, T)$, it is easy to see that T is μ_z-invariant and ergodic in μ_z, that is, $\mu_z(T^{-1}(E)) = \mu_z(E)$ for every $E \in B(\Re)$, and $T^{-1}(E) = E$ implies $\mu_z(E) \in \{0, 1\}$. Therefore, by Halmos [2], the Birkhoff's individual ergodic theorem holds for ψ, and, consequently, (3.2) is proved. □

The following theorem is a fuzzy generalization of the maximal ergodic theorem.

Theorem 3.2 *Let* (Ω, M, m, U) *be a fuzzy dynamical system, and* x *be an F-observable on* (Ω, M) *with a finite mean value in* m*. Let* $S_k = \sum_{i=1}^{k-1} U^i \circ x$, $k = 1, \ldots, n$, *and* $f = \bigvee_{i=1}^{n} S_i((0, \infty))$*. Then* $\int_f x \, dm \geq 0$.

Proof As in the proof of the previous theorem, we get

$$[f] = h(f) = \bigvee_{i=1}^{n} h\big(S_i\big((0, \infty)\big)\big) = \bigvee_{i=1}^{n} z\big(s_i^{-1}\big((0, \infty)\big)\big) = z\left(\bigvee_{i=1}^{n} s_i^{-1}\big((0, \infty)\big)\right),$$

where h is defined by (3.1), $s_k(t) = \sum_{i=1}^{k-1} \psi(T^i(t))$, $t \in \Re$, $k = 1, \dots, n$, and z, T, and ψ have the same meanings as before. Hence,

$$\begin{aligned}\int_f x\,dm &= \int x \cdot x_f\,dm = \int h \circ (x \cdot x_f)\,d\mu = \int (h \circ x) \cdot (h \circ x_f)\,d\mu \\ &= \int (h \circ x) \cdot \bar{x}_{[f]}\,d\mu = \int_{[f]} h \circ x\,d\mu = \int_A \psi(t)\,d\mu_z(t),\end{aligned}$$

where $\bar{x}_{[f]}$ is the question observable of $[f]$, that is, $\bar{x}_{[f]}(\{0\}) = [f]'$, $\bar{x}_{[f]}(\{1\}) = [f]$, and $A = \{t \in \Re : \max(0, s_1(t), \dots, s_n(t)) > 0\}$. Applying the maximal ergodic theorem to the classical dynamical system $(\Re, B(\Re), \mu_z, T)$, we obtain that $\int_A \psi(t)\,d\mu_z(t) \geq 0$. The proof is finished. □

4 Conclusions

In the classical theory, an event is understood as an exactly defined phenomenon, and from the mathematical point of view, it is a classical set. In practice, however, we often encounter events that are described imprecisely, vaguely, so-called fuzzy events. That is why various proposals for a fuzzy generalization of the dynamical system of classical ergodic theory have been created (*e.g.*, in [7, 9–12]). In this note, we contributed to the extension of our study concerning fuzzy dynamical systems introduced by Markechová in [7]. We presented generalizations of Birkhoff's individual ergodic theorem and the maximal ergodic theorem from the classical ergodic theory to the case of fuzzy dynamical systems.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors contributed equally and significantly in writing this article. The authors read and approved the final manuscript.

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