Cui and Wang *Advances in Difference Equations* (2016) 2016:115
DOI 10.1186/s13662-016-0843-1

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH
Open Access

# Nonlocal stochastic integro-differential equations driven by fractional Brownian motion

Jing Cui[1*] and Zhi Wang[2]

[*]Correspondence: jcui123@126.com
[1]Department of Statistics, Anhui Normal University, South Huajin Rd., Wuhu, 241000, P.R. China
Full list of author information is available at the end of the article

**Abstract**

In this paper, we study the existence of mild solutions for nonlocal stochastic integro-differential equations driven by fractional Brownian motions with Hurst parameter $H > \frac{1}{2}$ in a Hibert space. Sufficient conditions for the existence of mild solutions are derived by means of the Leray-Schauder nonlinear alternative. A special case of this result is given and an example is provided to illustrate the effectiveness of the proposed result.

**MSC:** 60H15; 60H20; 34K45

**Keywords:** stochastic integro-differential equations; fractional Brownian motion; nonlocal condition

## 1 Introduction

It is well known that fractional Brownian motion (fBm, for short) is a family of centered Gaussian processes with continuous sample paths indexed by the Hurst parameter $H \in (0,1)$. fBm admits stationary increments and self-similarity, and it has a long-memory when $H > \frac{1}{2}$. These compact and significant properties make fBm a natural candidate as a model for noise in a wide variety of physical phenomena, such as biological physics, condensed matter physics, telecommunication networks, mathematical finance, and so on (see [1] and the references therein). Therefore, it is interesting and important to investigate stochastic calculus with respect to fBm and related topics (we refer to [2–5] and references therein for a complete presentation of this subject).

Recently, stochastic differential equations driven by fBm have attracted a lot of attentions of works and the theory has been developed in different settings. Some interesting results of finite-dimensional stochastic differential equations with fractional noise have been investigated by Hu *et al.* [6], Fan [7], Liu and Yan [8, 9], and references therein; the case of stochastic differential equations driven by fBm in infinite-dimensional Hilbert spaces has also been studied extensively, for example, Boufoussi and Hajji [10] investigated the existence of neutral stochastic functional differential equations driven by fBm, Caraballo *et al.* [11] proved the existence and exponential behavior of mild solutions to stochastic delay evolution equations with fractional noise, and Duncan *et al.* [12] established the weak, strong and mild solutions to stochastic equations with multiplicative fractional noise.

Since it was introduced in 1990 by Byszewski and Lakshmikantham [13], the nonlocal Cauchy problems have been extensively studied in differential equations and dynamical systems [14–17]. It is demonstrated that the corresponding differential equations with nonlocal conditions more accurately describe the phenomena. For example, it is discovered in [16] that the nonlocal initial condition $x(0) + g(t_1, \ldots, t_n, x(t_1), \ldots, x(t_n)) = x_0$, $0 < t_1 < t_2 < \cdots < t_n \leq T$, has better effects in characterizing the diffusion phenomenon of a small amount of gas in a transparent tube than the classical Cauchy condition $x(0) = x_0$. In the infinite-dimensional framework, stochastic differential equations with nonlocal conditions driven by Brownian motion (*i.e.*, the case $H = \frac{1}{2}$) have received a lot of attention during the last years. For example, Muthukumar *et al.* [18] studied the controllability of fractional stochastic integro-differential equations with nonlocal conditions, the existence of solutions for stochastic functional differential equations has been discussed in [19, 20], the second-order stochastic functional differential equations with nonlocal conditions have been investigated in [21, 22] and references therein. For recent important results of stochastic differential equations in Hilbert spaces, we refer to [23, 24] and the references therein.

In contrast, for $H \neq \frac{1}{2}$, to the best of our knowledge, there is no work concerning the existence of mild solutions for stochastic evolution equations with nonlocal conditions. Therefore, the main objective of this paper is to fill this gap. Further, many existence results of stochastic differential equations with nonlocal conditions are valid only for the Lipschitz or compact assumptions on nonlocal items. The main purpose of this manuscript is to investigate the existence of nonlocal stochastic integro-differential equations driven by fractional Brownian motion, for which the nonlocal items are valid for non-Lipschitz and noncompact assumptions.

In this paper, we consider the existence of mild solutions for a class of stochastic integro-differential equations of the following form:

$$\begin{cases} dx(t) = A(t)[x(t) + \int_0^t B(t,s)x(s)\,ds + f(t,x(t))]\,dt + \sigma\,dB^H(t), & t \in J := [0.T], \\ x(0) + g(x) = x_0, \end{cases} \tag{1.1}$$

in a real separable Hilbert space $\mathcal{H}$ with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$, where $A(t)$ is a closed linear operator with a dense domain $D(A)$, which is independent of $t$, $B(t,s)$ is a bounded operator in $\mathcal{H}$. $B^H$ is a fBm with $H \in (\frac{1}{2}, 1)$ on a real separable Hilbert space $V$. $f: J \times \mathcal{H} \to \mathcal{H}$, $\sigma \in L_2^0(V,\mathcal{H})$, $g: C(J,\mathcal{H}) \to \mathcal{H}$ are appropriate functions specified later. Here, $L_2^0(V,\mathcal{H})$ denotes the space of all $Q$-Hilbert-Schmidt operators from $V$ into $\mathcal{H}$ (see Section 2 below). $x_0$ is an $\mathcal{F}_0$-measurable random variable independent of $B^H$ with finite second moment.

A brief outline of this paper is given. In Section 2, we present some basic notations and preliminaries; in Section 3, the existence result of system (1.1) is investigated by means of the Leray-Schauder nonlinear alternative, a special case of system (1.1) is also considered. An example is presented to illustrate the effectiveness of the main result.

## 2 Preliminaries

Throughout this paper, we assume that $H \in (\frac{1}{2}, 1)$ unless otherwise specified. In this section, our goal is to introduce some useful results about fBm and the corresponding stochastic integral taking values in a Hilbert space. For more details on this section, we refer to Hu [2], Mishura [5] and the references therein.

### 2.1 Fractional Brownian motion

We begin by recalling the definition of a fBm. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space and $H \in (0,1)$ be given. A fBm with Hurst parameter $H$ is a continuous centered Gaussian process $\{\beta^H(t); t \in \mathbb{R}\}$ with the covariance function

$$R_H(t,s) = E\bigl[\beta^H(s)\beta^H(t)\bigr] = \frac{1}{2}\bigl(s^{2H} + t^{2H} - |t-s|^{2H}\bigr), \quad s,t \in \mathbb{R}.$$

It is well known that fBm $\beta^H(t)$ with $H > \frac{1}{2}$ admits the Wiener integral representation of the following form:

$$\beta^H(t) = \int_0^t K_H(t,s)\,dB(s),$$

where $B$ is a standard Brownian motion and the kernel $K_H(t,s)$ is given by

$$K_H(t,s) = c_H \int_s^t (u-s)^{H-\frac{3}{2}} \left(\frac{u}{s}\right)^{H-\frac{1}{2}} du, \quad t > s,$$

with $c_H = \sqrt{\frac{H(2H-1)}{\beta(2-2H, H-\frac{1}{2})}}$, here $\beta(\cdot,\cdot)$ denotes the Beta function.

Let $T > 0$ be an arbitrary fixed horizon and $\phi \in L^2(0,T)$ be a deterministic function. It is well known from [25] that the Wiener integral of $\phi$ with respect to $\beta^H$ is given by

$$\int_0^T \phi(s)\,d\beta^H(s) = \int_0^T K_H^{\star}\phi(s)\,dB(s),$$

where $K_H^{\star}\phi(s) = \int_s^T \phi(r) \frac{\partial K_H}{\partial r}(r,s)\,dr$.

### 2.2 Stochastic integral in Hilbert space

We now introduce fBms with values in a Hilbert space and the corresponding stochastic integral. Let $V = (V, (\cdot,\cdot)_V, \|\cdot\|_V)$ be a real, separable Hilbert space. $\mathcal{L}(V,\mathcal{H})$ represents the space of all bounded linear operators from $V$ to $\mathcal{H}$. Let $\{e_n, n \in N\}$ be a complete orthonormal basis in $V$ and $Q \in \mathcal{L}(V,\mathcal{H})$ be an operator defined by $Qe_n = \lambda_n e_n$ with $\operatorname{Tr} Q = \sum_{n=1}^{\infty} \lambda_n < +\infty$, where $\{\lambda_n, n \in N\}$ are non-negative real numbers. A $V$-valued infinite-dimensional fBm with covariance $Q$ can be defined by

$$B^H(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} e_n \beta_n^H(t),$$

where $\beta_n^H$ are real, independent fBms with the same Hurst parameter $H \in (\frac{1}{2}, 1)$.

Let $L_2^0(V,\mathcal{H})$ denote the space of all $\Phi \in \mathcal{L}(V,\mathcal{H})$ such that $\Phi Q^{\frac{1}{2}}$ is a Hilbert-Schmidt operator. The norm is defined by

$$\|\Phi\|_{L_2^0}^2 = \sum_{n=1}^{\infty} \|\sqrt{\lambda_n}\Phi e_n\|^2.$$

Generally, $\Phi$ is called a $Q$-Hilbert-Schmidt operator from $V$ to $\mathcal{H}$.

**Definition 2.1** Let $\phi : [0, T] \to L_2^0(V, \mathcal{H})$ such that

$$\sum_{n=1}^{\infty} \left\| K_H^{\star}\left(\phi Q^{\frac{1}{2}} e_n\right) \right\|_{L_2^0}^2 < \infty.$$

Then the stochastic integral of $\phi$ with respect to the infinite-dimensional fBm $B^H$ is defined by

$$\int_0^t \phi(s)\,dB^H(s) := \sum_{n=1}^{\infty} \int_0^t \phi(s) Q^{\frac{1}{2}} e_n \,d\beta_n^H(s) = \sum_{n=1}^{\infty} \int_0^t \left(K_{\mathcal{H}}^{\star}\left(\phi Q^{\frac{1}{2}} e_n\right)\right)(s)\,dB(s).$$

**Lemma 2.1** [11] *If $\phi : [0, T] \to L_2^0(V, \mathcal{H})$ satisfying*

$$\sum_{n=1}^{\infty} \left\| \phi Q^{\frac{1}{2}} e_n \right\|_{L^{\frac{1}{H}}([0,T],\mathcal{H})} < \infty,$$

*and for $\tau, \sigma \in [0, T]$ with $\sigma > \tau$, then*

$$E\left\| \int_\tau^\sigma \phi(s)\,dB^H(s) \right\|^2 \le C_H(\sigma - \tau)^{2H-1} \sum_{n=1}^{\infty} \int_\tau^\sigma \left\| \phi(s) Q^{\frac{1}{2}} e_n \right\|^2 ds,$$

*where $C_H$ is a constant depending on $H$. If, in addition, $\sum_{n=1}^{\infty} \|\phi(t)Q^{\frac{1}{2}} e_n\|$ is uniformly convergent for $t \in [0, T]$, then*

$$E\left\| \int_\tau^\sigma \phi(s)\,dB^H(s) \right\|^2 \le C_H(\sigma - \tau)^{2H-1} \int_\tau^\sigma \left\| \phi(s) \right\|_{L_2^0}^2 ds.$$

**Remark 1** If $\{\lambda_n\}_{n\in\mathbb{N}}$ is a bounded sequence of non-negative real numbers such that the unclear operator $Q$ satisfies $Qe_n = \lambda e_n$, assuming that there exists a positive constant $k_\Phi$ such that

$$\left\| \Phi(t) \right\|_{L_2^0} \le k_\Phi, \quad \text{uniformly in } [0, T],$$

then it is obvious that $\sum_{n=1}^{\infty} \|\Phi(t)Q^{\frac{1}{2}} e_n\|$ is uniformly convergent for $t \in [0, T]$.

### 2.3 Deterministic integro-differential equations in Banach spaces

In this subsection, we recall some basic notations and properties needed in the sequel. For more details on this subsection, we refer to [26].

Let $(X, \|\cdot\|)$ be a Banach space, $C(I; X)$ denotes the space of all continuous functions from $I$ into $X$. We consider the following integral-differential equation with nonlocal condition:

$$\begin{cases} u'(t) = A(t)[u(t) + \int_0^t B(t, s)u(s)\,ds] + h(t, u(t)), \\ u(0) + \nu(x) = u_0, \quad t \in [0, T], u_0 \in X. \end{cases} \tag{2.1}$$

**Definition 2.2** [26] A resolvent operator for problem (2.1) is a bounded operator-valued function $R(t, s) \in \mathcal{L}(X, X)$, $0 \le s \le t \le T$, such that the properties

(a) $R(s,s) = I$, $0 \leq s \leq T$, $\|R(t,s)\| \leq Me^{\gamma(t-s)}$ for some constants $M$ and $\gamma$;
(b) for each $x \in X$, $R(t,s)x$ is strongly continuous in $s$ and $t$;
(c) for each $x \in X$, $R(t,s)x$ is continuous differentiable in $s \in [0,T]$, $t \in [s,T]$

and

$$\frac{\partial R}{\partial s}(t,s)x = -R(t,s)A(s)x - \int_s^t R(t,\tau)B(\tau,s)A(s)x\,d\tau,$$
$$\frac{\partial R}{\partial t}(t,s)x = A(t)R(t,s)x + \int_s^t B(t,\tau)A(\tau)R(\tau,s)x\,d\tau,$$

with $\frac{\partial R}{\partial t}(t,s)x$, $\frac{\partial R}{\partial s}(t,s)x$ are strongly continuous on $0 \leq s \leq t \leq T$.

We mention here that the resolvent operator $R(t,s)$ can be reduced from the evolution operator of the generator $A(t)$ under some suitable conditions (see [26] for the details).

**Definition 2.3** [26] A continuous function $u(\cdot): J \to X$ is said to be a mild solution to problem (2.1) if, for all $u_0 \in X$, it satisfies the following integral equation:

$$u(t) = R(t,0)\bigl[u_0 - v(x)\bigr] + \int_0^t R(t,s)h\bigl(s,u(s)\bigr)\,ds.$$

### 2.4 Hypotheses on stochastic system (1.1)

Let $C(J,\mathcal{H}) := \{x : J \to \mathcal{H}, x \text{ is continuous almost surely}\}$. The collection of all strongly measurable, square integrable, $\mathcal{H}$-valued random variables, denoted by $L^2(\Omega,\mathcal{H})$, is a Banach space with the norm $\|x\|_2 = (E\|x\|^2)^{\frac{1}{2}}$. Let $C(J,L^2(\Omega,H))$ be the Banach space of all continuous mappings from $J$ into $L^2(\Omega,H)$ such that $\sup_{t\in J} E|x(t)|_H^2 < \infty$. We denote by $\mathcal{C}$ the space of all $\mathcal{F}_t$-adapted stochastic processes $x \in C(J,L^2(\Omega,H))$ equipped with the norm $\|x\|_{\mathcal{C}} = (\sup_{s\in J} E|x(s)|_H^2)^{\frac{1}{2}}$, it is clear that $(\mathcal{C}, \|\cdot\|_{\mathcal{C}})$ is a Banach space.

In this work, we need the following assumptions:

($A_1$) The resolvent operator $R(t,s)$, $0 \leq s \leq t$, is compact and there exist some positive constants $M$, $\gamma$ such that $\|R(t,s)\| \leq Me^{\gamma(t-s)}$.

($A_2$) There exists a constant $K_f > 0$ such that for all $x,y \in \mathcal{H}$, $t \in [0,T]$,

$$\bigl\|f(t,x) - f(t,y)\bigr\|^2 \leq K_f\|x-y\|^2.$$

($A_3$) $\sigma \in L_2^0(V,\mathcal{H})$ with $\sum_{n=1}^{\infty} \|\sigma Q^{\frac{1}{2}} e_n\|_{L_2^0} < \infty$.

($A_4$) $g : C(J,\mathcal{H}) \to \mathcal{H}$ is square integrable and satisfies:

- (4a) there exists a constant $\alpha \in (0,T)$ such that $g(\xi) = g(\eta)$ for any $\xi,\eta \in C(J,\mathcal{H})$ with $\xi = \eta$ on $[\alpha,T]$;
- (4b) there exists a nondecreasing function $\Gamma : [0,+\infty) \to (0,+\infty)$ such that

$$E\bigl|g(x)\bigr|_H^2 \leq \Gamma\bigl(\|x\|_{\mathcal{C}}^2\bigr).$$

For simplicity, we also assume that $f(t,0) = 0$ for all $t \in J$.

At the end of this section, we recall the well-known Leray-Schauder nonlinear alternative, which is employed to prove our results.

**Lemma 2.2** [27] *Let $X$ be a closed and convex subset of a Banach space $B$. Assume that $\mathcal{U}$ is a relatively open subset of $X$ with $0 \in \mathcal{U}$ and $\phi : \bar{\mathcal{U}} \to X$ is a compact map, then either*
(a) *$\phi$ has a fixed point in $\bar{\mathcal{U}}$, or*
(b) *there is a point $u \in \partial\mathcal{U}$ and $\lambda \in (0,1)$ with $u \in \lambda\phi(u)$.*

## 3 Main results

In this section, we state and establish our main result. We first present the definition of the mild solution for system (1.1).

**Definition 3.1** A $\mathcal{H}$-valued, $\mathcal{F}_t$-adapted stochastic process $x(t)$ is called a mild solution of (1.1), if, for all $t \in [0,T]$, $x(t)$ satisfies

$$x(t) = R(t,0)\big[x_0 - g(x)\big] + \int_0^t R(t,s)f\big(s,x(s)\big)\,ds + \int_0^t R(t,s)\sigma\,dB^H(s).$$

**Theorem 3.1** *Let assumptions $(A_1)$-$(A_4)$ be satisfied. Then the system* (1.1) *has at least one mild solution provided that there exists a constant $N_0 > 0$ such that*

$$\frac{N_0}{[6M^2(E|x_0|_H^2 + \Gamma(N_0)) + 3M^2C_H T^{2H-1}\|\sigma\|^2_{\mathcal{L}_2^0}(2\gamma)^{-1}]e^{(3M^2K_fT+2\gamma)T}} > 1. \tag{3.1}$$

*Proof* We first introduce an equivalent norm $\|\cdot\|_{\tilde{\mathcal{C}}}$ in the space $\mathcal{C}$ defined by

$$\|x\|^2_{\tilde{\mathcal{C}}} := \sup_{t\in J} e^{-Lt}E\big|x(t)\big|^2, \quad \text{where } L = 2M^2TK_f + 2\gamma.$$

It is routine to check that $\tilde{\mathcal{C}} := (\mathcal{C}, \|\cdot\|_{\tilde{\mathcal{C}}})$ is a Banach space. Let $u \in \mathcal{C}$ be fixed. For $t \in J$, $x \in \tilde{\mathcal{C}}$, define a mapping $P_u$ on $\tilde{\mathcal{C}}$ by

$$(P_ux)(t) = R(t,0)\big[x_0 - g(u)\big] + \int_0^t R(t,s)f\big(s,x(s)\big)\,ds + \int_0^t R(t,s)\sigma\,dB^H(s). \tag{3.2}$$

Since $R(t,s)$ is strongly continuous, by the assumptions on $f$, $g$, and $\sigma$, it is easy to check that $P_u$ maps $\tilde{\mathcal{C}}$ into itself. For $x,y \in \tilde{\mathcal{C}}$, by $(A_1)$ and Hölder's inequality, we have

$$\begin{aligned}
e^{-Lt}E\big|(P_ux)(t) - (P_uy)(t)\big|^2 &= e^{-Lt}E\left\|\int_0^t R(t,s)\big[f\big(s,x(s)\big) - f\big(s,y(s)\big)\big]\,ds\right\|^2 \\
&\le e^{-Lt}E\left[\int_0^t \big\|R(t,s)\big[f\big(s,x(s)\big) - f\big(s,y(s)\big)\big]\big\|\,ds\right]^2 \\
&\le e^{-Lt}M^2T\int_0^t e^{2\gamma(t-s)}E\big\|f\big(s,x(s)\big) - f\big(s,y(s)\big)\big\|^2\,ds \\
&\le e^{-Lt}M^2TK_f\int_0^t e^{2\gamma(t-s)}E\big\|x(s) - y(s)\big\|^2\,ds \\
&\le M^2TK_f\int_0^t e^{(2\gamma-L)(t-s)}e^{-Ls}E\big\|x(s) - y(s)\big\|^2\,ds \\
&\le \frac{M^2TK_f}{L-2\gamma}\sup_{s\in J} e^{-Ls}E\big\|x(s) - y(s)\big\|^2,
\end{aligned}$$

recalling the choice of $L$, we have

$$e^{-Lt}E\big|(P_u x)(t)-(P_u y)(t)\big|^2 \le \frac{1}{2}\sup_{s\in J} e^{-Ls}E\big\|x(s)-y(s)\big\|^2,$$

which implies that $\|P_u x - P_u y\|^2_{\tilde{\mathcal{C}}} \le \frac{1}{2}\|x-y\|^2_{\tilde{\mathcal{C}}}$. By the Banach contraction mapping principle, it follows that $P_u$ has a unique fixed point $x_u \in \tilde{\mathcal{C}}$ such that

$$(x_u)(t) = R(t,0)\big[x_0 - g(u)\big] + \int_0^t R(t,s)f\big(s,x_u(s)\big)\,ds + \int_0^t R(t,s)\sigma\,dB^H(s).$$

Based on this fact, for some $\alpha \in (0,T)$, we set

$$\bar{u}(t) := \begin{cases} u(t), & t\in(\alpha,T]; \\ u(\alpha), & t\in[0,\alpha]. \end{cases}$$

From (3.2), it follows that

$$x_{\bar{u}}(t) = R(t,0)\big[x_0 - g(\bar{u})\big] + \int_0^t R(t,s)f\big(s,x_{\bar{u}}(s)\big)\,ds + \int_0^t R(t,s)\sigma\,dB^H(s).$$

Let $\mathcal{C}_\alpha := C([\alpha,T],L^2(\Omega,H))$. Define a map $\Phi : \mathcal{C}_\alpha \to \mathcal{C}_\alpha$ by

$$(\Phi u)(t) = x_{\bar{u}}(t), \quad t\in[\alpha,T].$$

We will show that the operator $\Phi$ satisfies all the conditions of Lemma 2.2. The proof will be divided into the following steps.

*Step 1.* $\Phi$ maps bounded sets into bounded sets in $C_\alpha$. Let $r > 0$ and

$$B_r(\alpha) := \Big\{x\in\mathcal{C}_\alpha : \sup_{t\in[\alpha,T]} E\big|x(t)\big|^2 \le r\Big\}.$$

It is obvious that $B_r(\alpha)$ is a bounded closed convex subset in $\mathcal{C}_\alpha$. We now show that there exists a constant $L_0 > 0$ such that $\|\Phi v\|^2_{\mathcal{C}} \le L_0$ for each $v \in B_r(\alpha)$.

Let $v \in B_r(\alpha)$ and $t \in J$. Recalling the definition of $\Phi$, by the assumptions $(A_1)$-$(A_4)$ together with Hölder's inequality we have

$$\begin{aligned} e^{-2\gamma t}E\big\|x_{\bar{v}}(t)\big\|^2 \le{}& 3e^{-2\gamma t}E\big\|R(t,0)\big[x_0 - g(\bar{v})\big]\big\|^2 \\ &+ 3e^{-2\gamma t}E\bigg\|\int_0^t R(t,s)f\big(s,x_{\bar{v}}(s)\big)\,ds\bigg\|^2 + 3e^{-2\gamma t}E\bigg\|\int_0^t R(t,s)\sigma\,dB^H(s)\bigg\|^2 \\ \le{}& 6M^2\big[E\|x_0\|^2 + E\big\|g(\bar{v})\big\|^2\big] + 3e^{-2\gamma t}M^2 T\int_0^t e^{2\gamma(t-s)}E\big\|f\big(s,x_{\bar{v}}(s)\big)\big\|^2\,ds \\ &+ 3e^{-2\gamma t}M^2 C_H T^{2H-1}\int_0^t e^{2\gamma(t-s)}\|\sigma\|^2_{\mathcal{L}_2^0}\,ds \\ \le{}& L_1 + 3M^2 T\int_0^t e^{-2\gamma s}E\big\|x_{\bar{v}}(s)\big\|^2\,ds, \end{aligned}$$

where $L_1 = 6M^2[E\|x_0\|^2 + \Gamma(r)] + \frac{3M^2C_H T^{2H-1}\|\sigma\|^2_{\mathcal{L}_2^0}}{2\gamma}$. This gives, by Gronwall's inequality,

$$\|\Pi v\|^2_{\mathcal{C}} = \|x_{\bar{v}}\|^2_{\mathcal{C}} \le L_1 e^{3M^2T+2\gamma T} := L_0.$$

*Step 2.* $\Phi$ maps bounded sets into equicontinuous sets of $\mathcal{C}_\alpha$. Let $v \in B_r(\alpha)$ and $\alpha \le t_1 < t_2 \le T$, we have

$$\begin{aligned}
&E\big\|(\Phi v)(t_2) - (\Phi v)(t_1)\big\|^2 \\
&\quad\le 3\big\|R(t_2,0) - R(t_1,0)\big\|^2 E\big\|x_0 - g(\bar{v}))\big\|^2 \\
&\qquad + 3E\bigg\|\int_0^{t_1}\big[R(t_2,s) - R(t_1,s)\big]f\big(s,x_{\bar{v}}(s)\big)\,ds + \int_{t_1}^{t_2} R(t_2,s)f\big(s,x_{\bar{v}}(s)\big)\,ds\bigg\|^2 \\
&\qquad + 3E\bigg\|\int_0^{t_1}\big[R(t_2,s) - R(t_1,s)\big]\sigma\,dB^H(s) + \int_{t_1}^{t_2} R(t_2,s)\sigma\,dB^H(s)\bigg\|^2 \\
&\quad= I_1 + I_2 + I_3.
\end{aligned} \tag{3.3}$$

By the strong continuity of $R(t,s)$ and the assumption of $g$, we have

$$I_1 \le 6\big\|R(t_2,0) - R(t_1,0)\big\|^2\big[E\|x_0\|^2 + \Gamma(r)\big] \to 0, \quad \text{as } t_1 \to t_2.$$

For the term $I_2$, by Hölder's inequality and $(A_1)$ we have

$$\begin{aligned}
I_2 &\le 6t_1\int_0^{t_1}\big\|R(t_2,s) - R(t_1,s)\big\|^2 E\big\|f\big(s,x_{\bar{v}}(s)\big)\big\|^2\,ds \\
&\quad + 6M^2(t_2 - t_1)\int_{t_1}^{t_2} e^{2\gamma(t_2-s)}E\big\|f\big(s,x_{\bar{v}}(s)\big)\big\|^2\,ds,
\end{aligned} \tag{3.4}$$

applying properties (b) of Definition 2.2 we get

$$\lim_{t_1\to t_2}\big\|R(t_2,s) - R(t_1,s)\big\|^2 E\big\|f\big(s,x_{\bar{v}}(s)\big)\big\|^2 = 0,$$

by the assumption on $f$, we further derive that

$$\begin{aligned}
&\int_0^{t_1}\big\|R(t_2,s) - R(t_1,s)\big\|^2 E\big\|f\big(s,x_{\bar{v}}(s)\big)\big\|^2\,ds \\
&\quad= \int_0^{t_1}\big\|R(t_2,s) - R(t_1,s)\big\|^2 E\big\|f\big(s,x_{\bar{v}}(s)\big) - f(s,0)\big\|^2\,ds \\
&\quad\le 2M^2\int_0^{t_1}\big[e^{2\gamma(t_2-s)} + e^{2\gamma(t_1-s)}\big]\,ds \sup_{t\in[\alpha,T]} E\big\|x_{\bar{v}}(t)\big\|^2 < \infty,
\end{aligned}$$

we conclude, by the dominated convergence theorem, that the first term of (3.4) tends to 0 independently of $v$. Similarly, we can deduce that the second term of (3.4) also tends to 0 independently of $v$.

For $I_3$, by Lemma 2.1 and the properties of $R(t,s)$ we have

$$\begin{aligned} I_3 &\le 6C_H t_1^{2H-1}\int_0^{t_1}\bigl\|R(t_2,s)-R(t_1,s)\bigr\|^2\|\sigma\|^2_{\mathcal{L}_2^0}\,ds \\ &\quad + 6C_H(t_2-t_1)^{2H-1}\int_{t_1}^{t_2} e^{2\gamma(t_2-s)}\|\sigma\|^2_{\mathcal{L}_2^0}\,ds \\ &\to 0, \quad \text{as } t_1\to t_2, \end{aligned}$$

where $C_H$ is the constant in Lemma 2.1.

Thus, the right hand side of (3.3) tends to 0 independently of $\nu$ when $t_1 \to t_2$. From the assumption $(A_1)$, the compactness of $R(t,s)$, $t > s > 0$ implies the continuity in the uniform operator topology. Therefore, the set $\{\Phi\nu, \nu \in B_r(\alpha)\}$ is equicontinuous.

*Step 3.* $\Phi$ maps $B_r(\alpha)$ into a relatively compact set in $\mathcal{H}$. That is, for every $t \in [\alpha, T]$, the set $\Pi(t) := \{(\Phi\nu)(t), \nu \in B_r(\alpha)\}$ is relatively compact in $H$.

Let $t \in [\alpha, T]$ be fixed and $0 < \epsilon < t$. For $\nu \in B_r(\alpha)$, we define an operator $\Phi^\epsilon$ on $B_r(\alpha)$ by

$$\begin{aligned} \bigl(\Phi^\epsilon \nu\bigr)(t) &= R(t,0)\bigl[x_0 - g(\bar{\nu})\bigr] + \int_0^{t-\epsilon} R(t,s)f\bigl(s, x_{\bar{\nu}}(s)\bigr)\,ds \\ &\quad + \int_0^{t-\epsilon} R(t,s)\sigma\,dB^H(s). \end{aligned}$$

By the compactness of $R(t,s)$, $t,s > 0$, one sees that the set $\Phi^\epsilon(t) = \{(\Phi^\epsilon\nu)(t), \nu \in B_r(\alpha)\}$ is relatively compact in $H$ for each $0 < \epsilon < t$. Moreover, applying Hölder's inequality together with Lemma 2.1 we have

$$\begin{aligned} &E\bigl\|(\Phi\nu)(t) - \bigl(\Phi^\epsilon\nu\bigr)(t)\bigr\|^2 \\ &\quad\le 2E\biggl\|\int_{t-\epsilon}^t R(t,s)f\bigl(s,x_{\bar{\nu}}(s)\bigr)\,ds\biggr\|^2 + 2E\biggl\|\int_{t-\epsilon}^t R(t,s)\sigma\,dB^H(s)\biggr\|^2 \\ &\quad\le 2M^2\epsilon\int_{t-\epsilon}^t e^{2\gamma(t-s)}E\bigl\|f\bigl(s,x_{\bar{\nu}}(s)\bigr)\bigr\|^2\,ds + 2M^2C_H\epsilon^{2H-1}\int_{t-\epsilon}^t e^{2\gamma(t-s)}\|\sigma\|^2_{\mathcal{L}_2^0}\,ds, \end{aligned}$$

which implies that the relatively compact set $\{(\Phi^\epsilon\nu) : \nu \in B_r(\alpha)\}$ arbitrarily close to the set $\{(\Phi\nu) : \nu \in B_r(\alpha)\}$. Thus, the set $\{(\Phi\nu), \nu \in B_r(\alpha)\}$ is relatively compact in $H$.

*Step 4.* We show that there exists an open set $\mathbf{D} \subseteq \mathcal{C}_\alpha$ such that, for all $\nu \in \partial\mathbf{D}$, $\nu \notin \{\lambda(\Phi\nu) : \lambda \in (0,1)\}$.

Let $\nu \in \mathcal{C}_\alpha$ be a possible solution of $\nu = \lambda(\Phi\nu)$ for some $\lambda \in (0,1)$. Then, for each $t \in J$, we have

$$\begin{aligned} \nu(t) &= \lambda(\Phi\nu)(t) = \lambda x_{\bar{\nu}}(t) \\ &= \lambda R(t,0)\bigl[x_0 - g(\bar{\nu})\bigr] + \lambda\int_0^t R(t,s)f\bigl(s,x_{\bar{\nu}}(s)\bigr)\,ds + \lambda\int_0^t R(t,s)\sigma\,dB^H(s). \end{aligned}$$

It is clear that

$$\|\nu\|^2_{\mathcal{C}} \le \|\Phi\nu\|^2_{\mathcal{C}} = \|x_{\bar{\nu}}\|^2_{\mathcal{C}}. \tag{3.5}$$

On the other hand, by the assumptions on $R(t,s)$, $f$, and $g$, we can use a similar argument to before, yielding

$$\begin{aligned}e^{-2\gamma t}E\big\|x_{\bar{v}}(t)\big\|^2 &\le 6M^2\big[E\|x_0\|^2+\Gamma\big(\|\bar{v}\|_{\mathcal{C}}^2\big)\big]+3M^2T\int_0^t e^{-2\gamma s}E\big\|f\big(s,x_{\bar{v}}(s)\big)\big\|^2\,ds\\&\quad+3M^2C_HT^{2H-1}\int_0^t e^{-2\gamma s}\|\sigma\|_{\mathcal{L}_2^0}^2\,ds\\&\le 6M^2\big[E\|x_0\|^2+\Gamma\big(\|\bar{v}\|_{\mathcal{C}}^2\big)\big]+3M^2HT^{2H-1}(2\gamma)^{-1}\|\sigma\|_{\mathcal{L}_2^0}^2\\&\quad+3M^2TK_f\int_0^t e^{-2\gamma s}E\big\|x_{\bar{v}}(s))\big\|^2\,ds.\end{aligned}$$

Applying Gronwall's inequality it follows that

$$E\big|x_{\bar{v}}(t)\big|_H^2\le\big[6M^2\big(E\|x_0\|^2+\Gamma\big(\|\bar{v}\|_{\mathcal{C}}^2\big)\big)+3M^2C_HT^{2H-1}(2\gamma)^{-1}\|\sigma\|_{\mathcal{L}_2^0}^2\big]e^{3M^2K_fT^2+2\gamma T}.$$

Recalling (3.5), we have

$$\|v\|_{\mathcal{C}}^2\le\big[6M^2\big(E\|x_0\|^2+\Gamma\big(\|\bar{v}\|_{\mathcal{C}}^2\big)\big)+3M^2C_HT^{2H-1}(2\gamma)^{-1}\|\sigma\|_{\mathcal{L}_2^0}^2\big]e^{3M^2K_fT^2+2\gamma T}.$$

That is,

$$\frac{\|v\|_{\mathcal{C}}^2}{[6M^2(E\|x_0\|^2+\Gamma(\|\bar{v}\|_{\mathcal{C}}^2))+3M^2C_HT^{2H-1}(2\gamma)^{-1}\|\sigma\|_{\mathcal{L}_2^0}^2]e^{3M^2K_fT^2+2\gamma T}}\le 1.$$

By assumption (3.1), there exists a constant $M^\star$ such that $\|v\|_{\mathcal{C}}^2\neq M^\star$.

Set

$$\mathbf{D}=\Big\{v\in C_\alpha:\sup_{t\in[\alpha,T]}E\big\|v(s)\big\|^2<M^\star\Big\}.$$

It is obvious that $\mathbf{D}$ is an open subset of $B_r(\alpha)$ for all $r\ge M^\star$. We infer that, for all $v\in\partial\mathbf{D}$, $v\notin\{\lambda(\Phi v):\lambda\in(0,1)\}$.

From step 1 to step 4 together with the Arzela-Ascoli theorem, it suffices to show that $\Phi:\bar{\mathbf{D}}\to\mathcal{C}_\alpha$ is a compact operator, where $\bar{\mathbf{D}}$ is the closure of $\mathbf{D}$. As a consequence of Lemma 2.2, it follows that $\Phi$ has a fixed point $v_0\in\bar{\mathbf{D}}$. Let $y=x_{v_0}$. By (3.2), we have

$$\begin{aligned}y(t)&=R(t,0)\big[x_0-g(v_0)\big]+\int_0^t R(t,s)f\big(s,x_{v_0}(s)\big)\,ds+\int_0^t R(t,s)\sigma\,dB^H(s)\\&=R(t,0)\big[x_0-g(v_0)\big]+\int_0^t R(t,s)f\big(s,y(s)\big)\,ds+\int_0^t R(t,s)\sigma\,dB^H(s).\end{aligned}$$

Noting that for $t\in[\alpha,T]$,

$$y(t)=x_{v_0}(t)=x_{\bar{v}_0}(t)=(\Phi v_0)(t)=v_0(t),$$

which indicates, by the assumptions on $g$, that $g(v_0)=g(y)$. We conclude that $y(t)$ is a mild solution of system (1.1). The proof is completed. □

We conclude this section with a comment on a special case of the nonlocal Cauchy problem, namely, where $g(x)$ is given by $g(x) = \sum_{i=0}^{k} g_i(x(t_i))$. More precisely, consider the following stochastic integral-differential equation with nonlocal condition:

$$\begin{cases} dx(t) = A(t)[x(t) + \int_0^t B(t,s)x(s)\,ds + f(t,x(t))]\,dt + \sigma\,dB^H(t), \\ \quad t \in J := [0,T], \\ x(0) + \sum_{i=0}^{k} g_i(x(t_i)) = x_0, \quad 0 < t_1 < t_2 < \cdots < t_k < T, k \in \mathbb{N}^+. \end{cases} \tag{3.6}$$

We need the following assumptions.

$(A_5)$ For every $i$, $i = 1,2,\ldots,k$, $g_i(\cdot) : C(J,\mathcal{H}) \to \mathcal{H}$ satisfy:

- (5a) there exists a constant $\alpha \in (0,T)$ such that $g_i(\xi) = g_i(\eta)$ for any $\xi, \eta \in C(J,\mathcal{H})$ with $\xi = \eta$ on $[\alpha, T]$;
- (5b) there exist some positive constants $m_i$, $n_i$, $i = 1,2,\ldots,k$ such that

$$\left|g_i(x)\right|^2 \le m_i\|x\|^2 + n_i, \qquad 6kM^2K^\star \sum_{i=1}^{k} m_i < 1,$$

where $K^\star = e^{6M^2T^2K_f + 2\gamma T}$.

**Theorem 3.2** *Under the assumptions* $(A_1)$-$(A_3)$ *and* $(A_5)$, *the system* (3.6) *has at least one mild solution.*

*Proof* Since the proof resembles the arguments of Theorem 3.1, we sketch it only.

We define a map $g$ on $\mathcal{H}$ by

$$g(u) = \sum_{i=0}^{k} g_i\bigl(u(t_i)\bigr), \quad 0 < t_1 < t_2 < \cdots < t_k < T, k \in \mathbb{N}^+.$$

Let $L$, $\tilde{\mathcal{C}}$, $P_u$, $\bar{u}$, $\Phi$, and $B_r(\alpha)$ be as in Theorem 3.1. Thus, $P_u$ has a unique fixed point $x_u$. By (3.2), we have

$$\begin{aligned}(x_{\bar{u}})(t) &= R(t,0)\left[x_0 - \sum_{i=0}^{k} g_i\bigl(\bar{u}(t_i)\bigr)\right] \\ &\quad + \int_0^t R(t,s)f\bigl(s, x_{\bar{u}}(s)\bigr)\,ds + \int_0^t R(t,s)\sigma\,dB^H(s).\end{aligned}$$

By using the assumptions on $f$, $g$, and $\sigma$, we can deduce that

$$\begin{aligned}&e^{-2\gamma t}E\bigl\|x_{\bar{u}}(t)\bigr\|^2 \\ &\quad\le 6M^2\left[E\|x_0\|^2 + k\sum_{i=1}^{k} E\bigl\|g_i\bigl(\bar{u}(t_i)\bigr)\bigr\|^2\right] \\ &\qquad + 3M^2T\int_0^t e^{-2\gamma s}E\bigl\|f\bigl(s, x_{\bar{u}}(s)\bigr)\bigr\|^2\,ds + 3M^2C_HT^{2H-1}\int_0^t e^{-2\gamma s}\|\sigma\|^2_{\mathcal{L}_2^0}\,ds\end{aligned}$$

$$\leq 6M^2\left[E\|x_0\|^2 + k\sum_{i=1}^{k}\left(m_i\|\bar{u}\|_{\mathcal{C}}^2 + n_i\right)\right] + 3M^2C_HT^{2H-1}(2\gamma)^{-1}\|\sigma\|_{\mathcal{L}_2^0}^2$$
$$+ 6M^2T(2\gamma)^{-1}\sup_{s\in J}E\left\|f(s,0)\right\|^2 + 6M^2TK_f\int_0^t e^{-2\gamma s}E\left\|x_{\bar{u}}(s)\right\|^2\,ds.$$

By Gronwall's inequality it follows that

$$E\left\|x_{\bar{u}}(t)\right\|^2 \leq \left(\bar{L} + 6M^2k\sum_{i=1}^{k}m_i\|\bar{u}\|_{\mathcal{C}}^2\right)K^\star,$$

where

$$\bar{L} = 6M^2\left[E\|x_0\|^2 + k\sum_{i=1}^{k}n_i + T(2\gamma)^{-1}\sup_{s\in J}E\left\|f(s,0)\right\|^2\right] + 3M^2C_HT^{2H-1}(2\gamma)^{-1}\|\sigma\|_{\mathcal{L}_2^0}^2.$$

Noting that

$$\|u\|_{\mathcal{C}}^2 \leq \|x_{\bar{u}}\|_{\mathcal{C}}^2,$$

we further derive that

$$\|u\|_{\mathcal{C}}^2 \leq \frac{K^\star\bar{L}}{1 - 6M^2kK^\star\sum_{i=1}^{k}m_i}.$$

Let $\epsilon > 0$ be fixed. Set $M^\star = \frac{K^\star\bar{L}}{1-6M^2kK^\star\sum_{i=1}^{k}m_i} + \epsilon$ and

$$D = \left\{v \in C\left([\alpha,T],\mathcal{H}\right) : \sup_{t\in[\alpha,T]}E\left\|v(t)\right\|^2 < M^\star\right\}.$$

Then $D$ is an open subset of $B_r(\alpha)$ for all $r \geq M^\star$. In the sequel, we can employ similar arguments to step 1-step 4 of Theorem 3.1 and prove the existence of mild solutions for system (3.6), we omit it here. This completes the proof. □

## 4 An example

As an application, we present an example to illustrate our results. Considering the following stochastic integro-differential equation with nonlocal condition:

$$\begin{cases} \frac{\partial x(t,z)}{\partial t} = \frac{\partial^2}{\partial z^2}a(t)x(t,z) + \int_0^t B(t,s)x(t,z)\,ds + \sin tx(t,z) + \frac{dB^H(t)}{dt}, \\ \quad x \in [0,\pi], t \in [0,T], \\ x(t,0) = x(t,\pi) = 0, \quad t \in [0,T], \\ x(0,z) = \sum_{i=1}^{k}h_i(z)x(t_i,z), \quad z \in [0,\pi], \end{cases} \tag{4.1}$$

where $0 < t_1 < t_2 < \cdots < t_k < T$, $B^H(t)$ stands for a cylindrical fBm defined on a complete probability space $(\Omega,\mathcal{F},P,\{\mathcal{F}_t\})$, $h_i \in L^2([0,\pi])$, and $a(t)$ are continuous functions.

To rewrite the stochastic differential equation in the abstract form (1.1), let $\mathcal{H} = K = V = L^2([0,\pi])$ with the norm $\|\cdot\|$. The operator $A(t)$ is defined by $A(t)x(z) = a(t)\frac{\partial^2 x}{\partial z^2}$ with the domain

$$\mathcal{D}(A) = \{x \in \mathcal{H}, x, x'' \text{ are absolutely continuous}, x'' \in \mathcal{H} \text{ and } x(0) = x(\pi) = 0\}.$$

Then $A(t)$ generates an evolution operator and $R(t,s)$ can be reduced by this evolution operator such that $(A_1)$ is satisfied (see [26] for details).

Let

$$f(t, x(t,\cdot)) = \sin tx(t,\cdot), \qquad g_i(x)(\cdot) = h_i(\cdot)x(t_i,\cdot), \qquad \sigma = I.$$

Then system (4.1) has an abstract formulation given by (3.6) and the assumptions $(A_2)$, $(A_3)$, and $(A_5)$ are satisfied with $K_f = 1$, $m_i = \sup_{z\in[0,\pi]} \|h_i(z)\|^2$, $n_i = 0$. By Theorem 3.2, the system (4.1) has a mild solution on $[0,T]$ provided that $6kM^2K^\star \sum_{i=1}^k m_i < 1$.

## 5 Conclusion

In this paper, we have investigated the existence of mild solutions for a class of nonlocal stochastic integro-differential equations driven by fractional Brownian motion in a Hilbert space. By employing the Leray-Schauder nonlinear alternative, the existence of mild solutions is proved, a special case of this result is given. An example is presented to illustrate our theoretical results. In a sequel of this paper we will study the asymptotic behavior of the mild solutions, and we are also interested in studying the existence and asymptotic stability of stochastic evolution equations driven by fractional Brownian motion with the Hurst parameter $H \in (0, \frac{1}{2})$.

#### Competing interests

The authors declare to have no competing interests.

#### Authors' contributions

All authors participated in drafting and checking the manuscript, and approved the final manuscript.

#### Author details

[1]Department of Statistics, Anhui Normal University, South Huajin Rd., Wuhu, 241000, P.R. China. [2]School of Sciences, Ningbo University of Technology, Fenghua Rd., Ningbo, 315211, P.R. China.

#### Acknowledgements

The authors are grateful to the anonymous referees and the associate editor for valuable comments and suggestions to improve this paper. The Project is sponsored by NSFC (11571071, 11271020, 11401010), Mathematical Tianyuan Foundation of China (11426036), Zhejiang Provincial Natural Science Foundation (LQ16A010006) and Natural Science Foundation of Ningbo Municipality (2015A610158).

Received: 20 January 2016 Accepted: 15 April 2016 Published online: 26 April 2016

#### References

1. Christodoulou-Volos, C, Siokis, FM: Long range dependence in stock market returns. Appl. Financ. Econ. **16**, 1331-1338 (2006)
2. Biagini, F, Hu, YZ, et al.: Stochastic Calculus for Fractional Brownian Motion and Applications. Springer, London (2010)
3. Hu, YZ: Integral transformations and anticipative calculus for fBms. Mem. Am. Math. Soc. **175**, 825 (2005)
4. Hu, YZ, Nualart, D, Xu, FJ: Central limit theorem for an additive functional of the fractional Brownian motion. Ann. Probab. **32**(1A), 168-203 (2014)
5. Mishura, YS: Stochastic Calculus for Fractional Brownian Motion and Related Processes. Springer, Berlin (2008)
6. Hu, YZ, Lu, F, Nualart, D: Feynman-Kac formula for the heat equation driven by fractional noise with Hurst parameter $H < \frac{1}{2}$. Ann. Probab. **40**(3), 1048-1063 (2012)
7. Fan, X: Integration by parts formula and applications for SDEs driven by fractional Brownian motions. Stoch. Anal. Appl. **33**(2), 199-212 (2015)
8. Liu, JF, Yan, LT: Solving a nonlinear fractional stochastic partial differential equation with fractional noise. J. Theor. Probab. **75**(16), 1-41 (2014)
9. Liu, JF, Yan, LT: On a semilinear stochastic partial differential equation with double-parameter fractional noises. Sci. China Math. **57**(4), 855-872 (2014)

10. Boufoussi, B, Hajji, S: Neutral stochastic functional differential equations driven by a fractional Brownian motion in a Hilbert space. Stat. Probab. Lett. **82**(8), 1549-1558 (2012)
11. Caraballo, T, Garrido-Atienza, MJ, Taniguchi, T: The existence and exponential behavior of solutions to stochastic delay evolution equations with a fractional Brownian motion. Nonlinear Anal. **74**(11), 3671-3684 (2011)
12. Duncan, TE, Maslowski, B, Pasik-Duncan, B: Stochastic equations in Hilbert space with a multiplicative fractional Gaussian noise. Stoch. Process. Appl. **115**(8), 1357-1383 (2005)
13. Byszewski, L, Lakshmikantham, V: Theorem about the existence and uniqueness of a solution of a nonlocal Cauchy problem in a Banach space. Appl. Anal. **40**, 11-19 (1990)
14. Byszewski, L: Theorems about the existence and uniqueness of solutions of a semilinear evolution nonlocal Cauchy problem. J. Math. Anal. Appl. **162**, 494-505 (1991)
15. Byszewski, L, Akca, H: Existence of solutions of a semilinear functional differential evolution nonlocal problem. Nonlinear Anal. **34**, 65-72 (1998)
16. Deng, K: Exponential decay of solutions of semilinear parabolic equations with nonlocal initial conditions. J. Math. Anal. Appl. **179**, 630-637 (1993)
17. Sobczyk, K: Stochastic Differential Equations with Applications to Physics and Engineering. Kluwer Academic, London (1991)
18. Muthukumar, P, Rajivganthi, C: Approximate controllability of fractional order neutral stochastic integro-differential system with nonlocal conditions and infinite delay. Taiwan. J. Math. **17**(5), 1693-1713 (2013)
19. Balasubramaniam, P, Park, JY, Kumara, AVA: Existence of solutions for semilinear neutral stochastic functional differential equations with nonlocal conditions. Nonlinear Anal. **71**, 1049-1058 (2009)
20. Yan, ZM: On a nonlocal problems for partial stochastic functional integro-differential equations in Hilbert spaces. Electron. J. Math. Anal. Appl. **1**(2), 212-229 (2013)
21. Balasubramaniam, P, Park, JY: Nonlocal Cauchy problem for second order stochastic evolution equations in Hilbert space. Dyn. Syst. Appl. **16**, 713-728 (2007)
22. Cui, J, Yan, LT: Existence results for impulsive neutral second-order stochastic evolution equations with nonlocal conditions. Math. Comput. Model. **57**, 2378-2387 (2013)
23. Balasubramaniam, P, Tamilalagan, P: Approximate controllability of a class of fractional neutral stochastic integro-differential inclusions with infinite delay by using Mainardis function. Appl. Math. Comput. **256**, 232-246 (2015)
24. Balasubramaniam, P, Kumaresan, N, et al.: Local and global existence of mild solution for impulsive fractional stochastic differential equation. Bull. Malays. Math. Sci. Soc. **38**, 867-884 (2015)
25. Nualart, D: Malliavin Calculus and Related Topics, 2nd edn. Springer, Berlin (2006)
26. Grimmer, R: Resolvent operators for integral equation in a Banach space. Transl. Am. Math. Soc. **273**(1), 333-349 (1982)
27. Granas, A, Dugundji, J: Fixed Point Theory. Springer, New York (2003)