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On a nonlocal boundary value problem for nonlinear second-order Hahn difference equation with two different q,ω-derivatives

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Abstract
In this paper, we study a nonlocal boundary value problem for a second-order Hahn difference equation. Our problem contains two Hahn difference operators with different numbers of q and ω. An existence and uniqueness result is proved by using the Banach fixed point theorem, and the existence of a positive solution is established by using the Krasnoselskii fixed point theorem.

MSC: 39A10; 39A13; 39A70

Keywords: Hahn difference equations; boundary value problems; positive solution; existence

1 Introduction

The quantum calculus, also known as the calculus without considering limits, deals with sets of nondifferentiable functions. There are many different types of quantum difference operators, for example, the Jackson q-difference operator, the forward (delta) difference operator, the backward (nabla) difference operator, and so on. These operators are found in many applications of mathematical areas such as orthogonal polynomials, basic hypergeometric functions, combinatorics, the calculus of variations, the theory of relativity, hypergeometric series, complex analysis, particle physics, and quantum mechanics. For some recent results and applications of the quantum calculus, see [1–9] and the references therein.

In 1949, Hahn [10] introduced the Hahn difference operator $D_{q,\omega}$,

$$D_{q,\omega}f(t) = \frac{f(qt+\omega)-f(t)}{t(q-1)+\omega}, \quad t \neq \omega_0 := \frac{\omega}{1-q}.$$

The Hahn difference operator is generalized to two well-known difference operators, the forward difference operator and the Jackson q-difference operator. Notice that, under appropriate conditions,

$$D_{q,\omega}f(t) = \Delta_\omega f(t) \quad \text{whenever } q=1, \qquad D_{q,\omega}f(t) = D_q f(t) \quad \text{whenever } \omega = 0, \quad \text{and}$$
$$D_{q,\omega}f(t) = f'(t) \quad \text{whenever } q=1, \omega \to 0.$$

The Hahn difference operator has been employed in many pieces of literature to construct families of orthogonal polynomials and to investigate some approximation problems; see [11–13] and the references therein.

Unfortunately, in the past, no one was interested in finding the right inverse of the Hahn difference operator. Until in 2009, Aldwoah [14, 15] (Ph.D. thesis supervised by M.H. Annaby and A.E. Hamza) defined the right inverse of $D_{q,\omega}$ in terms of both the Jackson q-integral containing the right inverse of D_q [16] and Nörlund sum involving the right inverse of Δ_ω [16].

In 2010, Malinowska and Torres [17, 18] introduced the Hahn quantum variational calculus. In 2013, Malinowska and Martins [19] studied the generalized transversality conditions for the Hahn quantum variational calculus. In the same year, Hamza et al. [20, 21] studied the theory of linear Hahn difference equations and investigated the existence and uniqueness results for the initial value problems for Hahn difference equations by using the method of successive approximations; moreover, they proved Gronwall's and Bernoulli's inequalities with respect to the Hahn difference operator and also established mean value theorems for this calculus.

In particular, the boundary value problem for Hahn difference equations has not been studied. The results mentioned are the motivation for this research. In this paper, we consider a nonlinear Hahn difference equation with nonlocal boundary value conditions of the form

$$\begin{aligned} &D^2_{q,\omega}x(t) + f\bigl(t, x(t), D_{p,\theta}x(pt+\theta)\bigr) = 0, \quad t \in [\omega_0, T]_{q,\omega}, \\ &x(\omega_0) = \varphi(x), \\ &x(T) = \lambda x(\eta), \quad \eta \in (\omega_0, T)_{q,\omega}, \end{aligned} \tag{1.1}$$

where $0 < q < 1$, $0 < \omega < T$, $\omega_0 := \frac{\omega}{1-q}$, $1 \le \lambda < \frac{T-\omega_0}{\eta-\omega_0}$, $p = q^m$, $m \in \mathbb{N}$, $\theta = \omega(\frac{1-p}{1-q})$, $f : [\omega_0, T]_{q,\omega} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function, and $\varphi : C([\omega_0, T]_{q,\omega}, \mathbb{R}) \to \mathbb{R}$ is a given functional.

In the next section, we briefly recall some definitions and lemmas used in this research. In Section 3, we prove the existence and uniqueness of a solution to problem (1.1) by the Banach fixed point theorem. In Sections 4-5, we establish some properties of the Green function and the existence of a positive solution to problem (1.1) by using the Krasnoselskii fixed point theorem. Finally, we provide an example to illustrate our results in the last section.

The following theorem is Krasnoselskii's fixed point theorem in a cone.

Theorem 1.1 ([22]) *Let E be a Banach space, and let $K \subset E$ be a cone. Let Ω_1, Ω_2 be open subsets of E with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, and let*

$$A : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \longrightarrow K$$

be a completely continuous operator such that

(i) $\|Au\| \le \|u\|$, $u \in K \cap \partial\Omega_1$, *and* $\|Au\| \ge \|u\|$, $u \in K \cap \partial\Omega_2$; *or*

(ii) $\|Au\| \ge \|u\|$, $u \in K \cap \partial\Omega_1$, *and* $\|Au\| \le \|u\|$, $u \in K \cap \partial\Omega_2$.

Then A has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

2 Preliminaries

We now give the notation, definitions, and lemmas used in the main results.

Definition 2.1 ([10]) For $0 < q < 1$, $\omega > 0$, and f defined on an interval $I \subseteq \mathbb{R}$ containing $\omega_0 := \frac{\omega}{1-q}$, the Hahn difference of f is defined by

$$D_{q,\omega}f(t) = \frac{f(qt+\omega) - f(t)}{t(q-1)+\omega} \quad \text{for } t \neq \omega_0$$

and $D_{q,\omega}f(\omega_0) = f'(\omega_0)$, provided that f is differentiable at ω_0. We call $D_{q,\omega}f$ the q,ω-derivative of f and say that f is q,ω-differentiable on I.

Let $a, b \in I \subseteq \mathbb{R}$ with $a < \omega_0 < b$, and $[k]_q = \frac{1-q^k}{1-q}$, $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. We define the q,ω-interval by

$$\begin{aligned}
[a,b]_{q,\omega} &:= \left\{q^k a + \omega[k]_q : k \in \mathbb{N}_0\right\} \cup \left\{q^k b + \omega[k]_q : k \in \mathbb{N}_0\right\} \cup \{\omega_0\} \\
&= [a,\omega_0]_{q,\omega} \cup [\omega_0,b]_{q,\omega} \\
&= (a,b)_{q,\omega} \cup \{a,b\} = [a,b)_{q,\omega} \cup \{b\} = (a,b]_{q,\omega} \cup \{a\}.
\end{aligned}$$

Observe that, for each $s \in [a,b]_{q,\omega}$, the sequence $\{q^k s + \omega[k]_q\}_{k=0}^{\infty}$ is uniformly convergent to ω_0.

If f is q,ω-differentiable n times on a q,ω-interval $I_{q,\omega}$, then we define the higher-order derivatives by

$$D_{q,\omega}^n f(s) := D_{q,\omega} D_{q,\omega}^{n-1} f(s),$$

where $D_{q,\omega}^0 f(s) := f(s)$, $s \in I_{q,\omega} \subset \mathbb{R}$.

Next, we introduce the right inverse of the operator $D_{q,\omega}$, which call the q,ω-integral operator.

Definition 2.2 ([14]) Let I be any closed interval of $\mathbb{R}$ containing a, b, and ω_0. For a function $f : I \to \mathbb{R}$, we define the q,ω-integral of f from a to b by

$$\int_a^b f(t)\, d_{q,\omega}t := \int_{\omega_0}^b f(t)\, d_{q,\omega}t - \int_{\omega_0}^a f(t)\, d_{q,\omega}t,$$

where

$$\int_{\omega_0}^x f(t)\, d_{q,\omega}t := \left[x(1-q) - \omega\right] \sum_{k=0}^{\infty} q^k f\left(xq^k + \omega[k]_q\right), \quad x \in I,$$

provided that the series converges at $x = a$ and $x = b$; we say that f is q,ω-integrable on $[a,b]$, and the sum on the right-hand side of the above equation is called the Jackson-Norlund sum.

Note that the actual domain of definition of f is $[a,b]_{q,\omega} \subset I$.

The following lemma is the fundamental theorem of Hahn calculus.

Lemma 2.1 ([14]) *Let $f : I \to \mathbb{R}$ be continuous at ω_0. Define*

$$F(x) := \int_{\omega_0}^{x} f(t)\, d_{q,\omega}t, \quad x \in I.$$

Then F is continuous at ω_0. Furthermore, $D_{q,\omega_0}F(x)$ exists for every $x \in I$, and

$$D_{q,\omega}F(x) = f(x).$$

Conversely,

$$\int_{a}^{b} D_{q,\omega}F(t)\, d_{q,\omega}t = F(b) - F(a) \quad \text{for all } a, b \in I.$$

Next, we give some auxiliary lemmas for simplifying calculations.

Lemma 2.2 *Let $0 < q < 1$, $\omega > 0$, and $f : I \to \mathbb{R}$ be continuous at ω_0. Then*

$$\int_{\omega_0}^{t}\int_{\omega_0}^{r} x(s)\, d_{q,\omega}s\, d_{q,\omega}r = \int_{\omega_0}^{t}\int_{qs+\omega}^{t} h(s)\, d_{q,\omega}r\, d_{q,\omega}s.$$

Proof Using the definition of the q,ω-integral, we have

$$\begin{aligned}
&\int_{\omega_0}^{t}\int_{\omega_0}^{r} x(s)\, d_{q,\omega}s\, d_{q,\omega}r \\
&\quad= \int_{\omega_0}^{t}\left[\big[r(1-q)-\omega\big]\sum_{k=0}^{\infty} q^k x\big(rq^k + \omega[k]_q\big)\right] d_{q,\omega}r \\
&\quad= \sum_{k=0}^{\infty} q^k \left[\int_{\omega_0}^{t}\big[r(1-q)-\omega\big] x\big(rq^k + \omega[k]_q\big)\, d_{q,\omega}r\right] \\
&\quad= \sum_{k=0}^{\infty} q^k \big[t(1-q)-\omega\big]\sum_{h=0}^{\infty} q^h \big[\big(tq^k + \omega[k]_q\big)(1-q) - \omega\big] x\big(\big(tq^k + \omega[k]_q\big)q^h + \omega[h]_q\big) \\
&\quad= \big[t(1-q)-\omega\big]\sum_{k=0}^{\infty}\sum_{h=0}^{\infty} q^{k+h}\big[t(1-q)q^k - \omega q^k\big] x\left(tq^{k+h} + \omega\left[\frac{1-q^k}{1-q}\right]q^h + \omega\left[\frac{1-q^h}{1-q}\right]\right) \\
&\quad= \big[t(1-q)-\omega\big]\sum_{k=0}^{\infty}\sum_{h=0}^{\infty} q^{2k+h}\big[t(1-q)-\omega\big] x\left(tq^{k+h} + \omega\left[\frac{1-q^{k+h}}{1-q}\right]\right) \\
&\quad= \big[t(1-q)-\omega\big]^2\sum_{k=0}^{\infty}\sum_{h=0}^{\infty} q^{2k+h} x\big(tq^{k+h} + \omega[k+h]_q\big) \\
&\quad= \big[t(1-q)-\omega\big]^2\sum_{h=0}^{\infty}\big[q^h x\big(tq^h + \omega[h]_q\big) + q^{h+2} x\big(tq^{h+1} + \omega[h+1]_q\big) + \cdots\big] \\
&\quad= \big[t(1-q)-\omega\big]^2\big[x(t) + q^2 x(tq+\omega) + q^4 x\big(tq^2 + \omega[2]_q\big) + \cdots \\
&\qquad + qx(tq+\omega) + q^3 x\big(tq^2 + \omega[2]_q\big) + q^5 x\big(tq^3 + \omega[3]_q\big) + \cdots \\
&\qquad + q^2 x\big(tq^2 + \omega[2]_q\big) + q^4 x\big(tq^3 + \omega[3]_q\big) + q^6 x\big(tq^4 + \omega[4]_q\big) + \cdots\big]
\end{aligned}$$

$$
\begin{aligned}
&= \big[t(1-q)-\omega\big]^2\big[x(t)+q(1+q)x(tq+\omega)+q^2\big(1+q+q^2\big)x\big(tq^2+\omega[2]_q\big)+\cdots\big] \\
&= \big[t(1-q)-\omega\big]^2\sum_{k=0}^{\infty} q^k[k+1]_q x\big(tq^k+\omega[k]_q\big) \\
&= \int_{\omega_0}^{t}\big[t-(qs+\omega)\big]x(s)\,d_{q,w}s \\
&= \int_{\omega_0}^{t}\bigg[\int_{\omega_0}^{t} x(s)\,d_{q,w}r-\int_{\omega_0}^{qs+\omega} x(s)\,d_{q,w}r\bigg]\,d_{q,w}s \\
&= \int_{\omega_0}^{t}\int_{qs+\omega}^{t} x(s)\,d_{q,\omega}r\,d_{q,\omega}s. \qquad\square
\end{aligned}
$$

Lemma 2.3 *Let* $0<q<1$ *and* $\omega>0$. *Then*

$$
\int_{\omega_0}^{t} d_{q,\omega}s = t-\omega_0 \quad \textit{and} \quad \int_{\omega_0}^{t}\big[t-(qs+\omega)\big]\,d_{q,\omega}s = \frac{(t-\omega_0)^2}{1+q}.
$$

Proof Using the definition of the q,ω-integral, we have

$$
\begin{aligned}
\int_{\omega_0}^{t} d_{q,\omega}s &= \big[t(1-q)-\omega\big]\sum_{k=0}^{\infty} q^k = (1-q)(t-\omega_0)\sum_{k=0}^{\infty} q^k \\
&= (1-q)(t-\omega_0)\bigg[\frac{1}{1-q}\bigg] = t-\omega_0
\end{aligned}
$$

and

$$
\begin{aligned}
\int_{\omega_0}^{t}\big[t-(qs+\omega)\big]\,d_{q,\omega}s &= \big[t(1-q)-\omega\big]\sum_{k=0}^{\infty} q^k\big[t-\big[q\big(tq^k+\omega[k]_q\big)+\omega\big]\big] \\
&= (1-q)(t-\omega_0)\sum_{k=0}^{\infty} q^k\big[t\big(1-q^{k+1}\big)-\omega[k+1]_q\big] \\
&= (1-q)(t-\omega_0)^2\sum_{k=0}^{\infty} q^k\big(1-q^{k+1}\big) \\
&= (1-q)(t-\omega_0)^2\bigg[\frac{1}{1-q}-\frac{q}{1-q^2}\bigg] = \frac{(t-\omega_0)^2}{1+q}.
\end{aligned}
$$

The proof is complete. □

The following lemma deals with the linear version of problem (1.1) and gives a representation of the solution.

Lemma 2.4 *Let* $1\le\lambda<\frac{T-\omega_0}{\eta-\omega_0}$, $h\in C([\omega_0,T]_{q,\omega},\mathbb{R})$ *be a given function, and* $\varphi: C([\omega_0,T]_{q,\omega},\mathbb{R})\to\mathbb{R}$ *be a given functional. Then the problem*

$$
\begin{aligned}
&D_{q,\omega}^2 x(t) = -h(t), \quad t\in[\omega_0,T]_{q,\omega}, \\
&x(\omega_0)=\varphi(x), \qquad x(T)=\lambda x(\eta), \qquad \eta\in(\omega_0,T)_{q,\omega},
\end{aligned}
\tag{2.1}
$$

has the unique solution

$$x(t) = \varphi(x) + \frac{(t-\omega_0)}{\Lambda}\Biggl[(\lambda-1)\varphi(x) - \lambda\int_{\omega_0}^{\eta}\bigl[\eta-(qs+\omega)\bigr]h(s)\,d_{q,\omega}s + \int_{\omega_0}^{T}\bigl[T-(qs+\omega)\bigr]h(s)\,d_{q,\omega}s\Biggr] - \int_{\omega_0}^{t}\bigl[t-(qs+\omega)\bigr]h(s)\,d_{q,\omega}s, \tag{2.2}$$

where

$$\Lambda = (T-\omega_0) - \lambda(\eta-\omega_0). \tag{2.3}$$

Proof By Lemmas 2.1 and 2.2 a general solution for (2.1) can be written as

$$\begin{aligned} x(t) &= C_1 + C_2(t-\omega_0) - \int_{\omega_0}^{t}\int_{\omega_0}^{r} h(s)\,d_{q,\omega}s\,d_{q,\omega}r \\ &= C_1 + C_2(t-\omega_0) - \int_{\omega_0}^{t}\int_{qs+\omega}^{t} h(s)\,d_{q,\omega}r\,d_{q,\omega}s \\ &= C_1 + C_2(t-\omega_0) - \int_{\omega_0}^{t}\bigl[t-(qs+\omega)\bigr]h(s)\,d_{q,\omega}s \end{aligned} \tag{2.4}$$

for $t \in [\omega_0, T]_{q,\omega}$.

From the conditions (2.1) we obtain

$$C_1 = \varphi(x), \tag{2.5}$$

$$C_2 = \frac{(\lambda-1)}{\Lambda}\varphi(x) - \frac{\lambda}{\Lambda}\int_{\omega_0}^{\eta}\bigl[\eta-(qs+\omega)\bigr]h(s)\,d_{q,\omega}s + \frac{1}{\Lambda}\int_{\omega_0}^{T}\bigl[T-(qs+\omega)\bigr]h(s)\,d_{q,\omega}s, \tag{2.6}$$

where Λ is defined by (2.3).

Substituting the constants C_1, C_2 into (2.4), we obtain (2.2). □

Lemma 2.5 *Problem* (2.1) *has the unique solution of the from*

$$x(t) = \biggl[1 + \frac{(\lambda-1)(t-\omega_0)}{\Lambda}\biggr]\varphi(x) + \int_{\omega_0}^{T} G(t, qs+\omega)h(s)\,d_{q,\omega}s, \tag{2.7}$$

where

$$G(t, qs+\omega) = \begin{cases} g_1(t, qs+\omega), & s \in [\omega_0, t]_{q,\omega} \cap [\omega_0, \eta]_{q,\omega}, \\ g_2(t, qs+\omega), & s \in [\eta, t]_{q,\omega}, \\ g_3(t, qs+\omega), & s \in [t, \eta]_{q,\omega}, \\ g_4(t, qs+\omega), & s \in [t, T]_{q,\omega} \cap [\eta, T]_{q,\omega}, \end{cases} \tag{2.8}$$

with $g_i(t,s)$, $1 \leq i \leq 4$, *defined as*

$$\begin{aligned}
g_1(t,qs+\omega) &:= \biggl[\frac{1}{\Lambda}(t-\omega_0)(T-\lambda\eta)-t\biggr]+(qs+\omega)\biggl[\frac{1}{\Lambda}(t-\omega_0)(\lambda-1)+1\biggr], \\
g_2(t,qs+\omega) &:= \biggl[\frac{1}{\Lambda}(t-\omega_0)T-t\biggr]+(qs+\omega)\biggl[1-\frac{(t-\omega_0)}{\Lambda}\biggr], \\
g_3(t,qs+\omega) &:= \frac{1}{\Lambda}(t-\omega_0)(T-\lambda\eta)+\frac{(qs+\omega)}{\Lambda}(t-\omega_0)(\lambda-1), \\
g_4(t,qs+\omega) &:= \frac{1}{\Lambda}(t-\omega_0)\bigl[T-(qs+\omega)\bigr].
\end{aligned} \tag{2.9}$$

Proof Suppose $t > \eta$. The unique solution of problem (2.1) can be written as

$$\begin{aligned}
x(t) &= \biggl[1+\frac{(\lambda-1)(t-\omega_0)}{\Lambda}\biggr]\varphi(x) \\
&\quad + \frac{(t-\omega_0)}{\Lambda}\int_{\omega_0}^{\eta}\biggl[\bigl[T-(qs+\omega)\bigr]-\lambda\bigl[\eta-(qs+\omega)\bigr]-\frac{\Lambda}{t-\omega_0}\bigl[t-(qs+\omega)\bigr]\biggr]h(s)\,d_{q,\omega}s \\
&\quad + \frac{(t-\omega_0)}{\Lambda}\int_{\eta}^{t}\biggl[\bigl[T-(qs+\omega)\bigr]-\frac{\Lambda}{t-\omega_0}\bigl[t-(qs+\omega)\bigr]\biggr]h(s)\,d_{q,\omega}s \\
&\quad + \frac{(t-\omega_0)}{\Lambda}\int_{t}^{T}\bigl[\bigl[T-(qs+\omega)\bigr]\bigr]h(s)\,d_{q,\omega}s \\
&= \biggl[1+\frac{(\lambda-1)(t-\omega_0)}{\Lambda}\biggr]\varphi(x) \\
&\quad + \int_{\omega_0}^{\eta}\biggl(\biggl[\frac{1}{\Lambda}(t-\omega_0)(T-\lambda\eta)-t\biggr]+(qs+\omega)\biggl[\frac{1}{\Lambda}(t-\omega_0)(\lambda-1)+1\biggr]\biggr)h(s)\,d_{q,\omega}s \\
&\quad + \int_{\eta}^{t}\biggl(\biggl[\frac{1}{\Lambda}(t-\omega_0)T-t\biggr]+(qs+\omega)\biggl[1-\frac{(t-\omega_0)}{\Lambda}\biggr]\biggr)h(s)\,d_{q,\omega}s \\
&\quad + \frac{(t-\omega_0)}{\Lambda}\int_{t}^{T}\bigl[T-(qs+\omega)\bigr]h(s)\,d_{q,\omega}s \\
&= \biggl[1+\frac{(\lambda-1)(t-\omega_0)}{\Lambda}\biggr]\varphi(x)+\int_{\omega_0}^{T}G(t,qs+\omega)h(s)\,d_{q,\omega}s
\end{aligned}$$

and similarly for $t < \eta$. The unique solution of problem (2.1) can be written as

$$\begin{aligned}
x(t) &= \biggl[1+\frac{(\lambda-1)(t-\omega_0)}{\Lambda}\biggr]\varphi(x) \\
&\quad + \int_{\omega_0}^{\eta}\biggl(\biggl[\frac{1}{\Lambda}(t-\omega_0)(T-\lambda\eta)-t\biggr]+(qs+\omega)\biggl[\frac{1}{\Lambda}(t-\omega_0)(\lambda-1)+1\biggr]\biggr)h(s)\,d_{q,\omega}s \\
&\quad + \int_{\eta}^{t}\biggl(\frac{1}{\Lambda}(t-\omega_0)(T-\lambda\eta)+\frac{(qs+\omega)}{\Lambda}(t-\omega_0)(\lambda-1)\biggr)h(s)\,d_{q,\omega}s \\
&\quad + \frac{(t-\omega_0)}{\Lambda}\int_{t}^{T}\bigl[T-(qs+\omega)\bigr]h(s)\,d_{q,\omega}s \\
&= \biggl[1+\frac{(\lambda-1)(t-\omega_0)}{\Lambda}\biggr]\varphi(x)+\int_{\omega_0}^{T}G(t,qs+\omega)h(s)\,d_{q,\omega}s.
\end{aligned}$$

This completes the proof. □

3 Existence and uniqueness of a solution for problem (1.1)

In this section, we present an existence and uniqueness result for problem (1.1). Let $\mathcal{C} = C([\omega_0, T]_{q,\omega}, \mathbb{R})$ be the Banach space of all continuous functions x with the norm

$$\|x\|_{\mathcal{C}} = \max\bigl\{\|x\|, \|D_{p,\theta}x\|\bigr\},$$

where $\|x\| = \max_{t\in[\omega_0,T]_{q,\omega}} |x(t)|$ and $\|D_{p,\theta}x\| = \max_{t\in[\omega_0,T]_{q,\omega}} |D_{p,\theta}x(pt+\theta)|$. Also, define the operator $\mathcal{F} : \mathcal{C} \to \mathcal{C}$ by

$$\begin{aligned}(\mathcal{F}x)(t) = \varphi(x) + \frac{(t-\omega_0)}{\Lambda}\biggl[&(\lambda-1)\varphi(x)\\ &- \lambda\int_{\omega_0}^{\eta}\bigl[\eta-(qs+\omega)\bigr]f\bigl(s, x(s), D_{p,\theta}x(ps+\theta)\bigr)\,d_{q,\omega}s\\ &+ \int_{\omega_0}^{T}\bigl[T-(qs+\omega)\bigr]f\bigl(s, x(s), D_{p,\theta}x(ps+\theta)\bigr)\,d_{q,\omega}s\biggr]\\ &- \int_{\omega_0}^{t}\bigl[t-(qs+\omega)\bigr]f\bigl(s, x(s), D_{p,\theta}x(ps+\theta)\bigr)\,d_{q,\omega}s, \end{aligned} \tag{3.1}$$

where $\Lambda \neq 0$ is defined by (2.3), $p = q^m$, $m \in \mathbb{N}$, and $\theta = \omega(\frac{1-p}{1-q})$.

Obviously, problem (1.1) has solutions if and only if the operator $\mathcal{F}$ has fixed points.

Theorem 3.1 *Assume that the following conditions hold*:

(H_1) *There exist constants* $\gamma_1, \gamma_2 > 0$ *such that*

$$\begin{aligned}&\bigl|f\bigl(t, x(t), D_{p,\theta}x(pt+\theta)\bigr) - f\bigl(t, y(t), D_{p,\theta}y(pt+\theta)\bigr)\bigr|\\ &\quad\le \gamma_1\bigl|x(t)-y(t)\bigr| + \gamma_2\bigl|D_{p,\theta}x(pt+\theta) - D_{p,\theta}y(pt+\theta)\bigr|\end{aligned}$$

for all $t \in [\omega_0, T]_{q,\omega}$ *and* $x, y \in \mathcal{C}$.

(H_2) *There exists a constant* $\ell > 0$ *such that*

$$\bigl|\varphi(x) - \varphi(y)\bigr| \le \ell\bigl\|x(t)-y(t)\bigr\|_{\mathcal{C}}$$

for each $x, y \in \mathcal{C}$.

(H_3) $\mathfrak{S} := \gamma\Omega + \ell\Phi < 1$, *where*

$$\begin{aligned}\gamma &= \max\{\gamma_1, \gamma_2\},\\ \Omega &= \frac{(T-\omega_0)}{|\Lambda|}\biggl[\frac{(T-\omega_0)^2 - \lambda(\eta-\omega_0)^2}{(1+q)}\biggr] + \frac{(T-\omega_0)^2}{1+q},\\ \Phi &= 1 + \frac{(\lambda-1)(T-\omega_0)}{|\Lambda|}.\end{aligned} \tag{3.2}$$

Then problem (1.1) *has a unique solution in* $[\omega_0, T]_{q,\omega}$.

Proof Denote $\mathcal{H}|x-y|(t) := |f(t,x(t),D_{p,\theta}x(pt+\theta)) - f(t,y(t),D_{p,\theta}y(pt+\theta))|$. Using Lemma 2.3, for all $t \in [\omega_0, T]_{q,\omega}$ and $x, y \in \mathcal{C}$, we have

$$\begin{aligned}
&\bigl|(\mathcal{F}x)(t) - (\mathcal{F}y)(t)\bigr| \\
&\quad\le \bigl|\varphi(x)-\varphi(y)\bigr| + \frac{(t-\omega_0)}{|\Lambda|}\biggl|(\lambda-1)\bigl|\varphi(x)-\varphi(y)\bigr| \\
&\qquad - \lambda\int_{\omega_0}^{\eta}\bigl[\eta-(qs+\omega)\bigr]\mathcal{H}|x-y|(s)\,d_{q,\omega}s + \int_{\omega_0}^{T}\bigl[T-(qs+\omega)\bigr]\mathcal{H}|x-y|(s)\,d_{q,\omega}s\biggr| \\
&\qquad + \int_{\omega_0}^{t}\bigl[t-(qs+\omega)\bigr]\mathcal{H}|x-y|(s)\,d_{q,\omega}s \\
&\quad\le \ell\|x-y\|_{\mathcal{C}} + \frac{(T-\omega_0)}{|\Lambda|}\biggl[(\lambda-1)\ell\|x-y\|_{\mathcal{C}} + \bigl(\gamma_1\bigl|x(t)-y(t)\bigr| + \gamma_2\bigl|D_{p,\theta}x(pt+\theta) \\
&\qquad - D_{p,\theta}y(pt+\theta)\bigr|\bigr)\biggl|\int_{\omega_0}^{T}\bigl[T-(qs+\omega)\bigr]\,d_{q,\omega}s - \lambda\int_{\omega_0}^{\eta}\bigl[\eta-(qs+\omega)\bigr]\,d_{q,\omega}s\biggr|\biggr] \\
&\qquad + \bigl(\gamma_1\bigl|x(t)-y(t)\bigr| + \gamma_2\bigl|D_{p,\theta}x(pt+\theta) - D_{p,\theta}y(pt+\theta)\bigr|\bigr)\int_{\omega_0}^{T}\bigl[T-(qs+\omega)\bigr]\,d_{q,\omega}s \\
&\quad\le \ell\|x-y\|_{\mathcal{C}}\biggl\{1 + \frac{(\lambda-1)(T-\omega_0)}{|\Lambda|}\biggr\} \\
&\qquad + \gamma\|x-y\|_{\mathcal{C}}\biggl\{\frac{T-\omega_0}{|\Lambda|}\biggl[\frac{(T-\omega_0)^2-\lambda(\eta-\omega_0)^2}{1+q}\biggr] + \frac{(T-\omega_0)^2}{1+q}\biggr\} \\
&\quad= \ell\|x-y\|_{\mathcal{C}}\Phi + \gamma\|x-y\|_{\mathcal{C}}\Omega \\
&\quad= \mathfrak{S}\|x-y\|_{\mathcal{C}}.
\end{aligned}$$

Taking the p,θ-derivative for (3.1) where $p = q^m$, $m \in \mathbb{N}$, and $\theta = \omega(\frac{1-p}{1-q})$, we obtain

$$\begin{aligned}
&\bigl|(D_{p,\theta}\mathcal{F}x)(pt+\theta) - (D_{p,\theta}\mathcal{F}y)(pt+\theta)\bigr| \\
&\quad\le \biggl|\frac{1}{[-(pt+\theta)(1-p)+\theta]}\biggl\{\frac{-(pt+\theta)(1-p)+\theta}{\Lambda}\biggl((\lambda-1)\bigl|\varphi(x)-\varphi(y)\bigr| \\
&\qquad - \lambda\int_{\omega_0}^{\eta}\bigl[\eta-(qs+\omega)\bigr]\mathcal{H}|x-y|(s)\,d_{q,\omega}s + \int_{\omega_0}^{T}\bigl[T-(qs+\omega)\bigr]\mathcal{H}|x-y|(s)\,d_{q,\omega}s\biggr) \\
&\qquad + \biggl(\int_{\omega_0}^{p(pt+\theta)+\theta}\bigl[p(pt+\theta)+\theta-(qs+\omega)\bigr]\mathcal{H}|x-y|(s)\,d_{q,\omega}s \\
&\qquad - \int_{\omega_0}^{pt+\theta}\bigl[(pt+\theta)-(qs+\omega)\bigr]\mathcal{H}|x-y|(s)\,d_{q,\omega}s\biggr)\biggr\}\biggr| \\
&\quad\le \frac{(\lambda-1)}{|\Lambda|}\ell\|x-y\|_{\mathcal{C}} + \frac{(\gamma_1|x(t)-y(t)| + \gamma_2|D_{p,\theta}x(pt+\theta) - D_{p,\theta}y(pt+\theta)|)}{\Lambda} \\
&\qquad \times\biggl|\int_{\omega_0}^{T}\bigl[T-(qs+\omega)\bigr]\,d_{q,\omega}s - \lambda\int_{\omega_0}^{\eta}\bigl[\eta-(qs+\omega)\bigr]\,d_{q,\omega}s\biggr| \\
&\qquad + \frac{(\gamma_1|x(t)-y(t)| + \gamma_2|D_{p,\theta}x(pt+\theta) - D_{p,\theta}y(pt+\theta)|)}{p(1-p)(t-\omega_0)} \\
&\qquad \times\biggl|\int_{\omega_0}^{pt+\theta}\bigl[(pt+\theta)-(qs+\omega)\bigr]\,d_{q,\omega}s - \int_{\omega_0}^{p^2t+(p+1)\theta}\bigl[p^2t+(p+1)\theta-(qs+\omega)\bigr]\,d_{q,\omega}s\biggr|
\end{aligned}$$

$$\begin{aligned}&\leq \ell\|x-y\|_{\mathcal{C}}\left\{\frac{\lambda-1}{|\Lambda|}\right\}+\gamma\|x-y\|_{\mathcal{C}}\left\{\left[\frac{(T-\omega_0)^2-\lambda(\eta-\omega_0)^2}{(1+q)|\Lambda|}\right]+\frac{p(1+p)(T-\omega_0)}{1+q}\right\}\\&<\mathfrak{S}\|x-y\|_{\mathcal{C}}.\end{aligned}$$

This implies that $\mathcal{F}$ is a contraction. Therefore, by the Banach fixed point theorem, $\mathcal{F}$ has a fixed point, which is a unique solution of problem (1.1) on $t\in[\omega_0,T]_{q,\omega}$. □

4 Properties of Green's function for problem (1.1)

We next prove that Green's function $G(t,s)$ in (2.8) satisfies a variety of properties that are necessary for considering the existence of a positive solution to problem (1.1). Firstly, we prove some necessary preliminary lemmas.

Theorem 4.1 ([21], Mean Value Theorem) *Let $f:I\to X$ be q,ω-differentiable on I. For every $s\in I$,*

$$\|f(b)-f(a)\|\leq\sup_{t\in I}\|D_{q,\omega}f(t)\|(b-a)$$

for all $a,b\in\{sq^k+\omega[k]_q\}_{k=0}^{\infty}$ and $a<b$.

Theorem 4.2 *Let f be q,ω-differentiable on $(a,b)_{q,\omega}$ and continuous on $[a,b]_{q,\omega}$. The following statements are true*:

(i) *If $D_{q,\omega}f(t)>0$ for all $t\in(a,b)_{q,\omega}$, then f is an increasing function on $[a,b]_{q,\omega}$.*
(ii) *If $D_{q,\omega}f(t)<0$ for all $t\in(a,b)_{q,\omega}$, then f is a decreasing function on $[a,b]_{q,\omega}$.*
(iii) *If $D_{q,\omega}f(t)=0$ for all $t\in(a,b)_{q,\omega}$, then f is a constant function on $[a,b]_{q,\omega}$.*

Proof Let $t_1,t_2\in[a,b]_{q,\omega}$, $t_1<t_2$. Since f is q,ω-differentiable on $(a,b)_{q,\omega}$ and continuous on $[a,b]_{q,\omega}$, we have that f is a continuous function on $(a,b)_{q,\omega}$.

By Theorem 4.1 there exists $t^*\in(a,b)_{q,\omega}$ such that $D_{q,\omega}f(t^*)=\frac{\|f(b)-f(a)\|}{b-a}$.

(i) If $D_{q,\omega}f(t)>0$ for all $t\in(a,b)_{q,\omega}$, then $D_{q,\omega}f(t^*)>0$, which implies that

$$f(t_2)-f(t_1)=(t_2-t_1)D_{q,\omega}f\bigl(t^*\bigr)>0.$$

So $f(t_2)>f(t_1)$ for all t_1, t_2, and hence f is increasing on $[a,b]_{q,\omega}$.

(ii) If $D_{q,\omega}f(t)<0$ for all $t\in(a,b)_{q,\omega}$, then $D_{q,\omega}f(t^*)<0$, which implies that

$$f(t_2)-f(t_1)=(t_2-t_1)D_{q,\omega}f\bigl(t^*\bigr)<0.$$

So $f(t_2)<f(t_1)$ for all t_1, t_2, and hence f is decreasing on $[a,b]_{q,\omega}$.

(iii) If $D_{q,\omega}f(t)=0$ for all $t\in(a,b)_{q,\omega}$, then $D_{q,\omega}f(t^*)=0$, which implies that

$$f(t_2)-f(t_1)=(t_2-t_1)D_{q,\omega}f\bigl(t^*\bigr)=0.$$

So $f(t_2)=f(t_1)$ for all t_1, t_2, and hence f is constant on $[a,b]_{q,\omega}$. □

Lemma 4.1 *We have that $\Lambda>0$ and $1+\frac{(\lambda-1)(t-\omega_0)}{\Lambda}$ is positive and strictly decreasing in t for $t\in[\omega_0,T]_{q,\omega}$. In addition,*

$$\min_{t\in[\eta,T]_{q,\omega}}\left[1+\frac{(\lambda-1)(t-\omega_0)}{\Lambda}\right]=\frac{T-\eta}{\Lambda}\quad\textit{and}$$

$$\max_{t\in[\omega_0,T]_{q,\omega}}\left[1+\frac{(\lambda-1)(t-\omega_0)}{\Lambda}\right]=\frac{\lambda(T-\eta)}{\Lambda}.$$

Proof Considering Λ in (2.3) and $1\leq\lambda<\frac{T-\omega_0}{\eta-\omega_0}$, we obtain

$$\Lambda=(T-\omega_0)-\lambda(\eta-\omega_0)>(T-\omega_0)-\left(\frac{T-\omega_0}{\eta-\omega_0}\right)(\eta-\omega_0)=0.$$

For the proof that $1+\frac{(\lambda-1)(t-\omega_0)}{\Lambda}>0$, $t\in[\omega_0,T]_{q,\omega}$, it is sufficient to show that

$$\begin{aligned}\Lambda+(\lambda-1)(t-\omega_0)&=\bigl[(T-\omega_0)-\lambda(\eta-\omega_0)\bigr]+(\lambda-1)(t-\omega_0)\\&=\bigl[(T-\omega_0)-(t-\omega_0)\bigr]-\lambda\bigl[(\eta-\omega_0)-(t-\omega_0)\bigr]\\&>\bigl[(T-\omega_0)-(t-\omega_0)\bigr]-\frac{T-\omega_0}{\eta-\omega_0}\bigl[(\eta-\omega_0)-(t-\omega_0)\bigr]\\&=\frac{(t-\omega_0)(T-\eta)}{\eta-\omega_0}\geq0.\end{aligned}$$

Next, we prove that $1+\frac{(\lambda-1)(t-\omega_0)}{\Lambda}$ is strictly decreasing in $t\in[\omega_0,T]_{q,\omega}$. Note that the q,ω-derivative with respect to t for $1+\frac{(\lambda-1)(t-\omega_0)}{\Lambda}$ is

$$\begin{aligned}{}_tD_{q,\omega}\left[1+\frac{(\lambda-1)(t-\omega_0)}{\Lambda}\right]&=\frac{\lambda-1}{(T-\omega_0)-\lambda(\eta-\omega_0)}<\frac{\lambda-1}{(\eta-\omega_0)-\lambda(\eta-\omega_0)}\\&=-\frac{1}{\eta-\omega_0}<0.\end{aligned}$$

By Theorem 4.2 we have that $1+\frac{(\lambda-1)(t-\omega_0)}{\Lambda}$ is strictly decreasing in $t\in[\omega_0,T]_{q,\omega}$.

Finally, observe that

$$\begin{aligned}&\min_{t\in[\eta,T]_{q,\omega}}\left[1+\frac{(\lambda-1)(t-\omega_0)}{\Lambda}\right]=\left[1+\frac{(\lambda-1)(t-\omega_0)}{\Lambda}\right]_{t=\eta}=\frac{T-\eta}{\Lambda}\quad\text{and}\\&\max_{t\in[\omega_0,T]_{q,\omega}}\left[1+\frac{(\lambda-1)(t-\omega_0)}{\Lambda}\right]=\left[1+\frac{(\lambda-1)(t-\omega_0)}{\Lambda}\right]_{t=T}=\frac{\lambda(T-\eta)}{\Lambda}.\end{aligned}$$

The proof is complete. □

Next, we show that Green's function given in (2.8) is positive.

Lemma 4.2 *Let $G(t,s)$ be Green's function given in* (2.8). *Then $G(t,s)\geq0$ for each $(t,s)\in[\omega_0,T]_{q,\omega}\times[\omega_0,T]_{q,\omega}$.*

Proof We aim to show that $g_i(t,qs+\omega)>0$ for all i, $1\leq i\leq4$, and for each admissible pair (t,s).

Firstly, we consider the function $g_4(t,qs+\omega)=\frac{1}{\Lambda}(t-\omega_0)[T-(qs+\omega)]$, $s\in[t,T]_{q,\omega}\cap[\eta,T]_{q,\omega}$. To guarantee that $g_4(t,qs+\omega)>0$, it suffices to show that

$$T-(qs+\omega)\geq T-(qT+\omega)>(T-\omega_0)(1-q)>0.\tag{4.1}$$

Thus, we conclude that $g_4(t,qs+\omega)>0$ on their respective domains.

Next, we consider the function $g_2(t, qs+\omega)$ for $s \in [\eta, T]_{q,\omega}$ and $t \in [\eta, T]_{q,\omega}$:

$$\begin{aligned} g_2(t, qs+\omega) &= \left(\frac{(t-\omega_0)T}{\Lambda} - t\right) + (qs+\omega)\left(1 - \frac{(t-\omega_0)}{\Lambda}\right) \\ &= \frac{(t-\omega_0)}{\Lambda}\bigl[T - (qs+\omega)\bigr] - \bigl[t - (qs+\omega)\bigr]. \end{aligned}$$

To guarantee that $g_2(t, qs+\omega) > 0$, it suffices to show that

$$\begin{aligned} \frac{(t-\omega_0)[T-(qs+\omega)]}{\Lambda[t-(qs+\omega)]} &> \frac{(t-\omega_0)[(T-\omega_0) - q(s-\omega_0)]}{(T-\omega_0)[(t-\omega_0) - q(s-\omega_0)]} \\ &= \frac{(T-\omega_0) - q(s-\omega_0)}{(T-\omega_0) - (\frac{T-\omega_0}{t-\omega_0})q(s-\omega_0)} > 1. \end{aligned} \tag{4.2}$$

So, we conclude that $g_2(t, qs+\omega) > 0$ on their respective domains.

We next consider the function $g_3(t, qs+\omega)$ for $s \in [t, \eta]_{q,\omega}$ and $t \in [\omega_0, \eta]_{q,\omega}$:

$$g_3(t, qs+\omega) = \frac{1}{\Lambda}(t-\omega_0)(T-\lambda\eta) + \frac{qs+\omega}{\Lambda}(t-\omega_0)(\lambda-1).$$

To guarantee that $g_3(t, qs+\omega) > 0$, it suffices to show that

$$\begin{aligned} &(T-\lambda\eta) + (\lambda-1)(qs+\omega) \\ &\quad= \bigl[T-(qs+\omega)\bigr] - \lambda\bigl[\eta-(qs+\omega)\bigr] \\ &\quad> \bigl[T-(qs+\omega)\bigr] - \frac{(1-q)T-\omega}{(1-q)\eta-\omega}\bigl[\eta-(qs+\omega)\bigr] \\ &\quad= \frac{1}{(1-q)\eta-\omega}\Bigl[\bigl[T-(qs+\omega)\bigr]\bigl((1-q)\eta-\omega\bigr) - \bigl((1-q)T-\omega\bigr)\bigl[\eta-(qs+\omega)\bigr]\Bigr] \\ &\quad= \frac{1}{(1-q)\eta-\omega}\Bigl[(T-\eta)q\bigl[s(1-q)-\omega\bigr]\Bigr] \\ &\quad> \frac{(T-\eta)q}{(1-q)\eta-\omega}\bigl[t(1-q)-\omega\bigr] \\ &\quad= \frac{(T-\eta)q}{\eta-\omega_0}[t-\omega_0] \geq 0. \end{aligned} \tag{4.3}$$

Hence, $g_3(t, qs+\omega) > 0$, as claimed.

Finally, we consider the function $g_1(t, qs+\omega)$ for $s \in [\omega_0, t]_{q,\omega} \cap [\omega_0, \eta]_{q,\omega}$:

$$\begin{aligned} g_1(t, qs+\omega) &= \left[\frac{1}{\Lambda}(t-\omega_0)(T-\lambda\eta) - t\right] + (qs+\omega)\left[\frac{1}{\Lambda}(t-\omega_0)(\lambda-1) + 1\right] \\ &= \frac{(t-\omega_0)}{\Lambda}\Bigl[T-(qs+\omega) - \lambda\bigl[\eta-(qs+\omega)\bigr]\Bigr] - t + (qs+\omega) \\ &= \frac{(t-\omega_0)}{\Lambda}\Bigl[(T-\omega_0) - q(s-\omega_0) - \lambda\bigl[(\eta-\omega_0) - q(s-\omega_0)\bigr]\Bigr] \\ &\quad - \bigl[(t-\omega_0) - q(s-\omega_0)\bigr]. \end{aligned}$$

To guarantee that $g_1(t, qs + \omega) > 0$, it suffices to show that

$$\frac{1}{\Lambda}\big[(T-\omega_0) - q(s-\omega_0) - \lambda\big[(\eta-\omega_0) - q(s-\omega_0)\big]\big] > \frac{(t-\omega_0) - q(s-\omega_0)}{(t-\omega_0)}. \tag{4.4}$$

We observe that

$$\begin{aligned}\mathcal{I}(\lambda) &:= \frac{1}{\Lambda}\big[(T-\omega_0) - q(s-\omega_0) - \lambda\big[(\eta-\omega_0) - q(s-\omega_0)\big]\big] \\ &= \frac{[(T-\omega_0) - q(s-\omega_0)] - \lambda[(\eta-\omega_0) - q(s-\omega_0)]}{(T-\omega_0) - \lambda(\eta-\omega_0)}\end{aligned} \tag{4.5}$$

is increasing in λ for $1 < \lambda < \frac{T-\omega_0}{\eta-\omega_0}$. Note that $\mathcal{I}(\lambda)$ is increasing for λ if and only if

$$\frac{(\eta-\omega_0)[(T-\omega_0) - q(s-\omega_0)]}{(T-\omega_0)[(\eta-\omega_0) - q(s-\omega_0)]} = \frac{(T-\omega_0) - q(s-\omega_0)}{(T-\omega_0) - q(\frac{T-\omega_0}{\eta-\omega_0})(s-\omega_0)} > 1. \tag{4.6}$$

Clearly, (4.6) implies that (4.4) also holds, and hence $g_1(t, qs + \omega) > 0$.

Consequently, from this it follows that $g_i(t, qs + \omega) > 0$ for each i, $1 \le i \le 4$. Therefore, $G(t, qs + \omega) > 0$. □

Lemma 4.3 *Let $G(t,s)$ be Green's function given in* (2.8). *Then for given $\eta \in (\omega_0, T)_{q,\omega}$ and $1 \le \lambda < \frac{T-\omega_0}{\eta-\omega_0}$, it follows that*

$$\max_{(t,s)\in[\omega_0,T]_{q,\omega}\times[\omega_0,T]_{q,\omega}} G(t, qs + \omega) = G(qs + \omega, qs + \omega). \tag{4.7}$$

Proof Our strategy is to consider the following two cases.

Case 1: $t < \eta$. We aim to show that ${}_tD_{q,\omega}g_1(t,s), {}_tD_{q,\omega}g_2(t,s) < 0$ and ${}_tD_{q,\omega}g_4(t,s) > 0$. Theorem 4.2 implies that g_1, g_2 are decreasing and g_4 is increasing in t, so $G(t, qs + \omega) \le G(qs + \omega, qs + \omega)$ for all $(t,s) \in [\omega_0, T]_{q,\omega} \times [\omega_0, T]_{q,\omega}$.

Case 2: $t > \eta$. We aim to show that ${}_tD_{q,\omega}g_1(t,s) < 0$ and ${}_tD_{q,\omega}g_3(t,s), {}_tD_{q,\omega}g_4(t,s) > 0$. Theorem 4.2 implies that g_1 is decreasing and g_3, g_4 are increasing in t, so $G(t, qs + \omega) \le G(qs + \omega, qs + \omega)$ for all $(t,s) \in [\omega_0, T]_{q,\omega} \times [\omega_0, T]_{q,\omega}$.

Firstly, for $g_4(t, qs + \omega)$, we have that

$$\begin{aligned}{}_tD_{q,\omega}g_4(t, qs + \omega) &= {}_tD_{q,\omega}\left\{\frac{1}{\Lambda}(t-\omega_0)\big[T - (qs + \omega)\big]\right\} \\ &= \frac{[(qt+\omega) - \omega_0] - [t - \omega_0]}{t(q-1) + \omega}\left[\frac{T - (qs+\omega)}{\Lambda}\right] \\ &= \left[\frac{T - (qs+\omega)}{\Lambda}\right] = \frac{1}{\Lambda}\big[(T-\omega_0) - q(s - \omega + 0)\big] > 0\end{aligned} \tag{4.8}$$

for all $s \in [t, T]_{q,\omega} \cap [\eta, T]_{q,\omega}$ and $t \in (\omega_0, T]_{q,\omega}$.

Later, for $g_3(t, qs + \omega)$, we have that

$$\begin{aligned}{}_tD_{q,\omega}g_3(t, qs + \omega) &= \frac{[(qt+\omega) - \omega_0] - [t - \omega_0]}{t(q-1) + \omega}\left[\frac{1}{\Lambda}\big((T - \lambda\eta) - (1-\lambda)(qs + \omega)\big)\right] \\ &= \frac{1}{\Lambda}\big[(T - \lambda\eta) - (1-\lambda)(qs + \omega)\big].\end{aligned} \tag{4.9}$$

From (4.3) we obtain that ${}_tD_{q,\omega}g_3(t, qs + \omega) > 0$ for all $s \in [t, \eta]_{q,\omega}$ and $t \in (\omega_0, \eta]_{q,\omega}$.

Next, we consider $g_2(t,s)$ and claim that ${}_tD_{q,\omega}g_2(t,qs+\omega) < 0$ for each admissible pair (t,s). To this end, noting that

$$g_2(t,qs+\omega) = \frac{(t-\omega_0)}{\Lambda}\big[T-(qs+\omega)\big] - \big[(t-\omega_0) - q(s-\omega+0)\big], \tag{4.10}$$

we obtain

$$\begin{aligned}
{}_tD_{q,\omega}g_2(t,qs+\omega) &= \frac{[(qt+\omega)-\omega_0]-[t-\omega_0]}{t(q-1)+\omega}\left[\frac{1}{\Lambda}\big[T-(qs+\omega)\big]-1\right] \\
&= \frac{1}{\Lambda}\big(T-(qs+\omega)\big)-1 \\
&= \frac{(T-\omega_0)-q(s-\omega_0)}{(T-\omega_0)-\lambda(\eta-\omega_0)}-1.
\end{aligned} \tag{4.11}$$

So, ${}_tD_{q,\omega}g_2(t,qs+\omega)$ is nonpositive when

$$\frac{(T-\omega_0)-q(s-\omega_0)}{(T-\omega_0)-\lambda(\eta-\omega_0)} < 1. \tag{4.12}$$

In addition, (4.12) is true if and only if

$$\lambda < \frac{q(s-\omega_0)}{\eta-\omega_0} \le \frac{q(t-\omega_0)}{\eta-\omega_0} < \frac{q(T-\omega_0)}{\eta-\omega_0} < \frac{T-\omega_0}{\eta-\omega_0}. \tag{4.13}$$

Clearly, (4.13) implies that (4.12) also holds. Hence, ${}_tD_{q,\omega}g_2(t,qs+\omega) < 0$ for all $s \in [\eta,t]_{q,\omega}$ and $t \in (\omega_0,T]_{q,\omega}$, as desired.

Finally, to claim that ${}_tD_{q,\omega}g_1(t,qs+\omega) < 0$ on its domain, we have that

$$\begin{aligned}
&{}_tD_{q,\omega}g_1(t,qs+\omega) \\
&\quad = {}_tD_{q,\omega}\left\{\left[\frac{1}{\Lambda}(t-\omega_0)(T-\lambda\eta)-t\right] + (qs+\omega)\left[\frac{1}{\Lambda}(t-\omega_0)(\lambda-1)+1\right]\right\} \\
&\quad = \frac{1}{\Lambda}\big[(T-\lambda\eta)+(\lambda-1)(qs+\omega)\big]-1 \\
&\quad = \frac{[(T-\omega_0)-q(s-\omega_0)]-\lambda[(\eta-\omega_0)-q(s-\omega_0)]}{(T-\omega_0)-\lambda(\eta-\omega_0)}-1 \\
&\quad = \frac{q(s-\omega_0)(\lambda-1)}{(T-\omega_0)-\lambda(\eta-\omega_0)} < \frac{q(s-\omega_0)(\lambda-1)}{(\eta-\omega_0)-\lambda(\eta-\omega_0)} \\
&\quad = -\frac{q(s-\omega_0)}{\eta-\omega_0} < 0
\end{aligned} \tag{4.14}$$

for all $s \in [\omega_0,t]_{q,\omega} \cap [\omega_0,\eta]_{q,\omega}$ and $t \in (\omega_0,T]_{q,\omega}$.

Now, note that

$$G(\omega_0,qs+\omega) = 0 = G(qs+\omega,qs+\omega) \quad \text{for all } s \in [\omega_0,T]_{q,\omega}.$$

Consequently, this implies that

$$\max_{(t,s)\in[\omega_0,T]_{q,\omega}\times[\omega_0,T]_{q,\omega}} G(t,qs+\omega) = G(qs+\omega,qs+\omega).$$

Observe that $G(qs+\omega,qs+\omega) = g_4(qs+\omega,qs+\omega) = q(s+\omega)[T-(qs+\omega)]$.

Thus, by the discussion in the first paragraph of this proof we deduce that (4.7) holds. The proof is complete. □

Lemma 4.4 *Let $G(t,s)$ be Green's function given in* (2.8). *Then it follows that*

$$\begin{aligned} \min_{(t,s)\in[\eta,T]_{q,\omega}\times[\omega_0,T]_{q,\omega}} G(t,qs+\omega) &\geq \sigma \max_{(t,s)\in[\omega_0,T]_{q,\omega}\times[\omega_0,T]_{q,\omega}} G(t,qs+\omega) \\ &= \sigma G(qs+\omega,qs+\omega), \end{aligned} \tag{4.15}$$

where σ satisfies the inequality $0<\sigma<1$, *and*

$$\sigma := \min\left\{\frac{\lambda(\eta-\omega_0)}{q(T-\omega_0)}, \frac{T-\eta}{(T-\omega_0)-q(\eta-\omega_0)}\right\}. \tag{4.16}$$

Proof We define

$$\tilde{g}_i(t,qs+\omega) := \frac{g_i(t,qs+\omega)}{g_k(qs+\omega,qs+\omega)},$$

where $k=3$ if $i=1,3$ and $k=4$ if $i=2,4$.

For $t<\eta$, we find that

$$\tilde{g}_3(t,qs+\omega) = \frac{t-\omega_0}{q(\eta-\omega_0)} > \frac{\eta-\omega_0}{q(\eta-\omega_0)} = \frac{1}{q} := \sigma_1, \tag{4.17}$$

$$\tilde{g}_4(t,qs+\omega) = \frac{t-\omega_0}{q(s-\omega_0)} > \frac{T-\omega_0}{q(T-\omega_0)} = \frac{1}{q} := \sigma_1. \tag{4.18}$$

If $t>\eta$, then we consider two cases of $\tilde{g}_1(t,qs+\omega)$ and $\tilde{g}_2(t,qs+\omega)$:

$$\begin{aligned} \tilde{g}_2(t,qs+\omega) &= \frac{\frac{t-\omega_0}{\Lambda}[T-(qs+\omega)]-[t-(qs+\omega)]}{\frac{qs+\omega-\omega_0}{\Lambda}[T-(qs+\omega)]} \\ &= \frac{t-\omega_0}{q(s-\omega_0)} - \frac{\Lambda[t-(qs+\omega)]}{q(s-\omega_0)[T-(qs+\omega)]} \\ &= \frac{q(s-\omega_0)[(T-t)-\lambda(\eta-\omega_0)]+\lambda(\eta-\omega_0)(t-\omega_0)}{q(s-\omega_0)[(T-\omega_0)-q(s-\omega_0)]} \\ &> \frac{T-t}{(T-\omega_0)-q(\eta-\omega_0)} + \frac{\lambda(\eta-\omega_0)}{q(s-\omega_0)} \\ &> \frac{T-t}{T-\omega_0} + \frac{\lambda(\eta-\omega_0)}{q(t-\omega_0)} > \frac{\lambda(\eta-\omega_0)}{q(T-\omega_0)} := \sigma_2, \end{aligned} \tag{4.19}$$

and

$$\begin{aligned} \tilde{g}_1(t,qs+\omega) &= \frac{\frac{t-\omega_0}{\Lambda}[(T-\lambda\eta)+(\lambda-1)(qs+\omega)]-[t-(qs+\omega)]}{\frac{qs+\omega-\omega_0}{\Lambda}[(T-\lambda\eta)+(\lambda-1)(qs+\omega)]} \\ &= \frac{t-\omega_0}{q(s-\omega_0)} - \frac{\Lambda[t-(qs+\omega)]}{q(s-\omega_0)[(T-\lambda\eta)+(\lambda-1)(qs+\omega)]} \\ &= \frac{t-\omega_0}{q(s-\omega_0)} - \left[\frac{(t-\omega_0)-q(s-\omega_0)}{q(s-\omega_0)}\right] \end{aligned}$$

$$
\begin{aligned}
&\times \left\{ \frac{(T-\omega_0)-\lambda(\eta-\omega_0)}{[(T-\omega_0)-q(s-\omega_0)]-\lambda[(\eta-\omega_0)-q(s-\omega_0)]} \right\} \\
&=: \frac{t-\omega_0}{q(s-\omega_0)} - \left[\frac{(t-\omega_0)-q(s-\omega_0)}{q(s-\omega_0)} \right] \times \mathcal{J}(\lambda). \qquad (4.20)
\end{aligned}
$$

Observe that $\mathcal{J}(\lambda) = \mathcal{I}^{-1}(\lambda)$, which implies that $\mathcal{J}(\lambda)$ is decreasing in λ, and we have

$$
\begin{aligned}
\tilde{g}_1(t, qs+\omega) &\geq \frac{t-\omega_0}{q(s-\omega_0)} - \left(\frac{(t-\omega_0)-q(s-\omega_0)}{q(s-\omega_0)} \right) \left[\frac{T-\omega_0}{(T-\omega_0)-q(s-\omega_0)} \right] \\
&= \frac{1}{q(s-\omega_0)} \left[(t-\omega_0) - (T-\omega_0) \left(\frac{(t-\omega_0)-q(s-\omega_0)}{(T-\omega_0)-q(s-\omega_0)} \right) \right] \\
&= \frac{T-t}{(T-\omega_0)-q(s-\omega_0)} \\
&> \frac{T-\eta}{(T-\omega_0)-q(\eta-\omega_0)} := \sigma_3. \qquad (4.21)
\end{aligned}
$$

Finally, note that since $\sigma_1 > 1$, $0 < \sigma_2 < \sigma_1$, and $0 < \sigma_3 < 1$, it follows that

$$
\sigma = \min\{\sigma_2, \sigma_3\} \leq \sigma_3 < 1. \qquad (4.22)
$$

We can conclude that $\min_{t\in[\eta,T]_{q,\omega}} G(t, qs+\omega) \geq \sigma \max_{t\in[\omega_0,T]_{q,\omega}} G(t, qs+\omega)$. □

Lemma 4.5 *Let φ be a nonnegative function. Then there exists $\sigma^* \in (0,1)$ such that*

$$
\begin{aligned}
&\min_{t\in[\eta,T]_{q,\omega}} \left\{ \int_{\omega_0}^{T} G(t, qs+\omega) f\big(s, x(s), (D_{p,\theta})x(ps+\theta)\big)\, d_{q,\omega}s \right. \\
&\qquad \left. + \left[1 - \frac{(1-\lambda)(t-\omega_0)}{\Lambda} \right] \varphi(x) \right\} \\
&\quad \geq \sigma^* \max_{t\in[\omega_0,T]_{q,\omega}} \left\{ \int_{\omega_0}^{T} G(t, qs+\omega) f\big(s, x(s), (D_{p,\theta})x(ps+\theta)\big)\, d_{q,\omega}s \right. \\
&\qquad \left. + \left[1 - \frac{(1-\lambda)(t-\omega_0)}{\Lambda} \right] \varphi(x) \right\}. \qquad (4.23)
\end{aligned}
$$

Proof Observe that by Lemma 4.4 there exists a constant $\sigma \in (0,1)$ such that

$$
\begin{aligned}
&\min_{t\in[\eta,T]_{q,\omega}} \int_{\omega_0}^{T} G(t, qs+\omega) f\big(s, x(s), (D_{p,\theta})x(ps+\theta)\big)\, d_{q,\omega}s \\
&\quad \geq \sigma \max_{t\in[\omega_0,T]_{q,\omega}} \int_{\omega_0}^{T} G(t, qs+\omega) f\big(s, x(s), (D_{p,\theta})x(ps+\theta)\big)\, d_{q,\omega}s. \qquad (4.24)
\end{aligned}
$$

Next, by Lemma 4.1 there exists a constant $\mathcal{S} > 0$ such that

$$
\min_{t\in[\eta,T]_{q,\omega}} \left[1 + \frac{(\lambda-1)(t-\omega_0)}{\Lambda} \right] = \frac{T-\eta}{\Lambda} = \mathcal{S} \quad \text{and} \qquad (4.25)
$$

$$
\max_{t\in[\omega_0,T]_{q,\omega}} \left[1 + \frac{(\lambda-1)(t-\omega_0)}{\Lambda} \right] = \frac{\lambda(T-\eta)}{\Lambda} = \lambda\mathcal{S}. \qquad (4.26)
$$

In particular, putting (4.25) and (4.26) together implies that by taking

$$\sigma_0 \quad \text{such that} \quad 0 < \frac{1}{\lambda} \le \sigma_0 < 1$$

it follows that

$$\begin{aligned} &\min_{t\in[\eta,T]_{q,\omega}} \left[1 + \frac{(\lambda-1)(t-\omega_0)}{\Lambda}\right]\varphi(x) \\ &\quad = \sigma_0 \max_{t\in[\omega_0,T]_{q,\omega}} \left[1 + \frac{(\lambda-1)(t-\omega_0)}{\Lambda}\right]\varphi(x). \end{aligned} \tag{4.27}$$

Finally, defining

$$\sigma^* := \min\{\sigma, \sigma_0\} \in (0,1), \tag{4.28}$$

we obtain (4.23). This completes the proof. □

Lemma 4.6 *Let G be Green's function given in* (2.8). *Then*

$$\begin{aligned} \int_{\omega_0}^{T} G(qs+\omega, qs+\omega)\, d_{q,\omega}s = {}& \frac{q(T-\omega_0)}{(1+q)\Lambda}\left\{T\left[T + \omega_0\left(1+q-q^2\right)\right]\right. \\ & \left. - \frac{1}{1+q+q^2}\left[T^2(1+q) - \omega_0^2 q\left(1+q^2\right)\right] - (T+q\omega_0)\right\}. \end{aligned}$$

Proof Using the definition of the q,ω-integral, for $s \in [\omega_0, T]_{q,\omega}$, we obtain

$$\begin{aligned} &\int_{\omega_0}^{T} G(qs+\omega, qs+\omega)\, d_{q,\omega}s \\ &\quad = \frac{1}{\Lambda}\int_{\omega_0}^{T} q(s+\omega)\left[T - (qs+\omega)\right] d_{q,\omega}s \\ &\quad = \frac{qT}{\Lambda}\int_{\omega_0}^{T} (s+\omega)\, d_{q,\omega}s - \frac{q}{\Lambda}\left[\int_{\omega_0}^{T} s(qs+\omega)\, d_{q,\omega}s + \omega\int_{\omega_0}^{T}(qs+\omega)\, d_{q,\omega}s\right] \\ &\quad = \frac{qT}{\Lambda}\left[T(1-q)-\omega\right]\sum_{k=0}^{\infty} q^k\left(Tq^k + \omega[k]_q + \omega\right) \\ &\qquad - \frac{q}{\Lambda}\left[\left[T(1-q)-\omega\right]\sum_{k=0}^{\infty} q^k\left[\left(Tq^k + \omega[k]_q\right)\left(\left(Tq^k+\omega[k]_q\right)q + \omega\right)\right]\right. \\ &\qquad \left. + \omega\left[T(1-q)-\omega\right]\sum_{k=0}^{\infty} q^k\left[\left(Tq^k+\omega[k]_q\right)q + \omega\right]\right] \\ &\quad = \frac{q(T-\omega_0)}{(1+q)\Lambda} \\ &\qquad \times \left\{T\left[T+\omega_0\left(1+q-q^2\right)\right] - \frac{1}{1+q+q^2}\left[T^2(1+q) - \omega_0^2 q\left(1+q^2\right)\right] - (T+q\omega_0)\right\}. \end{aligned}$$

This completes the proof. □

5 Existence of a positive solution for problem (1.1)

In this section, we consider the existence of at least one positive solution for problem (1.1) by appealing to the Krasnoselskii fixed point theorem in a cone.

Define the cone $\mathcal{P} \subseteq \mathcal{C}$ by

$$\mathcal{P} := \Big\{x \in \mathcal{C} : x(t) \geq 0, \min_{t\in[\eta,T]_{q,\omega}} x(t) \geq \sigma^* \big\|x(t)\big\|_{\mathcal{C}} \text{ and } \varphi(x) \geq 0\Big\}. \tag{5.1}$$

Consider nonlinear equation (1.1); then x solves (1.1) if and only if x is a fixed point of the operator $\mathcal{A} : \mathcal{P} \to \mathcal{P}$ defined by

$$\begin{aligned}(\mathcal{A}x)(t) :={}& \left[1 + \frac{(\lambda-1)(t-\omega_0)}{\Lambda}\right]\varphi(x) \\ &+ \int_{\omega_0}^{T} G(t, qs+\omega) f\big(s, x(s), D_{p,\theta}x(ps+\theta)\big)\, d_{q,\omega}s,\end{aligned} \tag{5.2}$$

where G is Green's function for problem (1.1), and $\mathcal{C}$ is the Banach space defined in Section 3.

Lemma 5.1 *Suppose that* $f : [\omega_0, T]_{q,\omega} \times [0,\infty) \times [0,\infty) \to [0,\infty)$ *and* $\varphi : C([\omega_0, T]_{q,\omega}, [0,\infty)) \to [0,\infty)$ *are continuous. Then the operator* $\mathcal{A} : \mathcal{P} \to \mathcal{P}$ *is completely continuous.*

Proof Since $G(t, qs+\omega) \geq 0$ for all $(t,s) \in [\omega_0, T]_{q,\omega} \times [\omega_0, T]_{q,\omega}$, we have $\mathcal{A} \geq 0$ for all $x \in \mathcal{P}$. For a constant $L > 0$, we define

$$B_L = \big\{x \in \mathcal{P} : \|x\|_{\mathcal{C}} < L\big\}$$

and let $M = \max_{(t,x)\in[\omega_0,T]_{q,\omega}\times B_L} |f(t, x(t), D_{p,\theta}x(pt+\theta))|$, $N = \sup_{x\in B_L} |\varphi(x)|$. Then, for $x \in B_L$, we obtain

$$\begin{aligned}&\big|(\mathcal{A}x)(t)\big| \\ &\quad= \left|\left[1 + \frac{(\lambda-1)(t-\omega_0)}{\Lambda}\right]\varphi(x) + \int_{\omega_0}^{T} G(t, qs+\omega) f\big(s, x(s), D_{p,\theta}x(ps+\theta)\big)\, d_{q,\omega}s\right| \\ &\quad\leq \frac{N\lambda(T-\eta)}{\Lambda} + M\int_{\omega_0}^{T} G(qs+\omega, qs+\omega)\, d_{q,\omega}s \\ &\quad= \frac{N\lambda(T-\eta)}{\Lambda} + \frac{qM(T-\omega_0)}{(1+q)\Lambda} \\ &\qquad\times \left|T\big[T + \omega_0\big(1+q-q^2\big)\big] - \frac{1}{1+q+q^2}\big[T^2(1+q) - \omega_0^2 q\big(1+q^2\big)\big] - (T + q\omega_0)\right| \\ &\quad=: \mathcal{K}.\end{aligned}$$

Similarly to the proof above and Theorem 3.1, we obtain

$$\begin{aligned}&\big|(D_{p,\theta}\mathcal{A}x)(pt+\theta)\big| \\ &\quad= \left|\frac{1}{(p-1)((pt+\theta)-\omega_0)}\left\{\frac{(pt+\theta)(p-1)+\theta}{\Lambda}\right.\right.\end{aligned}$$

$$
\begin{aligned}
&\times\bigg[(\lambda-1)\varphi(x)-\lambda\int_{\omega_0}^{\eta}G(pt+\theta,qs+\omega)f\big(s,x(s),D_{p,\theta}x(s)\big)\,d_{q,\omega}s \\
&+\int_{\omega_0}^{T}G(pt+\theta,qs+\omega)f\big(s,x(s),D_{p,\theta}x(s)\big)\,d_{q,\omega}s\bigg] \\
&-\int_{\omega_0}^{p(pt+\theta)+\theta}\big[p(pt+\theta)+\theta-(qs+\omega)\big]f\big(s,x(s),D_{p,\theta}x(s)\big)\,d_{q,\omega}s \\
&+\int_{\omega_0}^{pt+\theta}\big[(pt+\theta)-(qs+\omega)\big]f\big(s,x(s),D_{p,\theta}x(s)\big)\,d_{q,\omega}s\bigg\}\bigg| \\
<{}&\big|(\mathcal{A}x)(t)\big|.
\end{aligned}
\tag{5.3}
$$

Therefore, $\|(\mathcal{A}x)(t)\|_{\mathcal{C}}=\mathcal{K}$, and hence $\mathcal{A}(B_L)$ is uniformly bounded.

Next, we shall show that $\mathcal{A}(B_L)$ is equicontinuous. For $x\in B_L$ and $t_1,t_2\in[\omega_0,T]_{q,\omega}$ with $t_1<t_2$, there are three cases to consider.

Case 1: If $\eta\le t_1<t_2$, then by (2.7), letting $g_i(t,qs+\omega)=\frac{(t-\omega_0)}{\Lambda}\mathfrak{g}_i(t,qs+\omega)$, we obtain

$$
\begin{aligned}
&\big|(\mathcal{A}x)(t_2)-(\mathcal{A}x)(t_1)\big| \\
&\le|t_2-t_1|(\lambda-1)\frac{N}{\Lambda}+M\bigg|\int_{\omega_0}^{T}\big[G(t_2,qs+\omega)-G(t_1,qs+\omega)\big]\,d_{q,\omega}s\bigg| \\
&\le|t_2-t_1|\frac{N(\lambda-1)}{\Lambda}+M\bigg|\frac{(t_2-t_1)}{\Lambda}\int_{\omega_0}^{\eta}\mathfrak{g}_1(qs+\omega,qs+\omega)\,d_{q,\omega}s \\
&\quad+\bigg[\frac{(t_2-\omega_0)}{\Lambda}\int_{\eta}^{t_2}\mathfrak{g}_2(qs+\omega,qs+\omega)\,d_{q,\omega}s-\frac{(t_1-\omega_0)}{\Lambda}\int_{\eta}^{t_1}\mathfrak{g}_2(qs+\omega,qs+\omega)\,d_{q,\omega}s\bigg] \\
&\quad+\bigg[\frac{(t_2-\omega_0)}{\Lambda}\int_{t_2}^{T}\mathfrak{g}_4(qs+\omega,qs+\omega)\,d_{q,\omega}s-\frac{(t_1-\omega_0)}{\Lambda}\int_{t_1}^{T}\mathfrak{g}_4(qs+\omega,qs+\omega)\,d_{q,\omega}s\bigg]\bigg| \\
&=\frac{N(\lambda-1)}{\Lambda}|t_2-t_1|+\frac{M}{\Lambda}|t_2-t_1| \\
&\quad\times\bigg|\big[T-1-\lambda(\eta-1)\big]\int_{\omega_0}^{\eta}d_{q,\omega}s-\int_{\omega_0}^{\eta}\big[T-(qs+\omega)\big]\,d_{q,\omega}s \\
&\quad+\int_{\omega_0}^{T}\big[T-(qs+\omega)\big]\,d_{q,\omega}s\bigg| \\
&=\frac{|t_2-t_1|}{\Lambda}\bigg\{N(\lambda-1)+M\bigg|\big[T-1-\lambda(\eta-1)\big](\eta-\omega_0) \\
&\quad+\frac{\eta-\omega_0}{1+q}\big[(T-\omega_0)+q(T+\eta)\big]+\frac{(T-\omega_0)^2}{1+q}\bigg|\bigg\} \\
&=\frac{|t_2-t_1|}{\Lambda}\,\boldsymbol{\Xi}.
\end{aligned}
$$

Therefore, there exists a constant $\delta_1>0$ such that

$$
\big|(\mathcal{A}x)(t_2)-(\mathcal{A}x)(t_1)\big|<\frac{\epsilon}{2}\quad\text{whenever } |t_2-t_1|<\delta_1=\frac{\epsilon\Lambda}{2\boldsymbol{\Xi}}.
\tag{5.4}
$$

Case 2: If $t_1 < t_2 \leq \eta$, then by (2.7) we obtain

$$\begin{aligned}
&\big|(\mathcal{A}x)(t_2) - (\mathcal{A}x)(t_1)\big| \\
&\quad\leq |t_2 - t_1|(\lambda - 1)\frac{N}{\Lambda} + M\left|\int_{\omega_0}^{T}\big[G(t_2, qs+\omega) - G(t_1, qs+\omega)\big]\,d_{q,\omega}s\right| \\
&\quad\leq |t_2 - t_1|\frac{N(\lambda-1)}{\Lambda} + M\left|\left[\frac{(t_2-\omega_0)}{\Lambda}\int_{\omega_0}^{t_2}\mathfrak{g}_1(qs+\omega, qs+\omega)\,d_{q,\omega}s\right.\right. \\
&\qquad\left. - \frac{(t_1-\omega_0)}{\Lambda}\int_{\omega_0}^{t_1}\mathfrak{g}_1(qs+\omega, qs+\omega)\,d_{q,\omega}s\right] + \left[\frac{(t_2-\omega_0)}{\Lambda}\int_{t_2}^{\eta}\mathfrak{g}_3(qs+\omega, qs+\omega)\,d_{q,\omega}s\right. \\
&\qquad\left.\left. - \frac{(t_1-\omega_0)}{\Lambda}\int_{t_1}^{\eta}\mathfrak{g}_3(qs+\omega, qs+\omega)\,d_{q,\omega}s\right]\right| + \frac{(t_2-t_1)}{\Lambda}\int_{\eta}^{T}\mathfrak{g}_4(qs+\omega, qs+\omega)\,d_{q,\omega}s \\
&\quad= \frac{N(\lambda-1)}{\Lambda}|t_2 - t_1| + \frac{M}{\Lambda}|t_2 - t_1| \\
&\qquad\times\left|\int_{\omega_0}^{\eta}\big(\big[T - 1 - \lambda(\eta-1)\big] - \big[T - (qs+\eta)\big]\big)\,d_{q,\omega}s + \int_{\omega_0}^{T}\big[T - (qs+\omega)\big]\,d_{q,\omega}s\right| \\
&\quad= \frac{|t_2 - t_1|}{\Lambda}\left\{N(\lambda-1) + M\left|\big[T - 1 - \lambda(\eta-1)\big](\eta-\omega_0)\right.\right. \\
&\qquad\left.\left. + \frac{\eta-\omega_0}{1+q}\big[(T-\omega_0) + q(T+\eta)\big] + \frac{(T-\omega_0)^2}{1+q}\right|\right\} \\
&\quad= \frac{|t_2 - t_1|}{\Lambda}\mathbf{\Xi}.
\end{aligned}$$

Therefore, there exists a constant $\delta_2 > 0$ such that

$$\big|(\mathcal{A}x)(t_2) - (\mathcal{A}x)(t_1)\big| < \frac{\epsilon}{2} \quad \text{whenever } |t_2 - t_1| < \delta_2 = \frac{\epsilon\Lambda}{2\mathbf{\Xi}}. \tag{5.5}$$

Case 3: If $t_1 < \eta < t_2$ with $|t_2 - t_1| < \delta = \min\{\delta_1, \delta_2\}$, then from (5.4)-(5.5) it follows that

$$\begin{aligned}
\big|(\mathcal{A}x)(t_2) - (\mathcal{A}x)(t_1)\big| &\leq \big|(\mathcal{A}x)(t_2) - (\mathcal{A}x)(\eta)\big| + \big|(\mathcal{A}x)(\eta) - (\mathcal{A}x)(t_1)\big| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned} \tag{5.6}$$

Similarly to the proof above, by (5.3) we obtain

$$\big|(D_{p,\theta}\mathcal{A}x)(pt_2+\theta) - (D_{p,\theta}\mathcal{A}x)(pt_1+\theta)\big| < \big|(\mathcal{A}x)(t_2) - (\mathcal{A}x)(t_1)\big| < \epsilon. \tag{5.7}$$

Hence, we conclude that $|(\mathcal{A}x)(t_2) - (\mathcal{A}x)(t_1)| < \epsilon$ if $|t_2 - t_1| < \delta = \min\{\delta_1, \delta_2\}$ for $t_2, t_1 \in [\omega_0, T]_{q,\omega}$, that is, $\mathcal{A}(B_L)$ is equicontinuous. By the Arzelà-Ascoli theorem, $\mathcal{A} : \mathcal{C} \to \mathcal{C}$ is a completely continuous operator.

Finally, we apply Lemmas 4.2-4.4 to obtain

$$(\mathcal{A}x)(t) \geq 0 \quad \text{for all } t \in [\omega_0, T]_{q,\omega} \tag{5.8}$$

and, for $f \in \mathcal{P}$,

$$\begin{aligned}
\min_{t\in[\eta,T]_{q,\omega}} (\mathcal{A}x)(t) &\geq \min_{t\in[\eta,T]_{q,\omega}} \left\{ \left[1 + \frac{(\lambda-1)(t-\omega_0)}{\Lambda}\right]\varphi(x) \right. \\
&\quad \left. + \int_{\omega_0}^{T} G(t, qs+\omega) f\bigl(s, x(s), D_{p,\theta}x(ps+\theta)\bigr)\, d_{q,\omega}s \right\} \\
&= \sigma^* \left\{ \max_{t\in[\omega_0,T]_{q,\omega}} \left[1 + \frac{(\lambda-1)(t-\omega_0)}{\Lambda}\right]\varphi(x) \right. \\
&\quad \left. + \int_{\omega_0}^{T} G(qs+\omega, qs+\omega) f\bigl(s, x(s), D_{p,\theta}x(ps+\theta)\bigr)\, d_{q,\omega}s \right\} \\
&= \sigma^*(\mathcal{A}x)(qs+\omega).
\end{aligned} \tag{5.9}$$

Hence,

$$\min_{t\in[\eta,T]_{q,\omega}} (\mathcal{A}x)(t) \geq \sigma^* \|\mathcal{A}x\|_{\mathcal{C}}, \quad \text{that is, } \mathcal{A}\mathcal{P} \subset \mathcal{P}.$$

Consequence, it follows that $\mathcal{A} : \mathcal{P} \to \mathcal{P}$ is a completely continuous operator. □

The following notation is used in the sequel:

$$\begin{aligned}
\diamondsuit &:= \frac{1}{\int_{\omega_0}^{T} G(qs+\omega, qs+\omega)\, d_{q,\omega}s}, \\
\Theta &:= \frac{1}{\sigma^* \int_{\omega_0}^{T} G(qs+\omega, qs+\omega)\, d_{q,\omega}s}, \\
\Upsilon_1 &:= \frac{\Lambda}{\lambda(T-\eta)}, \quad \text{and} \\
\Upsilon_2 &:= \frac{\Lambda}{\sigma^*[\lambda(T-\eta)]}.
\end{aligned}$$

Next, we introduce some assumptions that will be helpful in the sequel.

(A_1) There exists a constant $r_1 > 0$ such that

$$f\bigl(t, x(t), D_{p,\theta}x(pt+\theta)\bigr) \leq \frac{1}{2}\diamondsuit r_1 \tag{5.10}$$

for all $t \in [\omega_0, T]_{q,\omega}$ and $0 \leq x \leq r_1$.

(A_2) There exists a constant $r_2 > 0$ with $r_2 < r_1$ such that

$$f\bigl(t, x(t), D_{p,\theta}x(pt+\theta)\bigr) \geq \frac{1}{2}\Theta r_2 \tag{5.11}$$

for all $t \in [\omega_0, T]_{q,\omega}$ and $\sigma^* r_2 \leq x \leq r_2$, where σ^* is defined in (4.28).

(A_3) There exists a constant $r_1 > 0$ such that

$$\varphi(x) \leq \frac{1}{2}\Psi_1 r_1 \tag{5.12}$$

for all $x \in \mathcal{P}$ and $0 \leq \|x\|_{\mathcal{C}} \leq r_1$.

(A_4) There exists a constant $r_2 > 0$ such that

$$\varphi(x) \geq \frac{1}{2}\Psi_2 r_2 \tag{5.13}$$

for all $x \in \mathcal{P}$ and $\sigma^* r_2 \leq \|x\|_{\mathcal{C}} \leq r_2$.

Now, we can prove the existence of at least one positive solution.

Theorem 5.1 *Suppose that conditions* (A_1)-(A_4) *hold. Let* $f(t,x) \in C([\eta, T]_{q,\omega} \times [0,\infty) \times [0,\infty), [0,\infty))$ *and* $\varphi(x) : C([\eta, T]_{q,\omega}, [0,\infty)) \to [0,\infty)$. *Then problem* (1.1) *has at least one positive solution, say* x^*, *where* $r_2 \leq \|x^*\|_{\mathcal{C}} \leq r_1$.

Proof Set $\Psi_1 = \{x \in C([\omega_0, T]_{q,\omega}) : \|x\|_{\mathcal{C}} < r_1\}$. Then, for $x \in \mathcal{P} \cap \partial\Psi_1$, we have

$$\begin{aligned}
(\mathcal{A}x)(t) &\leq \max_{t\in[\omega_0,T]_{q,\omega}} \biggl\{ \biggl[1 + \frac{(\lambda-1)(t-\omega_0)}{\Lambda}\biggr]\varphi(x) \\
&\quad + \int_{\omega_0}^{T} G(t, qs+\omega) f\bigl(s, x(s), D_{p,\theta}x(ps+\theta)\bigr)\, d_{q,\omega}s \biggr\} \\
&\leq \biggl(\frac{\lambda(T-\eta)}{\Lambda}\biggr)\frac{1}{2}\Upsilon_1 r_1 + \frac{1}{2}\diamondsuit r_1 \int_{\omega_0}^{T} G(qs+\omega, qs+\omega)\, d_{q,\omega}s \\
&= \frac{r_1}{2} + \frac{r_1}{2} = r_1.
\end{aligned}$$

Since $|(D_{p,\theta}\mathcal{A}x)(t)| < |(\mathcal{A}x)(t)| \leq r_1$, we have

$$\|\mathcal{A}x\|_{\mathcal{C}} \leq \|x\|_{\mathcal{C}} \quad \text{whenever } x \in \mathcal{P} \cap \partial\Psi_1. \tag{5.14}$$

Further, let $\Psi_2 = \{x \in C([\omega_0, T]_{q,\omega}) : \|x\|_{\mathcal{C}} < r_2\}$. Then, for $x \in \mathcal{P} \cap \partial\Psi_2$, using Lemma 4.4, we find that

$$\begin{aligned}
(\mathcal{A}x)(t) &\geq \min_{t\in[\eta,T]_{q,\omega}} \biggl\{ \biggl[1 + \frac{(\lambda-1)(t-\omega_0)}{\Lambda}\biggr]\varphi(x) \biggr\} \\
&\quad + \int_{\omega_0}^{T} \min_{t\in[\eta,T]_{q,\omega}} G(t, qs+\omega) f\bigl(s, x(s), D_{p,\theta}x(ps+\theta)\bigr)\, d_{q,\omega}s \\
&\geq \frac{1}{2}\Upsilon_2 r_2 \sigma^* \max_{t\in[\omega_0,T]_{q,\omega}} \biggl\{ \biggl[1 + \frac{(\lambda-1)(t-\omega_0)}{\Lambda}\biggr]\varphi(x) \biggr\} \\
&\quad + \frac{1}{2}\Theta r_2 \sigma^* \max_{t\in[\omega_0,T]_{q,\omega}} \int_{\omega_0}^{T} G(t, qs+\omega)\, d_{q,\omega}s \\
&= \frac{1}{2}\Upsilon_2 r_2 \sigma^* \biggl(\frac{\lambda(T-\eta)}{\Lambda}\biggr) + \frac{1}{2}\Theta r_2 \sigma^* \int_{\omega_0}^{T} G(qs+\omega, qs+\omega)\, d_{q,\omega}s \\
&= \frac{r_2}{2} + \frac{r_2}{2} = r_2.
\end{aligned}$$

Since $|(D_{p,\theta}\mathcal{A}x)(t)| < |(\mathcal{A}x)(t)| \leq r_2$, we have

$$\|\mathcal{A}x\|_{\mathcal{C}} \geq \|x\|_{\mathcal{C}} \quad \text{whenever } x \in \mathcal{P} \cap \partial\Psi_2. \tag{5.15}$$

We conclude by Theorem 1.1 that the operator $\mathcal{A}$ has a fixed point. This implies that problem (1.1) has a positive solution, say x^*, where $r_2 \leq \|x^*\|_{\mathcal{C}} \leq r_1$. □

6 Example

In this section, to illustrate our results, we consider an example.

Example Consider the following boundary value problem for the second-order Hahn difference equation

$$\begin{aligned} &D^2_{\frac{1}{3},\frac{3}{2}}x(t) + \frac{e^{-\cos^2(\pi t)}}{(t+20)^2} \cdot \frac{|x| + \arctan(\cos^2(\pi t))[D_{\frac{1}{243},\frac{121}{54}}x(\frac{1}{243}t + \frac{121}{54})]}{|x| + \frac{1}{e}} = 0, \\ &x\left(\frac{9}{4}\right) = \sum_{i=0}^{\infty} \frac{C_i|x(t_i)|}{1+|x(t_i)|}, \qquad t_i = 10\left(\frac{1}{3}\right)^i + \frac{3}{2}[i]_{\frac{1}{3}}, \\ &x(10) = \frac{4}{3}x\left(\frac{1109}{486}\right), \end{aligned} \tag{6.1}$$

where $t \in [\frac{9}{4}, 10]_{\frac{1}{3},\frac{3}{2}}$, and C_i are given positive constants with $\frac{1}{e^2} \leq \sum_{i=0}^{\infty} C_i \leq \frac{2}{e^2}$.

Set $q = \frac{1}{3}, \omega = \frac{3}{2}, \omega_0 = \frac{9}{4}, p = \frac{1}{3^5} = \frac{1}{243}, \theta = \frac{3}{2}[\frac{1-(\frac{1}{3})^5}{1-\frac{1}{3}}] = \frac{121}{54}, T = 10, \eta = 10(\frac{1}{3})^5 + \frac{3}{2}[5]_{\frac{1}{3}} = \frac{1109}{486}$, $\lambda = \frac{4}{3}$, $\varphi(x) = \sum_{i=0}^{\infty} \frac{C_i|x(t_i)|}{1+|x(t_i)|}$, and

$$\begin{aligned} f\bigl(t, x(t), D_{p,\theta}x(pt+\theta)\bigr) &= \frac{e^{-\cos^2(\pi t)}}{(t+20)^2} \\ &\quad \times \frac{|x| + \arctan(\cos^2(\pi t))[D_{\frac{1}{243},\frac{121}{54}}x(\frac{1}{243}t + \frac{121}{54})]}{|x| + \frac{1}{e}}. \end{aligned}$$

I. The existence and uniqueness of solution to problem (6.1). We can show that

$$1 \leq \lambda < \frac{30132}{5956} \approx 5, \qquad \Lambda \approx 7.707, \qquad \Omega \approx 90.362, \quad \text{and} \quad \Phi \approx 1.335.$$

Clearly,

$$\begin{aligned} &\bigl|f\bigl(t, x(t), D_{p,\theta}x(pt+\theta)\bigr) - f\bigl(t, y(t), D_{p,\theta}y(pt+\theta)\bigr)\bigr| \\ &\quad < \frac{|x(t)-y(t)|}{(t+20)^2} + \frac{\arctan(1)|D_{p,\theta}x(pt+\theta) - D_{p,\theta}y(pt+\theta)|}{(t+20)^2} \\ &\quad \leq 0.002\bigl|x(t)-y(t)\bigr| + 0.0016\bigl|D_{p,\theta}x(pt+\theta) - D_{p,\theta}y(pt+\theta)\bigr|, \end{aligned}$$

so that (H_1) holds with $\gamma_1 = 0.002$, $\gamma_2 = 0.0016$, and $\gamma = \max\{\gamma_1, \gamma_2\} = 0.002$, and

$$\begin{aligned} \bigl|\varphi(x) - \varphi(y)\bigr| &= \left|\sum_{i=0}^{\infty} \frac{C_i|x(t_i)|}{1+|x(t_i)|} - \sum_{i=0}^{\infty} \frac{C_i|y(t_i)|}{1+|y(t_i)|}\right| \\ &< \sum_{i=0}^{\infty} C_i\bigl|x(t_i) - y(t_i)\bigr| \leq \frac{2}{e^2}\|x-y\|, \end{aligned}$$

so that (H_2) holds with $\ell = \frac{2}{e^2}$.

Also, we can show that

$$\mathfrak{S} = \gamma\Omega + \ell\Phi \approx 0.542 < 1.$$

Hence, by Theorem 3.1 problem (6.1) has a unique solution.

II. The existence of at least one positive solution to problem (6.1). We can show that

$$\begin{aligned}
\sigma &= \min\left\{\frac{\lambda(\eta-\omega_0)}{q(T-\omega_0)}, \frac{T-\eta}{(T-\omega_0)-q(\eta-\omega_0)}\right\} \\
&= \min\{0.0165, 0.997\} = 0.0165, \\
&\frac{1}{\lambda} = 0.75 \le \sigma_0 < 1, \\
&\sigma^* = \min\bigl\{\sigma, \min\{\sigma_0\}\bigr\} = 0.0165, \\
&\diamondsuit = \frac{1}{\int_{\omega_0}^{T} G(qs+\omega, qs+\omega)\, d_{q,\omega}s} = 0.146, \\
&\Theta = \frac{1}{\sigma^* \int_{\omega_0}^{T} G(qs+\omega, qs+\omega)\, d_{q,\omega}s} = 8.850, \\
&\Upsilon_1 = \frac{\Lambda}{\lambda(T-\eta)} = 0.749, \quad \text{and} \quad \Upsilon_2 = \frac{\Lambda}{\sigma^*[\lambda(T-\eta)]} = 45.389.
\end{aligned}$$

Clearly,

$$\begin{aligned}
&\bigl|f\bigl(t, x(t), D_{p,\theta}x(pt+\theta)\bigr)\bigr| \le \frac{1+\frac{\pi}{4}}{(\frac{9}{2}+20)^2} = 0.0036 = 0.049\frac{\diamondsuit}{2} \\
&\quad \text{for } 0 \le x \le r_1 \le 0.049, \\
&\bigl|f\bigl(t, x(t), D_{p,\theta}x(pt+\theta)\bigr)\bigr| \ge \frac{1}{e(30)^2}\left[\frac{|x|}{|x|+1}\right] > \frac{1}{2e} = 0.0002 = 0.000045\frac{\Theta}{2} \\
&\quad \text{for } 0.00000074 = 0.000045\sigma^* \le x \le r_2 \le 0.000045, \\
&\varphi(x) \le \frac{2}{e^2} = 0.271 = 3.3457\frac{\Upsilon_1}{2} \quad \text{for } 0 \le x \le r_1 \le 3.3457, \quad \text{and} \\
&\varphi(x) \ge \frac{1}{e^2} = 0.135 = 0.0059\frac{\Upsilon_2}{2} \quad \text{for } 0.000097 = 0.0059\sigma^* \le x \le r_2 \le 0.0059.
\end{aligned}$$

Therefore, conditions (A_1)-(A_3) are satisfied. Consequently, by Theorem 5.1 problem (6.1) has at least one positive solution x^* such that $r_2 = 0.000097 \le \|x^*\|_{\mathcal{C}} \le 0.0036 = r_1$.

Competing interests
The author declares that he has no competing interests.

Author's contributions
The author declares that he carried out all the work in this manuscript and read and approved the final manuscript.

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