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The eigenvalues and sign-changing solutions of a fractional boundary value problem

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Abstract

In this paper, we are interested in the eigenvalues and its algebraic multiplicities of a fractional linear boundary value problem with mixed set of Neumann and Dirichlet boundary conditions. The research results are then applied to consider the sign-changing solutions of the corresponding nonlinear problem by fixed point index and Leray-Schauder degree. To date, no paper has appeared in the literature which discusses sign-changing solutions of fractional boundary value problems. This paper attempts to fill this gap in the literature.

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1 Introduction

With the development of science and technology, researchers have paid much attention to the fractional differential equations, it is extensively applied in various sciences, such as physics, mechanics, chemistry, engineering, astronomy, *etc.* There are a lot of research papers about the fractional differential equation boundary value problems; see [1–8] and the references therein. Most of them are devoted to the existence and multiplicity of positive solutions; see [1, 2, 6–8]. For example, in [1], the author considered the existence of positive solutions for a class of nonlinear boundary value problems of Caputo fractional equations with integral boundary conditions,

$$\begin{cases} {}^cD_t^\alpha y(t) + f(y(t)) = 0, & 0 < t < 1, \\ y(0) = y''(0) = 0, & y(1) = \lambda \int_0^1 y(s)\,ds. \end{cases}$$

In [7], the author considered the existence and multiplicity of positive solutions for a nonlinear boundary value problem involving Caputo's derivative

$$\begin{cases} {}^cD_t^\alpha y(t) = f(t, y(t)), & t \in (0, 1), \\ y(0) + y'(0) = 0, & y(1) + y'(1) = 0. \end{cases}$$

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To the best of the author's knowledge, although sign-changing solutions of integer boundary value problems with different conditions are extensively studied by computing the algebraic multiplicities of eigenvalues, see for example [9–13] and the references therein, to date, no paper has appeared in the literature which discusses sign-changing solutions of fractional boundary value problems due to the intrinsic distinction between the eigenvalues of fractional problems and the integer problems. For example, the eigenvalues of fractional differential equations have no periodicity.

Motivated by the above papers, first, we investigate the following eigenvalue problem with the mixed set of Neumann and Dirichlet boundary conditions,

$$\begin{cases} {}^cD_t^\alpha u(t) + \lambda u(t) = 0, & t \in (0,1), \\ u'(0) = 0, \qquad u(1) = 0. \end{cases} \tag{1.1}$$

Then we establish some existence results of sign-changing solutions for the following non-linear fractional boundary value problem with the same boundary conditions:

$$\begin{cases} {}^cD_t^\alpha u(t) + f(u(t)) = 0, & t \in (0,1), \\ u'(0) = 0, \qquad u(1) = 0, \end{cases} \tag{1.2}$$

where $1 < \alpha < 2$ is a real number and ${}^cD_t^\alpha$ is the Caputo fractional derivative, $f : \mathbb{R} \mapsto \mathbb{R}$.

For convenience in the presentation, throughout this paper, let

$$\beta_0 = \lim_{x \to 0} \frac{f(x)}{x}, \qquad \beta_\infty = \lim_{x \to \infty} \frac{f(x)}{x}.$$

And we always assume the following conditions are satisfied:

(H0) $f(x) \in C(\mathbb{R}, \mathbb{R}), f(\theta) = \theta, xf(x) > 0$ for all $x \in \mathbb{R} \setminus \{\theta\}$.

(H1) There exist two positive integers n_0 and n_1. And n_0, n_1 may be equal, with

$$\lambda_{2n_0} < \beta_0 < \lambda_{2n_0+1}, \qquad \lambda_{2n_1} < \beta_\infty < \lambda_{2n_1+1},$$

where $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_{n_\alpha}$ are the eigenvalues of (1.1), n_α is the number of eigenvalues.

(H2) There exists a positive constant number $C_0 > 0$ such that $|f(x)| < \Gamma(\alpha)C_0$ for all x with $|x| \le C_0$.

We shall organize the rest of this paper as follows. In Section 2, some basic definitions and preliminaries are given. Furthermore the eigenvalues and its algebraic multiplicities of (1.1) are considered. In Section 3, the sign-changing solutions of (1.2) are considered. An example will be given to illustrate the application in Section 4.

2 Some basic definitions and preliminaries

Definition 2.1 The Caputo fractional derivative of order $\alpha > 0$ for the function $y : (0, +\infty) \to \mathbb{R}$ is defined as

$$ {}^cD_t^\alpha y(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{y^{(m)}(s)}{(t-s)^{\alpha-m+1}} \, ds,$$

where $m - 1 < \alpha \le m$, and $y^{(m)}(t)$ exists.

Definition 2.2 The Riemann-Liouville fractional integral of order α for the function f is defined as

$$I_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\,ds, \quad \alpha > 0,$$

provided that the right side is point-wise defined on $(0, \infty)$.

Definition 2.3 The Mittag-Leffler function with two parameters is defined by the series expansion

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha > 0, \beta > 0, z \in \mathbb{C},$$

which is analytic on the whole complex plane.

Now we investigate the eigenvalue problem (1.1). From the Laplace transform of the Caputo fractional derivative [3]

$$\mathcal{L}\{^c D_t^{\alpha} u(t)\} = s^{\alpha} \mathcal{L}\{u(t)\} - \sum_{i=0}^{m-1} s^{\alpha-i-1} u^{(i)}(0), \quad m-1 < \alpha \leq m,$$

and $u'(0) = 0$, we have

$$\mathcal{L}\{^c D_t^{\alpha} u(t) + \lambda u(t)\} = s^{\alpha} \mathcal{L}\{u(t)\} - s^{\alpha-1} u(0) + \lambda \mathcal{L}\{u(t)\} = 0.$$

Hence

$$\mathcal{L}\{u(t)\} = u(0) \frac{s^{\alpha-1}}{s^{\alpha} + \lambda}.$$

From the inverse Laplace transform of the Mittag-Leffler function [14]

$$E_{\alpha,1}\left(-\lambda t^{\alpha}\right) = \mathcal{L}^{-1}\left\{ \frac{s^{\alpha-1}}{s^{\alpha} + \lambda} \right\},$$

we get

$$u(t) = u(0) E_{\alpha,1}\left(-\lambda t^{\alpha}\right).$$

By $u(1) = 0$, we know

$$E_{\alpha,1}(-\lambda) = 0.$$

Hence λ is the eigenvalue of (1.1) if and only if λ is a solution of $E_{\alpha,1}(-x) = 0$, and for all nonzero constants $C \in \mathbb{R}$, $u(t) = C E_{\alpha,1}(-\lambda t^{\alpha})$ are the eigenfunctions corresponding eigenvalue λ.

Then we consider the inverse problem of (1.1). It follows from the definition of the Caputo fractional derivative that u is an eigenfunction of (1.1) corresponding to the eigenvalue λ, if and only if u is a solution of the integral equation

$$u(t) = \int_0^1 \lambda G(t,s)u(s)\,ds, \tag{2.1}$$

where

$$G(t,s) = \begin{cases} \frac{(1-s)^{\alpha-1}-(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le s \le t \le 1, \\ \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le t \le s \le 1. \end{cases} \tag{2.2}$$

Define the operator T as follows:

$$(Tu)(t) = \int_0^t \frac{(1-s)^{\alpha-1} - (t-s)^{\alpha-1}}{\Gamma(\alpha)} u(s)\,ds + \int_t^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} u(s)\,ds.$$

Therefore, we know that $\lambda \neq 0$ is an eigenvalue of (1.1) if and only if $\frac{1}{\lambda}$ is an eigenvalue of operator T. That is, $\frac{1}{\lambda}$ is an eigenvalue of operator T if and only if λ is a solution of $E_{\alpha,1}(-x) = 0$. And $u(t) = CE_{\alpha,1}(-\lambda t^\alpha)$ ($C \neq 0$) are the eigenfunctions corresponding to the eigenvalue $\frac{1}{\lambda}$.

Let

$$\lambda_1 < \lambda_2 < \cdots < \lambda_{n_\alpha}$$

be the sequence of solutions to the equation $E_{\alpha,1}(-x) = 0$. From [5], we see that λ_j ($j = 1, 2, \ldots, n_\alpha$) are positive and n_α is finite. By computing, we can get

$$n_{1.1} = n_{1.2} = n_{1.3} = n_{1.4} = 1, \qquad n_{1.5} = 3, \qquad n_{1.6} = 5, \qquad n_{1.7} = 9, \qquad \ldots.$$

That is, when $\alpha = 1.1, 1.2, 1.3, 1.4$, the operator T has one eigenvalue, when $\alpha = 1.5$, T has three eigenvalues, when $\alpha = 1.6$, T has five eigenvalues, when $\alpha = 1.7$, T has nine eigenvalues, and so on. Furthermore we will consider the algebraic multiplicity of $\frac{1}{\lambda}$.

Lemma 2.1 *Assume $\frac{1}{\lambda}$ is the eigenvalue of T, that is, $E_{\alpha,1}(-\lambda) = 0$. Furthermore $E_{\alpha,1}^{(1)}(-\lambda) \neq 0$. Then the algebraic multiplicity of eigenvalue $\frac{1}{\lambda}$ for T is equal to 1.*

Proof It is obvious that

$$\ker(I - \lambda T) \subseteq \ker(I - \lambda T)^2.$$

Let $u \in \ker(I - \lambda T)^2$, if $u \notin \ker(I - \lambda T)$, then there exists a nonzero constant C such that

$$\big((I - \lambda T)u\big)(t) = CE_{\alpha,1}\big(-\lambda t^\alpha\big), \quad t \in (0,1),$$

since $CE_{\alpha,1}(-\lambda t^{\alpha})$ $(C \neq 0)$ are the eigenfunctions of operator T corresponding to the eigenvalue $\frac{1}{\lambda}$. By direct computation, we have

$$\begin{cases} D^{\alpha} u(t) + \lambda u(t) = -C\lambda E_{\alpha,1}(-\lambda t^{\alpha}), & t \in (0,1), \\ u'(0) = 0, \qquad u(1) = 0. \end{cases} \tag{2.3}$$

From the Laplace transform of the Caputo fractional derivative [3],

$$\mathcal{L}\{D^{\alpha} u(t)\} = s^{\alpha} \mathcal{L}\{u(t)\} - \sum_{i=0}^{m-1} s^{\alpha-i-1} u^{(i)}(0), \quad m-1 < \alpha \le m,$$

the Laplace transform of the Mittag-Leffler function [14],

$$\mathcal{L}\{t^{\mu-1} E_{\alpha,\mu}(-\lambda t^{\alpha})\} = \frac{s^{\alpha-\mu}}{s^{\alpha} + \lambda},$$

and $u'(0) = 0$, we have

$$s^{\alpha} \mathcal{L}\{u(t)\} - s^{\alpha-1} u(0) + \lambda \mathcal{L}\{u(t)\} = -\frac{C\lambda s^{\alpha-1}}{s^{\alpha} + \lambda}.$$

Hence

$$\mathcal{L}\{u(t)\} = u(0) \frac{s^{\alpha-1}}{s^{\alpha} + \lambda} - \frac{C\lambda s^{\alpha-1}}{(s^{\alpha} + \lambda)^2}.$$

From the inverse Laplace transform of the Mittag-Leffler function [14],

$$t^{m\rho+\beta-1} E_{\rho,\beta}^{(m)}(-\lambda t^{\rho}) = \mathcal{L}^{-1}\left\{ \frac{m! s^{\rho-\beta}}{(s^{\rho} + \lambda)^{m+1}} \right\} \quad \text{with } E_{\rho,\beta}^{(m)}(z) = \frac{d^m}{dz^m} E_{\rho,\beta}(z),$$

we can obtain

$$u(t) = u(0) E_{\alpha,1}(-\lambda t^{\alpha}) - C\lambda t^{\alpha} E_{\alpha,1}^{(1)}(-\lambda t^{\alpha}). \tag{2.4}$$

Let $u(1) = 0$, then we get

$$E_{\alpha,1}^{(1)}(-\lambda) = 0,$$

which is a contradiction. Hence $u \in \ker(I - \lambda T)$, that is,

$$\ker(I - \lambda T)^2 \subseteq \ker(I - \lambda T).$$

Therefore

$$\ker(I - \lambda T)^2 = \ker(I - \lambda T).$$

By

$$\dim \ker\left(\frac{1}{\lambda} I - T\right) = \dim \ker(I - \lambda T) = \dim \ker(I - \lambda T)^2 = \dim \ker\left(\frac{1}{\lambda} I - T\right)^2,$$

we see that the algebraic multiplicity of the eigenvalue $\frac{1}{\lambda}$ is equal to 1. This completes the proof. □

Lemma 2.2 *Assume $\frac{1}{\lambda}$ is the eigenvalue of T, that is, $E_{\alpha,1}(-\lambda) = 0$. Furthermore $E_{\alpha,1}^{(1)}(-\lambda) = E_{\alpha,2}^{(1)}(-\lambda) = 0$, $E_{\alpha,1}^{(2)}(-\lambda) \neq 0$. Then the algebraic multiplicity of the positive eigenvalue $\frac{1}{\lambda}$ for T is equal to 2.*

Proof By Lemma 2.1, we know that, if $E_{\alpha,1}^{(1)}(-\lambda) = 0$, then

$$\ker(I - \lambda T)^2 \nsubseteq \ker(I - \lambda T).$$

It is obvious that

$$\ker(I - \lambda T) \subseteq \ker(I - \lambda T)^2 \subseteq \ker(I - \lambda T)^3.$$

Then we need to show

$$\ker(I - \lambda T)^3 \subseteq \ker(I - \lambda T)^2.$$

Let $u \in \ker(I - \lambda T)^3$, if $u \notin \ker(I - \lambda T)^2$, then there exists a nonzero constant C such that

$$\big((I - \lambda T)^2 u\big)(t) = C E_{\alpha,1}\big(-\lambda t^\alpha\big), \quad t \in (0,1).$$

By direct computation, we have

$$\begin{cases} D^\alpha(D^\alpha u(t) + \lambda u(t)) + \lambda(D^\alpha u(t) + \lambda u(t)) = C\lambda^2 E_{\alpha,1}(-\lambda t^\alpha), & t \in (0,1), \\ u'(0) = 0, \qquad u(1) = 0. \end{cases} \quad (2.5)$$

Let $v = D^\alpha u(t) + \lambda u(t)$, by Lemma 2.1, we see that

$$D^\alpha u(t) + \lambda u(t) = v(t) = v(0)E_{\alpha,1}\big(-\lambda t^\alpha\big) + v'(0)tE_{\alpha,2}\big(-\lambda t^\alpha\big) + C\lambda^2 t^\alpha E_{\alpha,1}^{(1)}\big(-\lambda t^\alpha\big). \quad (2.6)$$

From the Laplace transform of the Caputo fractional derivative, the Laplace transform of the Mittag-Leffler function [3, 14] and $u'(0) = 0$, we have

$$s^\alpha \mathcal{L}\{u(t)\} - s^{\alpha-1}u(0) + \lambda_k \mathcal{L}\{u(t)\} = v(0)\frac{s^{\alpha-1}}{s^\alpha + \lambda} + v'(0)\frac{s^{\alpha-2}}{s^\alpha + \lambda} + \frac{C\lambda^2 s^{\alpha-1}}{(s^\alpha + \lambda)^2}.$$

Hence

$$\mathcal{L}\{u(t)\} = u(0)\frac{s^{\alpha-1}}{s^\alpha + \lambda} + v(0)\frac{s^{\alpha-1}}{(s^\alpha + \lambda)^2} + v'(0)\frac{s^{\alpha-2}}{(s^\alpha + \lambda)^2} + \frac{C\lambda^2 s^{\alpha-1}}{(s^\alpha + \lambda)^3}.$$

From the inverse Laplace transform of the Mittag-Leffler function [14], we can obtain

$$u(t) = u(0)E_{\alpha,1}\big(-\lambda t^\alpha\big) + v(0)t^\alpha E_{\alpha,1}^{(1)}\big(-\lambda t^\alpha\big) + v'(0)t^{\alpha+1}E_{\alpha,2}^{(1)}\big(-\lambda t^\alpha\big)$$
$$+ C\lambda t^{2\alpha}E_{\alpha,1}^{(2)}\big(-\lambda t^\alpha\big). \quad (2.7)$$

Let $u(1) = 0$, then we get

$$E_{\alpha,1}^{(2)}(-\lambda) = 0,$$

which is a contradiction. Hence

$$\ker(I - \lambda T)^3 \subseteq \ker(I - \lambda T)^2.$$

Therefore

$$\ker(I - \lambda T)^3 = \ker(I - \lambda T)^2.$$

That is, the algebraic multiplicity of the eigenvalue $\frac{1}{\lambda}$ is equal to 2. This completes the proof. □

Similarly to Lemma 2.1, Lemma 2.2, we can study the algebraic multiplicity of eigenvalue $\frac{1}{\lambda}$ for operator T by Laplace transforms. Then we will consider the sign-changing solutions of (1.2) by the algebraic multiplicity of the eigenvalue $\frac{1}{\lambda}$ for the operator T.

3 The existence of sign-changing solutions

Consider the Banach space

$$E = \left\{ u \in C^1[0,1] : u'(0) = 0, u(1) = 0 \right\}$$

with the norm $\|u\| = \max\{\|u\|_\infty, \|u'\|_\infty\}$, where $\|u\|_\infty = \max_{0 \le t \le 1} |u(t)|$. Let

$$P = \left\{ u \in E : u(t) \ge 0, \forall t \in [0,1] \right\}$$

be a cone of E. Define operators F and B as follows:

$$(Fu)(t) = f\big(u(t)\big), \quad t \in (0,1), u \in E$$

and

$$B = T \circ F.$$

Then u is a solution of (1.2) if and only if u is a solution of the operator equation

$$u = Bu.$$

By (H0), we can see that B, T are completely continuous.

Lemma 3.1 *Assume that* (H0) *hold, then the operator B is Fréchet differentiable at θ and ∞, and $B'(\theta) = \beta_0 T$, $B'(\infty) = \beta_\infty T$.*

Proof From $\beta_0 = \lim_{x \to 0} \frac{f(x)}{x}$, we have $\forall \varepsilon > 0$, $\exists \delta > 0$, $\forall 0 < |x| < \delta$, and we see that

$$\left| \frac{f(x)}{x} - \beta_0 \right| < \varepsilon,$$

that is, $|f(x) - \beta_0 x| < \varepsilon |x|$. Assume that $\|u\|_\infty \leq \delta$, then by (H0), we have

$$\left| (Bu - B\theta - \beta_0 Tu)(t) \right| = \left| \left(T(Fu - \beta_0)u \right)(t) \right| < \varepsilon \|u\|_\infty \int_0^1 G(t,s)\,ds$$

$$< \varepsilon \|u\| \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\,ds = \frac{\|u\|}{\Gamma(\alpha+1)} \varepsilon.$$

That implies

$$\|Bu - B\theta - \beta_0 Tu\|_\infty < \frac{\|u\|}{\Gamma(\alpha+1)} \varepsilon, \quad \forall u \in E.$$

Similarly, we can show that

$$\left\| (Bu - B\theta - \beta_0 Tu)'(t) \right\|_\infty < \frac{\|u\|}{\Gamma(\alpha)} \varepsilon.$$

Hence

$$\left\| Bu - B\theta - \beta_0 Tu \right\| < \max \left\{ \frac{1}{\Gamma(\alpha+1)}, \frac{1}{\Gamma(\alpha)} \right\} \|u\| \varepsilon = \frac{\|u\|}{\Gamma(\alpha)} \varepsilon.$$

Consequently

$$\lim_{\|u\| \to 0} \frac{\|Bu - B\theta - \beta_0 Tu\|}{\|u\|} = 0. \tag{3.1}$$

This means that B is Fréchet differentiable at θ, and $B'(\theta) = \beta_0 T$.

From $\beta_\infty = \lim_{x \to \infty} \frac{f(x)}{x}$, we have $\forall \varepsilon > 0$; let $N > 0$, when $|x| > N$, we have

$$\left| \frac{f(x)}{x} - \beta_\infty \right| < \varepsilon,$$

that is, $|f(x) - \beta_\infty x| < \varepsilon |x|$. Make $M = \max_{|x| \leq N} |f(x) - \beta_\infty x|$, then we have

$$\left| f(x) - \beta_\infty x \right| < \varepsilon |x| + M.$$

Hence, assume that $\|u\|_\infty > N$, by (H0), we see that

$$\left| (Bu - \beta_\infty Tu)(t) \right| = \left| T(Fu - \beta_\infty u)(t) \right| < \left(\varepsilon \|u\|_\infty + M \right) \int_0^t G(t,s)\,ds$$

$$< \left(\varepsilon \|u\| + M \right) \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\,ds = \frac{1}{\Gamma(\alpha+1)} \left(\varepsilon \|u\| + M \right).$$

That implies that

$$\|Bu - \beta_\infty Tu\|_\infty < \frac{1}{\Gamma(\alpha+1)} \left(\varepsilon \|u\| + M \right), \quad u \in E.$$

Similarly, we can show that

$$\left\|(Bu - \beta_\infty Tu)'\right\|_\infty < \frac{1}{\Gamma(\alpha)}\left(\varepsilon\|u\| + M\right), \quad u \in E.$$

Hence

$$\|Bu - \beta_\infty Tu\| < \frac{1}{\Gamma(\alpha)}\left(\varepsilon\|u\| + M\right).$$

Consequently

$$\lim_{\|u\|\to\infty} \frac{\|Bu - \beta_\infty Tu\|}{\|u\|} = 0. \tag{3.2}$$

Therefore B is Fréchet differentiable at ∞, and $B'(\infty) = \beta_\infty T$. $\qquad\square$

Lemma 3.2 *Assume that* (H0) *hold,* $u \in P \setminus \{\theta\}$ *is a solution of* (1.2), *then* $u \in \mathring{P}$.

Proof If $u \in P \setminus \{\theta\}$ is a solution of (1.2), then

$$u(t) = \int_0^1 G(t,s)f\left(u(s)\right) ds$$

$$= \int_0^t \frac{(1-s)^{\alpha-1} - (t-s)^{\alpha-1}}{\Gamma(\alpha)}f\left(u(s)\right) ds + \int_t^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}f\left(u(s)\right) ds,$$

$$u'(t) = -\frac{1}{\Gamma(\alpha)}\int_0^t (\alpha-1)(t-s)^{\alpha-2}f\left(u(s)\right) ds.$$

It is obvious that

$$u'(0) = 0, \qquad u(1) = 0, \qquad u(0) > 0, \qquad u'(1) < 0, \qquad u'(t) < 0.$$

From $u'(1) < 0$ we learn that there exist $\varepsilon > 0$, $\tau_1 > 0$, such that

$$u'(t) < -\tau_1, \quad \forall t \in [1-\varepsilon, 1]. \tag{3.3}$$

From $u(1) = 0$, $u'(t) < 0$, $\forall t \in (0,1]$ we learn that there exists $\tau_2 > 0$, such that

$$u(t) > \tau_2, \quad \forall t \in [0, 1-\varepsilon]. \tag{3.4}$$

Let $\tau = \min(\tau_1, \tau_2)$, then if $\|x - u\| < \tau$ for any $x \in E$, we can get $x(t) \geq 0$, $t \in [0,1]$ by (3.4), (3.5), that is, $x \in P$. Consequently $B(u, \tau) \subset P$ and $u \in \mathring{P}$. $\qquad\square$

Lemma 3.3 ([15]) *Let P be a solid cone of real Banach space E, Ω be a relatively bounded open set of P, $A : P \mapsto P$ be a completely continuous operator. If all fixed points of A are an interior point of P, there exists an open subset O of E such that $O \subset \Omega$ and $\deg(I - A, O, \theta) = i(A, \Omega, P)$.*

Theorem 3.4 *Suppose that* (H0)-(H2) *hold,* $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_{n_\alpha}$ *are the eigenvalues of* (1.1). *Furthermore $E_{\alpha,1}^{(1)}(-\lambda_j) \neq 0$, where $j = 1, 2, \ldots, \max\{2n_0, 2n_1\}$. Then the boundary value*

problem (1.2) *has at least two sign-changing solutions, two positive solutions and two negative solutions.*

Proof It follows from the definition of B that u is a solution of (1.2) if and only if u is the fixed point of the operator B. Then by (H2), we have, for any $u \in E$ with $\|u\| = C_0$,

$$
\begin{aligned}
\left|(Bu)(t)\right| &= \left| -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f\big(u(s)\big)\,ds + \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} f\big(u(s)\big)\,ds \right| \\
&< \Gamma(\alpha) C_0 \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\,ds \\
&< C_0,
\end{aligned}
$$

$$
\begin{aligned}
\left|(Bu)'(t)\right| &= \left| -\frac{1}{\Gamma(\alpha)} \int_0^t (\alpha-1)(t-s)^{\alpha-2} f\big(u(s)\big)\,ds \right| \\
&< \Gamma(\alpha) C_0 \frac{\alpha-1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-2}\,ds \\
&= C_0.
\end{aligned}
$$

Therefore $\|Bu\| < C_0$. By (H0) and $G(t,s) \geq 0$, we have $B(P) \subset P$. Hence

$$
i\big(B, P \cap B(\theta, C_0), P\big) = 1. \tag{3.5}
$$

By Lemma 3.3, we see that

$$
\deg\big(I - B, B(\theta, C_0), \theta\big) = 1. \tag{3.6}
$$

By Lemma 3.1, we can learn that $B'(\theta) = \beta_0 T$. Hence $\frac{\beta_0}{\lambda_j}$ $(j = 1, 2, \ldots, n_\alpha)$ are the eigenvalues of the operator $B'(\theta)$. By (H1), we know that $\frac{\beta_0}{\lambda_1} > 1$. Hence 1 is not an eigenvalue of $B'(\theta)$. Furthermore, $u(t) = CE_{\alpha,1}(-\lambda_1 t^\alpha)$ $(C \neq 0)$ is an eigenfunction of $B'(\theta)$ corresponding to the eigenvalue λ_1. And $E_{\alpha,1}(-\lambda_1 t^\alpha) \neq 0$ for any $t \in (0,1)$, since λ_1 is the smallest positive solution of $E_{\alpha,1}(-x) = 0$. Hence we can choose the suitable C to ensure $u(t) \geq 0$. Therefore, by Theorem 21.2, in [16], we know that there exist a small enough r_1 and a large enough R_1 such that

$$
\deg\big(I - B, U(\theta, r_1), \theta\big) = (-1)^{k_0}, \tag{3.7}
$$

$$
\deg\big(I - B, U(\theta, R_1), \theta\big) = (-1)^{k_1}, \tag{3.8}
$$

where k_0 is the sum of the algebraic multiplicities of the real eigenvalues of $B'(\theta)$ which are larger than 1, k_1 is the sum of the algebraic multiplicities of the real eigenvalues of $B'(\infty)$ which are larger than 1.

By Lemma 2.3.7 in [17], we see that there exist two constants r_0, R_0 $(0 < r_0 < C_0 < R_0)$ such that for any r_1, R_1 $(0 < r_1 < r_0 < C_0 < R_0 < R_1)$,

$$
i\big(B, P \cap U(\theta, r_1), P\big) = 0, \tag{3.9}
$$

$$
i\big(B, P \cap U(\theta, R_1), P\big) = 0. \tag{3.10}
$$

Hence, by (3.5), (3.9), (3.10), we see that

$$i\big(B, P \cap \big(U(\theta, R_1) \setminus \overline{U(\theta, C_0)}\big), P\big) = 0 - 1 = -1, \tag{3.11}$$

$$i\big(B, P \cap \big(U(\theta, C_0) \setminus \overline{U(\theta, r_1)}\big), P\big) = 1 - 0 = 1. \tag{3.12}$$

Therefore the operator B has at least two fixed points

$$u_1 \in P \cap \big(U(\theta, R_1) \setminus \overline{U(\theta, C_0)}\big), \qquad u_2 \in P \cap \big(U(\theta, C_0) \setminus \overline{U(\theta, r_1)}\big).$$

That is, u_1 and u_2 are positive solutions of the boundary value problem (1.2) and $r_1 < \|u_1\| \le C_0$ and $C_0 < \|u_2\| \le R_1$.

By (H0), we have $uf(u) > 0$ for all $u \in \mathbb{R} \setminus \{\theta\}$. Similarly, we see that boundary value problem (1.2) has two negative solutions $u_3, u_4 \in -P$ with

$$-u_3 \in P \cap \big(U(\theta, R_1) \setminus \overline{U(\theta, C_0)}\big), \qquad -u_4 \in P \cap \big(U(\theta, C_0) \setminus \overline{U(\theta, r_1)}\big)$$

and $r_1 < \|u_3\| \le C_0$, $C_0 < \|u_4\| \le R_1$.

By (3.11), (3.12), and Lemma 3.3. there exist two open subsets O_1, O_2 of E

$$O_1 \subset P \cap \big(U(\theta, C_0) \setminus \overline{U(\theta, r_1)}\big), \qquad O_2 \subset P \cap \big(U(\theta, R_1) \setminus \overline{U(\theta, C_0)}\big),$$

such that

$$\deg(I - B, O_1, \theta) = 1, \tag{3.13}$$

$$\deg(I - B, O_2, \theta) = -1. \tag{3.14}$$

Similarly, there exist two open subsets O_3, O_4 of E

$$O_3 \subset -P \cap \big(U(\theta, C_0) \setminus \overline{U(\theta, r_1)}\big), \qquad O_4 \subset -P \cap \big(U(\theta, R_1) \setminus \overline{U(\theta, C_0)}\big),$$

such that

$$\deg(I - B, O_3, \theta) = 1, \tag{3.15}$$

$$\deg(I - B, O_4, \theta) = -1, \tag{3.16}$$

by (H0).

By (H1) and Lemma 3.1, the number of eigenvalues of the operator $B'(\theta) = \beta_0 K$ which are larger than 1 is $2n_0$. From

$$E_{\alpha,1}^{(1)}(-\lambda_j) \ne 0,$$

$j = 1, 2, \ldots, \max\{2n_0, 2n_1\}$ and Lemma 2.1 we see that the algebraic multiplicity of positive eigenvalue $\frac{\beta_0}{\lambda_n}$ have algebraic multiplicity one. Hence $k_0 = 2n_0$, and

$$\deg\big(I - B, U(\theta, r_1), \theta\big) = (-1)^{k_0} = (-1)^{2n_0} = 1. \tag{3.17}$$

Similarly we can see that

$$\deg\big(I - B, U(\theta, R_1), \theta\big) = (-1)^{k_1} = (-1)^{2n_1} = 1. \tag{3.18}$$

From (3.6), (3.13), (3.15), and (3.17) we see that

$$\deg\big(I - B, U(\theta, C_0) \setminus \big(\overline{O_1} \cup \overline{O_3} \cup \overline{U(\theta, r_1)}\big), \theta\big) = 1 - 1 - 1 - 1 = -2. \tag{3.19}$$

By (3.19), we know B has at least one fixed point $u_5 \in U(\theta, C_0) \setminus (\overline{O_1} \cup \overline{O_3} \cup \overline{U(\theta, r_1)})$.

That is, boundary value problem (1.2) has a sign-changing solution u_5. Similarly, we get another different solution $u_6 \in U(\theta, C_0) \setminus (\overline{O_2} \cup \overline{O_4} \cup \overline{U(\theta, C_0)})$ by (3.6), (3.14), (3.16), and (3.18). This completes the proof. □

According to the method used in the proof of Theorem 3.4, we can give the following corollaries.

Corollary 3.5 *Let* (H0)-(H2) *hold,* $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_{n_\alpha}$ *are the eigenvalues of* (1.1), *there exists a positive integer* n_0 *such that* $\lambda_{2n_0} < \beta_0 < \lambda_{2n_0+1}$ *or* $\lambda_{2n_0} < \beta_\infty < \lambda_{2n_0+1}$ *and* $E_{\alpha,1}^{(1)}(-\lambda_j) \neq 0$, *where* $j = 1, 2, \ldots, 2n_0$. *Then the boundary value problem* (1.2) *have at least one sign-changing solution, one positive solution and one negative solution.*

Corollary 3.6 *Let* (H0)-(H2) *hold,* λ_1 *is the eigenvalue of* (1.1), $E_{\alpha,1}^{(1)}(-\lambda_1) \neq 0$.
  (1) *If* $\beta_\infty < \lambda_1 < \beta_0$ *or* $\beta_0 < \lambda_1 < \beta_\infty$, *then the boundary value problem* (1.2) *has at least one positive solution and one negative solution.*
  (2) *If* $\beta_0 > \lambda_1$, $\beta_\infty > \lambda_1$, *then the boundary value problem* (1.2) *has at least two positive solutions and two negative solutions.*

4 Example
Consider the following fractional differential equation:

$$\begin{cases} {}^c D_x^{1.5} u(t) + f(u(t)) = 0, & t \in (0,1), \\ u'(0) = 0, & u(1) = 0, \end{cases} \tag{4.1}$$

where

$$f(u) = \begin{cases} -\frac{1}{2}, & u \leq -\frac{1}{28}, \\ 14u, & -\frac{1}{28} < u < \frac{1}{28}, \\ \frac{1}{2}, & \frac{1}{28} \leq u. \end{cases} \tag{4.2}$$

We can find that
  (1) $\beta_0 = \lim_{u \to 0} \frac{f(u)}{u} = 14$;
  (2) $f(u) : \mathbb{R} \mapsto \mathbb{R}, f(\theta) = \theta$, $uf(u) > 0$ for all $t \in (0,1)$, $u \in \mathbb{R} \setminus \{\theta\}$;
  (3) from [4, 5], we see that $E_{1.5,1}(-x) = 0$ has three zero points, $x_1 = 2.11$, $x_2 = 13.765$, $x_3 = 24.243$, and $E_{\alpha,1}^{(1)}(-x_1) \neq 0$, $E_{\alpha,1}^{(1)}(-x_2) \neq 0$;
  (4) $\lambda_2 < \beta_0 < \lambda_3$;

Table 1 Sign-changing solutions u_1, u_4, positive solution u_2, negative solution u_3

t	u_1	u_2	u_3	u_4
0.0	−0.1025	0.3772	−0.3772	0.1025
0.1	−0.0910	0.3657	−0.3657	0.0910
0.2	−0.0692	0.3439	−0.3439	0.0692
0.3	−0.0411	0.3158	−0.3158	0.0411
0.4	−0.0077	0.2824	−0.2824	0.0077
0.5	0.0293	0.2446	−0.2446	−0.0293
0.6	0.0449	0.2028	−0.2028	−0.0449
0.7	0.0455	0.1573	−0.1573	−0.0455
0.8	0.0370	0.1084	−0.1084	−0.0370
0.9	0.0214	0.0564	−0.0564	−0.0214
1.0	0.0000	0.0000	0.0000	0.0000
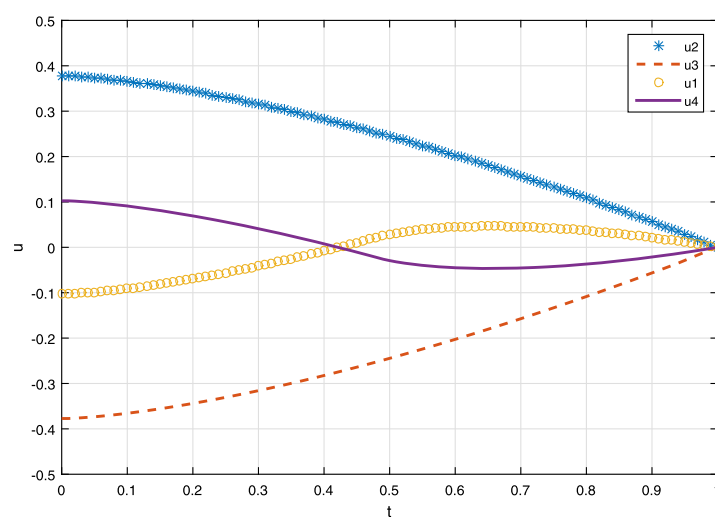
Figure 1 Sign-changing solutions u_1, u_4, positive solution u_2, negative solution u_3.

(5) let $C_0 = 1$, when $-\frac{1}{28} < u < \frac{1}{28}$, we have

$$\left| f(u) \right| = |14u| \leq \frac{1}{2} < \Gamma(1.5)$$

when $u \leq -\frac{1}{28}$ or $\frac{1}{28} \leq u$, we have

$$\left| f(u) \right| = \frac{1}{2} < \Gamma(1.5).$$

By Corollary 3.5, we see that problem (4.1) has at least one sign-changing solution u_1, one positive solution u_2, one negative solution u_3. By $f(-u) = -f(u)$, we see that $u_4 = -u_1$ is another sign-changing solution of (4.1). The numerical results of u_1, u_2, u_3 and u_4 are shown in Table 1, the graphs of u_1, u_2, u_3 and u_4 are shown in Figure 1.

5 Conclusion

In the paper, the existence of sign-changing solutions for a fractional boundary value problem is considered by the eigenvalues. Some new results as regards the eigenvalues and

their algebraic multiplicities are established. Finally, an example is presented to illustrate the application.

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