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Asymptotic behavior of third-order differential equations with nonpositive neutral coefficients and distributed deviating arguments

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Abstract

Using the Riccati transformation technique, we present several sufficient conditions that guarantee that all solutions to a third-order differential equation with nonpositive neutral coefficients and distributed deviating arguments are either oscillatory or converge to zero asymptotically. In particular, we establish Hille and Nehari type criteria. Two examples are given to demonstrate the practicability of the main results.

MSC: 34K11

Keywords: asymptotic behavior; third-order neutral differential equation; nonpositive neutral coefficient; distributed deviating argument; oscillation

1 Introduction

Third-order differential equations have attracted noticeable interests due to their potential applications in assorted fields, including physical sciences, technology, population dynamics, and so on. Recently, the qualitative theory of third-order differential equations has become an interesting topic, and there have been some results on the oscillatory and asymptotic behavior of third-order equations; see, for example, the monographs [1, 2], the papers [3–23], and the references therein. In particular, it is a necessary and valuable issue, either theoretically or practically, to investigate differential equations with distributed deviating arguments; see the papers by Tian *et al.* [21], Wang [24], and Wang and Cai [25]. On the basis of these background details, the objective of this paper is to analyze the oscillation and asymptotic properties of a class of third-order neutral differential equations

$$[r(t)(z''(t))^\alpha]' + \int_c^d q(t, \xi) f[x(\sigma(t, \xi))] d\xi = 0, \quad (1.1)$$

where $t \geq t_0 > 0$, $z(t) := x(t) - \int_a^b p(t, \mu)x[\tau(t, \mu)] d\mu$, $\alpha > 0$ is a quotient of odd positive integers, $r(t) \in C^1([t_0, \infty), (0, \infty))$, $\int_{t_0}^\infty r^{-1/\alpha}(t) dt = \infty$, $p(t, \mu) \in C([t_0, \infty) \times [a, b], \mathbb{R})$, $0 \leq \int_a^b p(t, \mu) d\mu \leq p_0 < 1$, $\tau(t, \mu) \in C([t_0, \infty) \times [a, b], \mathbb{R})$, $\tau(t, \mu) \leq t$, $\liminf_{t \rightarrow \infty} \tau(t, \mu) = \infty$ for $\mu \in [a, b]$, $q(t, \xi) \in C([t_0, \infty) \times [c, d], [0, \infty))$, $q(t, \xi)$ is not identically zero for large t , $\sigma(t, \xi) \in C([t_0, \infty) \times [c, d], \mathbb{R})$ is a nondecreasing function for ξ satisfying $\sigma(t, \xi) \leq t$ and

$\liminf_{t \rightarrow \infty} \sigma(t, \xi) = \infty$ for $\xi \in [c, d], f(x) \in C(\mathbb{R}, \mathbb{R})$, and there exists a positive constant k such that $f(x)/x^\alpha \geq k$ for all $x \neq 0$.

We assume that solutions of (1.1) exist for any $t \in [t_0, \infty)$. Our attention is restricted to those solutions of (1.1) that are not identically zero for large t . As usual, a solution of (1.1) is called oscillatory if it has arbitrarily large zeros on the interval $[t_0, \infty)$. Otherwise, it is termed nonoscillatory (*i.e.*, it is either eventually positive or eventually negative).

It is known that analysis of neutral differential equations is more difficult in comparison with that of ordinary differential equations, although certain similarities in the behavior of solutions of these two classes of equations are observed; see, for example, [1, 2, 6, 8–10, 12, 13, 16–18, 20–24, 26–30] and the references therein. Assuming that

$$r'(t) \geq 0 \tag{1.2}$$

and

$$0 \leq - \int_a^b p(t, \mu) d\mu \leq p_0 < 1,$$

asymptotic criteria for (1.1) have been reported in [20, 21, 23]. So far, there are few results dealing with the asymptotic properties of third-order differential equations with nonpositive coefficients; we refer the reader to [6, 18, 22]. In particular, Baculiková and Džurina [6] and Zhang *et al.* [22] established several Hille and Nehari type (see Agarwal *et al.* [4]) criteria for the equation

$$(r(t)[(x(t) - p(t)x(\tau(t)))''']')' + q(t)x(\sigma(t)) = 0$$

under the assumptions that $0 \leq p(t) \leq p_0 < 1$ and (1.2) holds.

It should be noted that condition (1.2) is a restrictive condition in the study of asymptotic behavior of third-order differential equations. To solve this problem without requiring (1.2), Li *et al.* [17] obtained some oscillation criteria for a third-order neutral delay differential equation

$$(r(t)(x(t) + p(t)x(\tau(t)))''')' + q(t)x(\sigma(t)) = 0$$

by employing the Riccati substitution

$$w(t) := \rho(t) \frac{r(t)z'(t)}{z'(t)},$$

where $0 \leq p(t) \leq p_0 < 1, z(t) := x(t) + p(t)x(\tau(t))$, and $\rho(t) \in C^1([t_0, \infty), (0, \infty))$. A natural question now is: *is it possible to establish asymptotic tests for (1.1) without requiring restrictive condition (1.2)?* Motivated by Baculiková and Džurina [6], Li *et al.* [17], and Zhang *et al.* [22], the principal goal of this paper is to give an affirmative answer to this question. In Section 2, some lemmas are provided to prove the main results. In Section 3, some oscillation results for (1.1) are obtained by using the Riccati transformation technique, and these results also can be applied to the cases where $r'(t) \leq 0$ or $r'(t)$ is oscillatory. In Section 4, two illustrative examples are included. All functional inequalities considered in the sequel are tacitly assumed to hold for all t large enough.

2 Several lemmas

Lemma 2.1 *Assume that $x(t)$ is an eventually positive solution of (1.1). Then there exists a $t_1 \geq t_0$ such that, for $t \geq t_1$, $z(t)$ has the following four possible cases:*

- (i) $z(t) > 0, z'(t) > 0, z''(t) > 0, (r(t)(z''(t))^\alpha)' \leq 0;$
- (ii) $z(t) > 0, z'(t) < 0, z''(t) > 0, (r(t)(z''(t))^\alpha)' \leq 0;$
- (iii) $z(t) < 0, z'(t) < 0, z''(t) > 0, (r(t)(z''(t))^\alpha)' \leq 0;$
- (iv) $z(t) < 0, z'(t) < 0, z''(t) < 0, (r(t)(z''(t))^\alpha)' \leq 0.$

Proof Let $x(t)$ be an eventually positive solution of (1.1). Then there exists a $t_1 \geq t_0$ such that, for $t \geq t_1$,

$$x(t) > 0, \quad x[\tau(t, \mu)] > 0, \quad \mu \in [a, b], \quad \text{and} \quad x[\sigma(t, \xi)] > 0, \quad \xi \in [c, d].$$

It follows from (1.1) and the definition of $z(t)$ that $x(t) \geq z(t)$ and

$$[r(t)(z''(t))^\alpha]' = - \int_c^d q(t, \xi) f[x(\sigma(t, \xi))] d\xi \leq 0.$$

Hence, $r(t)(z''(t))^\alpha$ is nonincreasing and of one sign, which implies that $z''(t)$ is also of one sign. Therefore, there exists a $t_2 \geq t_1$ such that, for $t \geq t_2$, $z''(t) < 0$ or $z''(t) > 0$.

Case 1. The condition $z''(t) < 0$ yields that there exists a constant $M > 0$ such that

$$r(t)(z''(t))^\alpha \leq -M < 0,$$

that is,

$$z''(t) \leq \frac{-M^{1/\alpha}}{r^{1/\alpha}(t)}.$$

Integrating this inequality from t_2 to t , we conclude that

$$z'(t) \leq z'(t_2) - M^{1/\alpha} \int_{t_2}^t r^{-1/\alpha}(s) ds.$$

Letting $t \rightarrow \infty$, we have that $z'(t) \rightarrow -\infty$, and so $z'(t) < 0$ eventually. Note that the conditions $z''(t) < 0$ and $z'(t) < 0$ imply that $z(t) < 0$. Thus, we get case (iv).

Case 2. Assume that $z''(t) > 0$. Then $z'(t)$ is of one sign. If $z'(t) > 0$, then $z(t) > 0$. If $z'(t) < 0$, then $z(t) > 0$ or $z(t) < 0$. Hence, we have three possible cases (i), (ii), and (iii) when $z''(t) > 0$. The proof is complete. □

Lemma 2.2 *Assume that $x(t)$ is an eventually positive solution of (1.1) and the corresponding $z(t)$ satisfies case (i) in Lemma 2.1. Then there exist two numbers $t_1 \geq t_0$ and $t_2 > t_1$ such that, for $t \geq t_2$,*

$$z(t) \geq \frac{\int_{t_2}^t \int_{t_1}^s r^{-1/\alpha}(u) du ds}{\int_{t_1}^t r^{-1/\alpha}(u) du} z'(t)$$

and $z'(t) / \int_{t_1}^t r^{-1/\alpha}(s) ds$ is nonincreasing eventually.

Proof Let $z(t)$ satisfy case (i) in Lemma 2.1. Then

$$\begin{aligned} z'(t) &= z'(t_1) + \int_{t_1}^t z''(s) ds = z'(t_1) + \int_{t_1}^t \frac{(r(s)(z''(s))^\alpha)^{1/\alpha}}{r^{1/\alpha}(s)} ds \\ &\geq z''(t)r^{1/\alpha}(t) \int_{t_1}^t r^{-1/\alpha}(s) ds. \end{aligned}$$

Hence, we deduce that

$$\left(\frac{z'(t)}{\int_{t_1}^t r^{-1/\alpha}(s) ds} \right)' \leq 0,$$

which implies that $z'(t)/\int_{t_1}^t r^{-1/\alpha}(s) ds$ is nonincreasing eventually, and so

$$\begin{aligned} z(t) &= z(t_2) + \int_{t_2}^t z'(s) ds = z(t_2) + \int_{t_2}^t \frac{z'(s)}{\int_{t_1}^s r^{-1/\alpha}(u) du} \int_{t_1}^s r^{-1/\alpha}(u) du ds \\ &\geq \frac{\int_{t_2}^t \int_{t_1}^s r^{-1/\alpha}(u) du ds}{\int_{t_1}^t r^{-1/\alpha}(u) du} z'(t). \end{aligned}$$

This completes the proof. □

Lemma 2.3 *Let $x(t)$ be an eventually positive solution of (1.1) and assume that the corresponding $z(t)$ satisfies case (ii) in Lemma 2.1. If*

$$\int_{t_0}^\infty \int_v^\infty \left[\frac{1}{r(u)} \int_u^\infty \int_c^d q(s, \xi) d\xi ds \right]^{1/\alpha} du dv = \infty, \tag{2.1}$$

then $\lim_{t \rightarrow \infty} x(t) = 0$.

Proof It follows from property (ii) that there exists a finite constant $l \geq 0$ such that $\lim_{t \rightarrow \infty} z(t) = l$. We claim that $l = 0$. Otherwise, assume that $l > 0$. By the definition of $z(t)$, $x(t) \geq z(t) > l$. An application of (1.1) yields

$$\begin{aligned} [r(t)(z''(t))^\alpha]' &\leq -k \int_c^d q(t, \xi) x^\alpha[\sigma(t, \xi)] d\xi \leq -k \int_c^d q(t, \xi) z^\alpha[\sigma(t, \xi)] d\xi \\ &\leq -kl^\alpha \int_c^d q(t, \xi) d\xi. \end{aligned}$$

Integrating the latter inequality from t to ∞ , we have

$$r(t)(z''(t))^\alpha \geq kl^\alpha \int_t^\infty \int_c^d q(s, \xi) d\xi ds,$$

which implies that

$$z''(t) \geq lk^{1/\alpha} \left(\frac{1}{r(t)} \int_t^\infty \int_c^d q(s, \xi) d\xi ds \right)^{1/\alpha}.$$

Integrating this inequality from t to ∞ and then integrating the resulting inequality from t_1 to ∞ , we conclude that

$$z(t_1) \geq l k^{1/\alpha} \int_{t_1}^{\infty} \int_v^{\infty} \left[\frac{1}{r(u)} \int_u^{\infty} \int_c^d q(s, \xi) d\xi ds \right]^{1/\alpha} du dv,$$

which is a contradiction to (2.1). Hence, $l = 0$ and $\lim_{t \rightarrow \infty} z(t) = 0$.

Next, we prove that $x(t)$ is bounded. If not, then there exists a sequence $\{t_m\}$ such that $\lim_{m \rightarrow \infty} t_m = \infty$ and $\lim_{m \rightarrow \infty} x(t_m) = \infty$, where $x(t_m) := \max\{x(s); t_0 \leq s \leq t_m\}$. Since $\liminf_{t \rightarrow \infty} \tau(t, \mu) = \infty$, $\tau(t_m, \mu) > t_0$ for all sufficiently large m . By $\tau(t, \mu) \leq t$, we conclude that

$$x(\tau(t_m, \mu)) = \max\{x(s); t_0 \leq s \leq \tau(t_m, \mu)\} \leq \max\{x(s); t_0 \leq s \leq t_m\} = x(t_m),$$

and so

$$\begin{aligned} z(t_m) &= x(t_m) - \int_a^b p(t_m, \mu) x[\tau(t_m, \mu)] d\mu \geq x(t_m) - \int_a^b p(t_m, \mu) x(t_m) d\mu \\ &\geq (1 - p_0)x(t_m), \end{aligned}$$

which yields $\lim_{m \rightarrow \infty} z(t_m) = \infty$. This contradicts $\lim_{t \rightarrow \infty} z(t) = 0$. Therefore, $x(t)$ is bounded, and hence we may suppose that $\limsup_{t \rightarrow \infty} x(t) = a_0$, where $0 \leq a_0 < \infty$. Then, there exists a sequence $\{t_k\}$ such that $\lim_{k \rightarrow \infty} t_k = \infty$ and $\lim_{k \rightarrow \infty} x(t_k) = a_0$. Assuming now that $a_0 > 0$ and letting $\varepsilon := a_0(1 - p_0)/(2p_0)$, we have $x(\tau(t_k, \mu)) < a_0 + \varepsilon$ eventually, and thus

$$0 = \lim_{k \rightarrow \infty} z(t_k) \geq \lim_{k \rightarrow \infty} (x(t_k) - p_0(a_0 + \varepsilon)) = \frac{a_0(1 - p_0)}{2} > 0,$$

which is a contradiction. Thus, $a_0 = 0$ and $\lim_{t \rightarrow \infty} x(t) = 0$. The proof is complete. □

3 Main results

In what follows, we let

$$\rho'_+(t) := \max\{0, \rho'(t)\}, \quad q_*(t) := \int_c^d q(t, \xi) d\xi, \quad \text{and} \quad \sigma_*(t) := \sigma(t, c),$$

where the meaning of $\rho(t)$ will be explained later.

Theorem 3.1 *Assume that condition (2.1) is satisfied. If there exists a function $\rho(t) \in C^1([t_0, \infty), (0, \infty))$ such that, for all sufficiently large $t_1 \geq t_0$ and for some $t_3 > t_2 > t_1$,*

$$\limsup_{t \rightarrow \infty} \int_{t_3}^t \left(k\rho(s)q_*(s)G(s) - \frac{1}{(\alpha + 1)^{\alpha+1}} \frac{r(s)(\rho'_+(s))^{\alpha+1}}{\rho^\alpha(s)} \right) ds = \infty, \tag{3.1}$$

where

$$G(t) := \left(\frac{\int_{t_2}^{\sigma_*(t)} \int_{t_1}^v r^{-1/\alpha}(u) du dv}{\int_{t_1}^t r^{-1/\alpha}(u) du} \right)^\alpha, \tag{3.2}$$

then every solution $x(t)$ of (1.1) is either oscillatory or converges to zero as $t \rightarrow \infty$.

Proof Suppose to the contrary that (1.1) has a nonoscillatory solution $x(t)$. Without loss of generality, we may assume that $x(t)$ is eventually positive (since the proof of the case where $x(t)$ is eventually negative is similar). By Lemma 2.1, we observe that, for $t \geq t_1 \geq t_0$, $z(t)$ satisfies four possible cases (i), (ii), (iii), or (iv) (as those of Lemma 2.1). We consider each of the four cases separately.

Assume first that case (i) is satisfied. For $t \geq t_1$, define the Riccati transformation $\omega(t)$ by

$$\omega(t) := \rho(t) \frac{r(t)(z''(t))^\alpha}{(z'(t))^\alpha}. \tag{3.3}$$

Then $\omega(t) > 0$ for $t \geq t_1$. Differentiation of (3.3) yields

$$\begin{aligned} \omega'(t) &= \rho'(t) \frac{r(t)(z''(t))^\alpha}{(z'(t))^\alpha} + \rho(t) \left(\frac{r(t)(z''(t))^\alpha}{(z'(t))^\alpha} \right)' \\ &= \frac{\rho'(t)}{\rho(t)} \omega(t) + \rho(t) \frac{(r(t)(z''(t))^\alpha)'}{(z'(t))^\alpha} - \alpha \rho(t) r(t) \left(\frac{z''(t)}{z'(t)} \right)^{\alpha+1}. \end{aligned} \tag{3.4}$$

It follows from (1.1) and (i) that

$$(r(t)(z''(t))^\alpha)' \leq -kz^\alpha(\sigma(t, c)) \int_c^d q(t, \xi) d\xi = -kq_*(t)z^\alpha(\sigma_*(t)). \tag{3.5}$$

Using (3.3) and (3.5) in (3.4), we deduce that

$$\begin{aligned} \omega'(t) &\leq \frac{\rho'(t)}{\rho(t)} \omega(t) - k\rho(t)q_*(t) \left(\frac{z(\sigma_*(t))}{z'(t)} \right)^\alpha - \alpha \rho(t)r(t) \left(\frac{\omega(t)}{r(t)\rho(t)} \right)^{1+1/\alpha} \\ &= \frac{\rho'(t)}{\rho(t)} \omega(t) - k\rho(t)q_*(t) \left(\frac{z(\sigma_*(t))}{z'(t)} \right)^\alpha - \frac{\alpha \omega^{1+1/\alpha}(t)}{(r(t)\rho(t))^{1/\alpha}}. \end{aligned} \tag{3.6}$$

Since $\sigma_*(t) \leq t$ and $z'(t)/\int_{t_1}^t r^{-1/\alpha}(s) ds$ is nonincreasing (see Lemma 2.2), we have

$$\frac{z'(\sigma_*(t))}{\int_{t_1}^{\sigma_*(t)} r^{-1/\alpha}(s) ds} \geq \frac{z'(t)}{\int_{t_1}^t r^{-1/\alpha}(s) ds},$$

that is,

$$\frac{z'(\sigma_*(t))}{z'(t)} \geq \frac{\int_{t_1}^{\sigma_*(t)} r^{-1/\alpha}(s) ds}{\int_{t_1}^t r^{-1/\alpha}(s) ds}. \tag{3.7}$$

It follows now from Lemma 2.2 and (3.7) that

$$\left(\frac{z(\sigma_*(t))}{z'(t)} \right)^\alpha = \left(\frac{z(\sigma_*(t))}{z'(\sigma_*(t))} \frac{z'(\sigma_*(t))}{z'(t)} \right)^\alpha \geq G(t), \tag{3.8}$$

where $G(t)$ is defined by (3.2). Substituting (3.8) into (3.6), we get

$$\omega'(t) \leq -k\rho(t)q_*(t)G(t) + \frac{\rho'(t)}{\rho(t)} \omega(t) - \frac{\alpha \omega^{1+1/\alpha}(t)}{(r(t)\rho(t))^{1/\alpha}}. \tag{3.9}$$

Set

$$v := \omega(t), \quad A := \frac{\alpha}{(r(t)\rho(t))^{1/\alpha}}, \quad \text{and} \quad B := \frac{\rho'_+(t)}{\rho(t)}.$$

Using the inequality (see [15])

$$Bv - Av^{1+1/\alpha} \leq \frac{\alpha^\alpha}{(\alpha + 1)^{\alpha+1}} \frac{B^{\alpha+1}}{A^\alpha}, \quad A > 0, \tag{3.10}$$

we have

$$\frac{\rho'_+(t)}{\rho(t)} \omega(t) - \frac{\alpha \omega^{1+1/\alpha}(t)}{(r(t)\rho(t))^{1/\alpha}} \leq \frac{1}{(\alpha + 1)^{\alpha+1}} \frac{r(t)(\rho'_+(t))^{\alpha+1}}{\rho^\alpha(t)}.$$

Substituting the latter inequality into (3.9), we conclude that

$$\omega'(t) \leq -k\rho(t)q_*(t)G(t) + \frac{1}{(\alpha + 1)^{\alpha+1}} \frac{r(t)(\rho'_+(t))^{\alpha+1}}{\rho^\alpha(t)}.$$

Integrating this inequality from t_3 ($t_3 > t_2$) to t , we arrive at

$$\int_{t_3}^t \left(k\rho(s)q_*(s)G(s) - \frac{1}{(\alpha + 1)^{\alpha+1}} \frac{r(s)(\rho'_+(s))^{\alpha+1}}{\rho^\alpha(s)} \right) ds \leq \omega(t_3),$$

which contradicts (3.1).

Suppose that case (ii) is satisfied. By Lemma 2.3, $\lim_{t \rightarrow \infty} x(t) = 0$.

If case (iii) or case (iv) holds, then $\lim_{t \rightarrow \infty} z(t) = c_0 < 0$ (possibly $c_0 = -\infty$) or $\lim_{t \rightarrow \infty} z(t) = -\infty$, respectively. Proceeding similarly as in the proof of Lemma 2.3, we conclude that $x(t)$ and $z(t)$ are bounded. Hence, c_0 is finite, and case (iv) does not occur. Similar analysis to that in Lemma 2.3 leads to the conclusion that $\lim_{t \rightarrow \infty} x(t) = 0$. This completes the proof. \square

Letting $\rho(t) = t$ and $\rho(t) = 1$, we can derive the following results from Theorem 3.1.

Corollary 3.1 *Let condition (2.1) hold. If for all sufficiently large $t_1 \geq t_0$ and for some $t_3 > t_2 > t_1$,*

$$\limsup_{t \rightarrow \infty} \int_{t_3}^t \left(ksq_*(s)G(s) - \frac{1}{(\alpha + 1)^{\alpha+1}} \frac{r(s)}{s^\alpha} \right) ds = \infty,$$

where $G(t)$ is as in (3.2), then the conclusion of Theorem 3.1 remains intact.

Corollary 3.2 *Let condition (2.1) be satisfied. If for all sufficiently large $t_1 \geq t_0$ and for some $t_3 > t_2 > t_1$,*

$$\int_{t_3}^\infty q_*(s)G(s) ds = \infty,$$

where $G(t)$ is defined by (3.2), then the conclusion of Theorem 3.1 remains intact.

In what follows, we establish Hille and Nehari type criteria for (1.1). To this end, we introduce the following lemma.

Lemma 3.1 *Let $x(t)$ be an eventually positive solution of (1.1). Define*

$$\begin{aligned} \omega(t) &:= \frac{r(t)(z'(t))^\alpha}{(z'(t))^\alpha}, & (3.11) \\ \bar{p} &:= \liminf_{t \rightarrow \infty} k \left(\int_{t_1}^t r^{-1/\alpha}(s) ds \right)^\alpha \int_t^\infty q_*(s)G(s) ds, \\ \bar{q} &:= \liminf_{t \rightarrow \infty} \frac{k \int_{t_3}^t \left(\int_{t_1}^s r^{-1/\alpha}(u) du \right)^{\alpha+1} q_*(s)G(s) ds}{\int_{t_1}^t r^{-1/\alpha}(u) du}, \\ \bar{r} &:= \liminf_{t \rightarrow \infty} \left(\int_{t_1}^t r^{-1/\alpha}(s) ds \right)^\alpha \omega(t), \quad \text{and} \quad \bar{R} := \limsup_{t \rightarrow \infty} \left(\int_{t_1}^t r^{-1/\alpha}(s) ds \right)^\alpha \omega(t), \end{aligned}$$

where $G(t)$ is defined by (3.2), $t_1 \geq t_0$ is sufficiently large, and $t_3 > t_2 > t_1$.

(I) *Let $\bar{p} < \infty$, $\bar{q} < \infty$, and suppose that the corresponding $z(t)$ satisfies case (i) in Lemma 2.1. Then*

$$\bar{p} \leq \bar{r} - \bar{r}^{1+1/\alpha} \leq \frac{\alpha^\alpha}{(\alpha + 1)^{\alpha+1}} \quad \text{and} \quad \bar{p} + \bar{q} \leq 1. \tag{3.12}$$

(II) *If $\bar{p} = \infty$ or $\bar{q} = \infty$, then $z(t)$ does not have property (i) in Lemma 2.1.*

Proof Part (I). Assume that $x(t)$ is an eventually positive solution of (1.1) and the corresponding $z(t)$ satisfies (i). By (3.11), we have $\omega(t) > 0$ and

$$\omega'(t) = \frac{(r(t)(z'(t))^\alpha)' }{(z'(t))^\alpha} - \alpha r(t) \left(\frac{z'(t)}{z'(t)} \right)^{\alpha+1}.$$

As in the proof of Theorem 3.1, we get (3.5) and (3.8), and so

$$\begin{aligned} \omega'(t) &\leq -kq_*(t) \left(\frac{z(\sigma_*(t))}{z'(t)} \right)^\alpha - \alpha r(t) \left(\frac{\omega(t)}{r(t)} \right)^{1+1/\alpha} \\ &\leq -kq_*(t)G(t) - \frac{\alpha \omega^{1+1/\alpha}(t)}{r^{1/\alpha}(t)}. \end{aligned} \tag{3.13}$$

On the other hand, we conclude that

$$\frac{r^{1/\alpha}(t)z''(t)}{z'(t)} \leq \frac{1}{\int_{t_1}^t r^{-1/\alpha}(s) ds}$$

due to the proof of Lemma 2.2. Hence,

$$\omega(t) \leq \left(\int_{t_1}^t r^{-1/\alpha}(s) ds \right)^{-\alpha},$$

which implies that $0 \leq \bar{r} \leq \bar{R} \leq 1$ and $\lim_{t \rightarrow \infty} \omega(t) = 0$. Integrating (3.13) from t to ∞ , we obtain

$$\omega(t) \geq \int_t^\infty kq_*(s)G(s) ds + \int_t^\infty \frac{\alpha\omega^{1+1/\alpha}(s)}{r^{1/\alpha}(s)} ds. \tag{3.14}$$

Multiplying (3.14) by $(\int_{t_1}^t r^{-1/\alpha}(s) ds)^\alpha$, we deduce that

$$\begin{aligned} \omega(t) \left(\int_{t_1}^t r^{-1/\alpha}(s) ds \right)^\alpha &\geq \left(\int_{t_1}^t r^{-1/\alpha}(s) ds \right)^\alpha \int_t^\infty kq_*(s)G(s) ds \\ &\quad + \left(\int_{t_1}^t r^{-1/\alpha}(s) ds \right)^\alpha \int_t^\infty \frac{\alpha\omega^{1+1/\alpha}(s) \left(\int_{t_1}^s r^{-1/\alpha}(u) du \right)^{\alpha+1}}{r^{1/\alpha}(s) \left(\int_{t_1}^s r^{-1/\alpha}(u) du \right)^{\alpha+1}} ds, \end{aligned}$$

that is,

$$\begin{aligned} \omega(t) \left(\int_{t_1}^t r^{-1/\alpha}(s) ds \right)^\alpha &\geq \left(\int_{t_1}^t r^{-1/\alpha}(s) ds \right)^\alpha \int_t^\infty kq_*(s)G(s) ds \\ &\quad + \left(\int_{t_1}^t r^{-1/\alpha}(s) ds \right)^\alpha \int_t^\infty \left(w(s) \left(\int_{t_1}^s r^{-1/\alpha}(u) du \right)^\alpha \right)^{1+1/\alpha} h(s) ds, \end{aligned}$$

where

$$h(t) := - \left(\left(\frac{1}{\int_{t_1}^t r^{-1/\alpha}(u) du} \right)^\alpha \right)'.$$

Now, for any $\varepsilon > 0$, there exists a $t_3 > t_2$ such that, for $t \geq t_3$,

$$\omega(t) \left(\int_{t_1}^t r^{-1/\alpha}(s) ds \right)^\alpha \geq \bar{r} - \varepsilon,$$

which yields

$$\omega(t) \left(\int_{t_1}^t r^{-1/\alpha}(s) ds \right)^\alpha \geq \left(\int_{t_1}^t r^{-1/\alpha}(s) ds \right)^\alpha \int_t^\infty kq_*(s)G(s) ds + (\bar{r} - \varepsilon)^{1+1/\alpha}. \tag{3.15}$$

Applications of (3.15) and the definitions of \bar{r} and \bar{p} imply that

$$\bar{r} \geq \bar{p} + (\bar{r} - \varepsilon)^{1+1/\alpha}.$$

Since ε is arbitrary, we conclude that

$$\bar{r} \geq \bar{p} + \bar{r}^{1+1/\alpha}. \tag{3.16}$$

Next, we prove that

$$\bar{p} + \bar{q} \leq 1.$$

Multiplying (3.13) by $(\int_{t_1}^t r^{-1/\alpha}(u) du)^{\alpha+1}$ and integrating the resulting inequality from t_3 to t , we have

$$\int_{t_3}^t \left(\int_{t_1}^s r^{-1/\alpha}(u) du \right)^{\alpha+1} \omega'(s) ds \leq - \int_{t_3}^t \left(\int_{t_1}^s r^{-1/\alpha}(u) du \right)^{\alpha+1} kq_*(s)G(s) ds - \int_{t_3}^t \left(\int_{t_1}^s r^{-1/\alpha}(u) du \right)^{\alpha+1} \frac{\alpha\omega^{1+1/\alpha}(s)}{r^{1/\alpha}(s)} ds.$$

Integrating by parts, we deduce that

$$\left(\int_{t_1}^t r^{-1/\alpha}(u) du \right)^{\alpha+1} \omega(t) \leq \left(\int_{t_1}^{t_3} r^{-1/\alpha}(u) du \right)^{\alpha+1} \omega(t_3) - \int_{t_3}^t \left(\int_{t_1}^s r^{-1/\alpha}(u) du \right)^{\alpha+1} kq_*(s)G(s) ds + \int_{t_3}^t H(s) ds,$$

where

$$H(t) := (\alpha + 1)r^{-1/\alpha}(t) \left(\int_{t_1}^t r^{-1/\alpha}(u) du \right)^\alpha \omega(t) - \alpha r^{-1/\alpha}(t) \left(\int_{t_1}^t r^{-1/\alpha}(u) du \right)^{\alpha+1} \omega^{1+1/\alpha}(t).$$

Using inequality (3.10) with

$$\nu := \omega(t), \quad A := \alpha r^{-1/\alpha}(t) \left(\int_{t_1}^t r^{-1/\alpha}(u) du \right)^{\alpha+1}, \quad \text{and} \\ B := (\alpha + 1)r^{-1/\alpha}(t) \left(\int_{t_1}^t r^{-1/\alpha}(u) du \right)^\alpha,$$

we have

$$H(t) \leq r^{-1/\alpha}(t).$$

Thus, we arrive at

$$\left(\int_{t_1}^t r^{-1/\alpha}(u) du \right)^\alpha \omega(t) \leq \frac{(\int_{t_1}^{t_3} r^{-1/\alpha}(u) du)^{\alpha+1} \omega(t_3)}{\int_{t_1}^t r^{-1/\alpha}(u) du} - \frac{\int_{t_3}^t (\int_{t_1}^s r^{-1/\alpha}(u) du)^{\alpha+1} kq_*(s)G(s) ds}{\int_{t_1}^t r^{-1/\alpha}(u) du} + \frac{\int_{t_3}^t r^{-1/\alpha}(u) du}{\int_{t_1}^t r^{-1/\alpha}(u) du}. \tag{3.17}$$

Taking the lim sup of both sides of the latter inequality as $t \rightarrow \infty$, we have

$$\bar{R} \leq 1 - \bar{q}. \tag{3.18}$$

It follows from (3.16) and (3.18) that

$$\bar{p} \leq \bar{r} - \bar{r}^{1+1/\alpha} \leq \bar{r} \leq \bar{R} \leq 1 - \bar{q}.$$

Moreover, by inequality (3.10),

$$\bar{r} - \bar{r}^{1+1/\alpha} \leq \frac{\alpha^\alpha}{(\alpha + 1)^{\alpha+1}}.$$

Therefore, the desired inequalities in (3.12) hold. This completes the proof of Part (I).

Part (II). Let $x(t)$ be an eventually positive solution of (1.1). We show that $z(t)$ does not have property (i). Assume the contrary. Suppose first that $\bar{p} = \infty$. Inequality (3.14) implies that

$$\omega(t) \left(\int_{t_1}^t r^{-1/\alpha}(s) ds \right)^\alpha \geq \left(\int_{t_1}^t r^{-1/\alpha}(s) ds \right)^\alpha \int_t^\infty kq_*(s)G(s) ds.$$

Taking the \liminf of both sides of the latter inequality as $t \rightarrow \infty$, we arrive at

$$1 \geq \bar{r} \geq \infty,$$

which is a contradiction. Assume now that $\bar{q} = \infty$. An application of inequality (3.17) yields

$$0 \leq \bar{R} \leq -\infty,$$

which is also a contradiction. The proof of Part (II) is complete. □

On the basis of Lemma 3.1, we easily derive the following result with a proof similar to that of Theorem 3.1.

Theorem 3.2 *Assume that condition (2.1) is satisfied. If for all sufficiently large $t_1 \geq t_0$ and for some $t_3 > t_2 > t_1$,*

$$\liminf_{t \rightarrow \infty} \left(\int_{t_1}^t r^{-1/\alpha}(s) ds \right)^\alpha \int_t^\infty q_*(s)G(s) ds > \frac{\alpha^\alpha}{k(\alpha + 1)^{\alpha+1}} \tag{3.19}$$

or

$$\bar{p} + \bar{q} > 1, \tag{3.20}$$

where $G(t)$ is defined by (3.2), \bar{p} and \bar{q} are as in Lemma 3.1, then the conclusion of Theorem 3.1 remains intact.

4 Examples

The following examples illustrate applications of the main results in this paper.

Example 4.1 For $t \geq 1$, consider the third-order differential equation

$$\left(x(t) - \frac{1}{2} \int_0^{\pi/2} x(t - \mu) d\mu \right)''' + \frac{1}{4} \int_{-3\pi}^{-2\pi} x\left(t + \frac{\xi}{2}\right) d\xi = 0. \tag{4.1}$$

Let $\alpha = 1$, $a = 0$, $b = \pi/2$, $c = -3\pi$, $d = -2\pi$, $k = 1$, $r(t) = 1$, $p(t, \mu) = 1/2$, $\tau(t, \mu) = t - \mu$, $q(t, \xi) = 1/4$, and $\sigma(t, \xi) = t + \xi/2$. Note that

$$\int_{t_0}^\infty r^{-1/\alpha}(s) ds = \int_1^\infty ds = \infty, \quad \int_a^b p(t, \mu) d\mu = \int_0^{\pi/2} \frac{1}{2} d\mu = \frac{\pi}{4} < 1,$$

$$\sigma_*(t) = \sigma(t, -3\pi) = t - \frac{3\pi}{2},$$

and

$$G(t) = \frac{\int_{t_2}^{t-3\pi/2} \int_{t_1}^v r^{-1/\alpha}(u) du dv}{\int_{t_1}^t r^{-1/\alpha}(u) du} = \frac{t^2/2 - (3\pi/2 + t_1)t + \beta}{t - t_1}, \quad \beta = \frac{9\pi^2}{8} - \frac{t_2^2}{2} + \frac{3\pi}{2}t_1 + t_1t_2.$$

Furthermore, $q_*(t) = \pi/4$ and

$$\int_{t_3}^\infty q_*(s)G(s) ds = \frac{\pi}{8} \int_{t_3}^\infty \frac{s^2 - (3\pi + 2t_1)s + 2\beta}{s - t_1} ds = \infty.$$

Hence, by Corollary 3.2, every solution $x(t)$ of (4.1) is either oscillatory or converges to zero as $t \rightarrow \infty$. As a matter of fact, $x(t) = \sin t$ is an oscillatory solution to (4.1).

Example 4.2 For $t \geq 1$ and $q_0 > 0$, consider the third-order differential equation

$$\left(x(t) - \int_1^2 \frac{\mu}{t+1} x\left(\frac{t+\mu}{3}\right) d\mu\right)''' + \int_0^1 \frac{2q_0\xi}{t^3} x\left(\frac{t+\xi}{2}\right) d\xi = 0. \tag{4.2}$$

Let $\alpha = 1, a = 1, b = 2, c = 0, d = 1, k = 1, r(t) = 1, p(t, \mu) = \mu/(t + 1), \tau(t, \mu) = (t + \mu)/3, q(t, \xi) = 2q_0\xi/t^3$, and $\sigma(t, \xi) = (t + \xi)/2$. Note that

$$\int_{t_0}^\infty r^{-1/\alpha}(s) ds = \int_1^\infty ds = \infty, \quad \int_a^b p(t, \mu) d\mu = \int_1^2 \frac{\mu}{t+1} d\mu = \frac{3}{2(t+1)} \leq \frac{3}{4},$$

and

$$\int_1^\infty \int_v^\infty \int_u^\infty \int_0^1 \frac{2q_0\xi}{s^3} d\xi ds du dv = \infty, \quad \sigma_*(t) = \sigma(t, 0) = \frac{t}{2}.$$

Moreover,

$$G(t) = \frac{\int_{t_2}^{t/2} \int_{t_1}^v r^{-1/\alpha}(u) du dv}{\int_{t_1}^t r^{-1/\alpha}(u) du} = \frac{1}{8} \frac{t^2 - 4t_1t + \gamma}{t - t_1}, \quad \gamma = 8t_1t_2 - 4t_2^2,$$

$q_*(t) = q_0t^{-3}$, and

$$\begin{aligned} & \liminf_{t \rightarrow \infty} \left(\int_{t_1}^t r^{-1/\alpha}(s) ds\right)^\alpha \int_t^\infty q_*(s)G(s) ds \\ &= \liminf_{t \rightarrow \infty} \frac{q_0(t - t_1)}{8} \left(\int_t^\infty \frac{1}{s(s - t_1)} ds - 4t_1 \int_t^\infty \frac{1}{s^2(s - t_1)} ds\right. \\ & \quad \left. + \gamma \int_t^\infty \frac{1}{s^3(s - t_1)} ds\right) = \frac{q_0}{8}. \end{aligned}$$

Using Theorem 3.2, every solution $x(t)$ of (4.2) is either oscillatory or converges to zero as $t \rightarrow \infty$ if $q_0 > 2$.

Remark 4.1 Observe that Theorems 3.1 and 3.2 cannot distinguish solutions of (1.1) with different behaviors. It is not easy to obtain sufficient conditions that ensure that all solutions $x(t)$ of (1.1) just satisfy $\lim_{t \rightarrow \infty} x(t) = 0$ and do not oscillate. Neither is it possible to utilize the technique exploited in this work for proving that all solutions of (1.1) are oscillatory. Therefore, two interesting problems for future research can be formulated as follows.

- (P1) Suggest a different method to establish asymptotic criteria that ensure that all solutions of (1.1) tend to zero asymptotically.
- (P2) Is it possible to establish sufficient conditions that guarantee that all solutions of (1.1) are oscillatory?

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All three authors contributed equally to this work. They all read and approved the final version of the manuscript.

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Acknowledgements

The authors express their sincere gratitude to the editors and two anonymous referees for careful reading of the manuscript and valuable suggestions that helped to improve the paper. This research is supported by NNSF of P.R. China (Grant Nos. 61503171, 61403061, and 11447005), CPSF (Grant No. 2015M582091), NSF of Shandong Province (Grant No. ZR2012FL06), DSRF of Linyi University (Grant No. LYDX2015BS001), and the AMEP of Linyi University, P.R. China.

Received: 12 November 2015 Accepted: 6 April 2016 Published online: 12 April 2016

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