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Quasi-periodic wave solutions and asymptotic behavior for an extended $(2 + 1)$ -dimensional shallow water wave equation

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Abstract

Based on Riemann theta function and bilinear Bäcklund transformation, quasi-periodic wave solutions are constructed for an extended $(2 + 1)$ -dimensional shallow water wave equation. A detail asymptotic analysis procedure to the one- and two-periodic wave solutions are presented, and the asymptotic properties of this type of solutions are proved. It is shown that the quasi-periodic wave solutions converge to the soliton solutions under small amplitude limits.

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1 Introduction

Nonlinear evolution equations (NLEEs) have attracted much interest in the past few decades since they appear in many areas of scientific fields such as fluid mechanics, plasma physics, solid-state physics, and mathematical biology [1–4]. The investigation of solutions for NLEEs plays an important role in the study of nonlinear physical phenomena, and many effective methods have been discovered. Successful numerical methods include the decomposition method [5] and the spectral method [6–11] developed in recent years. Various analytic methods such as inverse scattering method, Darboux transformation, Bäcklund transformation, Hirota method, and algebro-geometrical approach [12–16] have been presented for NLEEs. Among the mentioned methods, the algebro-geometrical approach presents quasi-periodic or algebro-geometric solutions to many NLEEs. However, the approach needs Lax pair representations and involves complicated calculus on Riemann surfaces. Based on bilinear forms, Nakamura proposed a straightforward way to construct a kind of quasi-periodic solutions of nonlinear equations [17, 18], where the quasi-periodic wave solutions of the Korteweg-de Vries equation (KdV equation) and the Boussinesq equation were obtained by using the Riemann theta function. Recently, Hon and Fan have developed this method to investigate $(2 + 1)$ -dimensional Bogoyavlenskii's breaking soliton equation [19], the discrete Toda lattice [20], and the asymmetrical Nizhnik-Novikov-Veselov equation [21]. Ma [22] constructed one-periodic and two-

periodic wave solutions to a class of $(2 + 1)$ -dimensional Hirota bilinear equations. However, little work has been done on quasi-periodic solutions for systems that involve coupled Hirota’s bilinear equations.

In this paper, we focus our study on the following extended $(2 + 1)$ -dimensional shallow water wave equation:

$$u_{xt} - 4u_{xy}u_x - 2u_{2x}u_y + u_{3xy} + \alpha u_{xy} = 0, \tag{1.1}$$

where α is a constant. According to Wang and Chen [23], the bilinear Bäcklund transformation of Eq. (1.1) can be written as

$$\begin{aligned} (D_x^2 - \lambda)f \cdot g &= 0, \\ (D_t + D_x^2 D_y + 3\lambda D_y + \alpha D_y - \nu)f \cdot g &= 0, \end{aligned} \tag{1.2}$$

where λ and ν are arbitrary constants. Equation (1.2) is a type of coupled bilinear equations; it is more difficult to be dealt with than a single bilinear equation due to the appearance of two functions and two equations. As a reduction of Eq. (1.1), the shallow water wave equation can be used to describe the propagation of ocean waves in shallow water and has been investigated in many literatures [24–26]. So, Eq. (1.1) may be helpful for understanding the behavior of real ocean waves. In Ref. [27], the integrability and multiple soliton solutions of Eq. (1.1) are investigated with the aid of the simplified Hereman method and the Cole-Hopf transformation method. Wang and Chen [23] used the binary Bell polynomial approach to construct bilinear equation, bilinear Bäcklund transformation, Lax pair, and Darboux covariant Lax pair for this equation. However, to the best of our knowledge, quasi-periodic wave solutions for Eq. (1.1) have not been studied yet. Therefore, one objective of this paper is to construct one- and two-periodic wave solutions for the extended $(2 + 1)$ -dimensional shallow water wave equation (1.1). Another objective of the paper is to investigate the asymptotic behavior of the quasi-periodic wave solutions. The organization of this paper is as follows. In Section 2, we briefly introduce some main points on the Riemann theta function and get the one- and two-soliton solutions for Eq. (1.1) by using the Hirota bilinear method. In Sections 3 and 4, we apply the Riemann theta function to construct one- and two-periodic wave solutions for Eq. (1.1), respectively. Furthermore, a detail asymptotic analysis procedure to the quasi-periodic wave solutions is presented and demonstrates that the quasi-periodic solutions tend to the known soliton solutions for Eq. (1.1). Finally, some conclusions are given in Section 5.

2 Soliton solutions and the Riemann theta function

The quasi-periodic wave solutions of Eq. (1.1) are constructed based on the multidimensional Riemann theta function

$$\vartheta(\xi, \epsilon, s|\tau) = \sum_{n \in \mathbb{Z}^N} e^{-\pi \langle \tau(n+s), n+s \rangle + 2\pi i \langle \xi + \epsilon, n+s \rangle}. \tag{2.1}$$

Here $n = (n_1, \dots, n_N)^T \in \mathbb{Z}^N$ are the integer-valued vectors, $\xi = (\xi_1, \dots, \xi_N)^T \in \mathbb{C}^N$ are complex phase variables, $s = (s_1, \dots, s_N)^T$ and $\epsilon = (\epsilon_1, \dots, \epsilon_N)^T$ are complex vectors, and $\langle \cdot, \cdot \rangle$ is

the standard inner product on R^n . The parameter $\tau = (\tau_{ij})$ is a positive definite and real-valued symmetric $N \times N$ matrix, which we call the period matrix of the Riemann theta function.

For simplicity, we have $\vartheta(\xi, \tau) = \vartheta(\xi, 0, 0|\tau)$ and $\vartheta(\xi + \epsilon, \tau) = \vartheta(\xi, \epsilon, 0|\tau)$. The periodicity of the Riemann theta function is defined as follows.

Definition 1 [19] A function $g(x, t)$ on $C^N \times C$ is said to be quasi-periodic in t with fundamental periods $T_1, \dots, T_k \in C$ if T_1, \dots, T_k are linearly dependent over Z and there exists a function $G(t, t) \in C^N \times C^k$ such that, for all $(y_1, \dots, y_k) \in C^k$,

$$\begin{aligned} G(x, y_1, \dots, y_j + T_j, \dots, y_k) &= G(x, y_1, \dots, y_j, \dots, y_k), \\ G(x, t, \dots, t, \dots, t) &= g(x, t). \end{aligned} \tag{2.2}$$

In particular, $g(x, t)$ becomes periodic with T if and only if $T_j = m_j T$.

Proposition 1 [28] Let e_j be the j th column of the $N \times N$ identity matrix I_N , τ_j be the j th column of τ , and τ_{jj} be the (j, j) entry of τ . Then the theta function $\vartheta(\xi, \tau)$ has the periodic property

$$\vartheta(\xi + e_j + i\tau_j, \tau) = \exp(-2\pi i\xi_j + \pi \tau_{jj})\vartheta(\xi, \tau). \tag{2.3}$$

The vectors $\{e_j, j = 1, \dots, N\}$ and $\{i\tau_j, j = 1, \dots, N\}$ can be regarded as periods of the theta function $\vartheta(\xi, \tau)$ with multipliers 1 and $\exp(-2\pi i\xi_j + \pi \tau_{jj})$, respectively.

Let $\vartheta(\xi, \epsilon', 0|\tau)$ and $\vartheta(\xi, \epsilon, 0|\tau)$ be two Riemann theta functions, where $\epsilon = (\epsilon_1, \dots, \epsilon_N)^T$, $\epsilon' = (\epsilon'_1, \dots, \epsilon'_N)^T$, and $\xi = (\xi_1, \dots, \xi_N)^T$ with $\xi_j = k_j x + l_j y + m_j t + \xi_j^{(0)}$, $j = 1, 2, \dots, N$. For a polynomial operator $H(D_x, D_t, D_n)$ with respect to D_x , D_t , and D_n , the following useful formula holds:

$$\begin{aligned} H(D_x, D_y, D_t)\vartheta(\xi, \epsilon', 0|\tau) \cdot \vartheta(\xi, \epsilon, 0|\tau) \\ = \sum_{\mu} C(\epsilon', \epsilon, \mu)\vartheta(2\xi, \epsilon' + \epsilon, \mu/2|2\tau), \end{aligned} \tag{2.4}$$

where

$$\begin{aligned} C(\epsilon', \epsilon, \mu) &= \sum_{n \in Z^N} H(4\pi i\langle n - \mu/2, k \rangle, 4\pi i\langle n - \mu/2, l \rangle, 4\pi i\langle n - \mu/2, m \rangle) \\ &\times \exp\{-2\pi i\langle \tau(n - \mu/2), n - \mu/2 \rangle + 2\pi i\langle n - \mu/2, \epsilon' - \epsilon \rangle\}. \end{aligned} \tag{2.5}$$

Formulae (2.4) and (2.5) show that once

$$C(\epsilon', \epsilon, \mu) = 0 \tag{2.6}$$

for all possible combinations $\mu_1 = 0, 1; \dots; \mu_N = 0, 1$, then $\vartheta(\xi, \epsilon', 0|\tau)$ and $\vartheta(\xi, \epsilon, 0|\tau)$ are quasi-periodic solutions of the bilinear equation

$$H(D_x, D_y, D_t)\vartheta(\xi, \epsilon', 0|\tau) \cdot \vartheta(\xi, \epsilon, 0|\tau) = 0.$$

Proposition 2 [29] *Let $C(\mu)$ be given in (2.5), and make a choice such that $\epsilon'_j - \epsilon_j = \pm \frac{1}{2}$, $j = 1, 2, \dots, N$. Then:*

(1) *If $H(D_x, D_y, D_t)$ is an even function in the sense that*

$$H(-D_x, -D_y, -D_t) = H(D_x, D_y, D_t),$$

then $C(\mu)$ vanishes automatically for the case where $\sum_{j=1}^N \mu_j$ is an odd number, namely

$$C(\mu)|_\mu = 0 \quad \text{for} \quad \sum_{j=1}^N \mu_j = 1 \pmod{2}. \tag{2.7}$$

(2) *If $H(D_x, D_y, D_t)$ is an odd function in the sense that*

$$H(-D_x, -D_y, -D_t) = -H(D_x, D_y, D_t),$$

then $C(\mu)$ vanishes automatically for the case where $\sum_{j=1}^N \mu_j$ is an even number, namely

$$C(\mu)|_\mu = 0 \quad \text{for} \quad \sum_{j=1}^N \mu_j = 0 \pmod{2}. \tag{2.8}$$

Proposition 2 plays an important role in constructing quasi-periodic wave solutions for coupled bilinear equations.

Next, we need to get the one- and two-soliton solutions for the extended (2 + 1)-dimensional shallow water wave equation (1.1). By the dependent variable transformation [23]

$$u = -2(\ln f)_x, \tag{2.9}$$

the bilinear Bäcklund transformation for Eq. (1.1) was obtained as Eq. (1.2). In order to get the quasi-periodic wave solutions of Eq. (1.1), we take $v = 0$, that is, the bilinear Bäcklund transformation for Eq. (1.1) is

$$\begin{aligned} H_1(D_x, D_y, D_t) &= (D_x^2 - \lambda)f \cdot g = 0, \\ H_2(D_x, D_y, D_t) &= (D_t + D_x^2 D_y + 3\lambda D_y + \alpha D_y)f \cdot g = 0. \end{aligned} \tag{2.10}$$

By using the Hirota bilinear method we can easily get the soliton solutions of Eq. (1.1). We start with a simple solution $f = 1$, from which we can get the original solution $u = -2(\ln f)_x = 0$. Setting $\lambda = \frac{\tilde{k}_1^2}{4}$ and substituting $f = 1$ into Eq. (2.10), we readily obtain

$$g_1 = e^{\frac{\eta_1}{2}} + e^{-\frac{\eta_1}{2}}, \quad \eta_1 = \tilde{k}_1 x + \tilde{l}_1 y - (\tilde{k}_1^2 + \alpha)\tilde{l}_1 t + \eta_1^{(0)}, \tag{2.11}$$

where \tilde{k}_1, \tilde{l}_1 , and $\eta_1^{(0)}$ are arbitrary constants. The one-soliton solution can be written as

$$u_1 = -2 \left[\ln \left(1 + e^{\tilde{k}_1 x + \tilde{l}_1 y - (\tilde{k}_1^2 + \alpha)\tilde{l}_1 t + \eta_1^{(0)}} \right) \right]_x. \tag{2.12}$$

Taking $f = e^{\frac{\eta_1}{2}} + e^{-\frac{\eta_1}{2}}$, Eq. (2.10) can be written in the form

$$\begin{aligned} (D_x^2 - \lambda)(e^{\frac{\eta_1}{2}} + e^{-\frac{\eta_1}{2}}) \cdot g_2 &= 0, \\ (D_t + D_x^2 D_y + 3\lambda D_y + \alpha D_y)(e^{\frac{\eta_1}{2}} + e^{-\frac{\eta_1}{2}}) \cdot g_2 &= 0. \end{aligned} \tag{2.13}$$

Setting

$$\begin{aligned} g_2 &= \beta_1 \left(e^{\frac{\eta_1 + \eta_2}{2}} + e^{-\frac{\eta_1 + \eta_2}{2}} \right) + \beta_2 \left(e^{\frac{\eta_1 + \eta_2}{2}} + e^{-\frac{\eta_1 + \eta_2}{2}} \right), \\ \eta_j &= \tilde{k}_j x + \tilde{l}_j y - (\tilde{k}_j^2 + \alpha) \tilde{l}_j t + \eta_j^{(0)}, \quad j = 1, 2, \end{aligned} \tag{2.14}$$

where β_1 and β_2 are arbitrary constants, and substituting (2.14) into Eq. (2.13), we have

$$\lambda = \frac{\tilde{k}_2^2}{4}, \quad \beta_1 = \tilde{k}_1 - \tilde{k}_2, \quad \beta_2 = -(\tilde{k}_1 + \tilde{k}_2).$$

In this way, the two-soliton solution for Eq. (1.1) reads

$$u_2 = -2 \left[\ln(1 + e^{\eta_1} + e^{\eta_2} + e^{\eta_1 + \eta_2 + A_{12}}) \right]_x, \tag{2.15}$$

where $e^{A_{12}} = \left(\frac{\tilde{k}_1 - \tilde{k}_2}{\tilde{k}_1 + \tilde{k}_2} \right)^2$.

3 One-periodic waves and asymptotic properties

In this section, we consider the one-periodic wave solutions for Eq. (1.1) with $N = 1$ in the Riemann theta function (2.1). Setting $f(x, y, t) = \vartheta(\xi, 0, 0 | \tau)$ and $g(x, y, t) = \vartheta(\xi, \frac{1}{2}, 0 | \tau)$, $f(x, y, t)$ and $g(x, y, t)$ can be written as the following Fourier series in n :

$$\begin{aligned} f(x, y, t) &= \vartheta(\xi, 0, 0 | \tau) = \sum_{n=-\infty}^{\infty} e^{2\pi i n \xi - \pi n^2 \tau}, \\ g(x, y, t) &= \vartheta\left(\xi, \frac{1}{2}, 0 | \tau\right) = \sum_{n=-\infty}^{\infty} e^{2\pi i n (\xi + \frac{1}{2}) - \pi n^2 \tau}, \end{aligned} \tag{3.1}$$

where $\xi = kx + ly + mt + \xi^{(0)}$ is the phase variable, and $\tau > 0$ is the parameter.

3.1 Construction of one-periodic waves

In order to make the theta function (3.1) be a solution of bilinear equation (2.10), we substitute Eq. (3.1) into Eq. (2.10); then, for $i = 1, 2$,

$$\begin{aligned} &H_i(D_x, D_y, D_t) f \cdot g \\ &= \sum_{\mu} \left[\sum_{n \in \mathbb{Z}} H_i \left(4\pi i \left\langle n - \frac{\mu}{2}, k \right\rangle, 4\pi i \left\langle n - \frac{\mu}{2}, l \right\rangle, 4\pi i \left\langle n - \frac{\mu}{2}, m \right\rangle \right) \right. \\ &\quad \times \exp \left\{ -2\pi \left\langle \tau \left(n - \frac{\mu}{2} \right), n - \frac{\mu}{2} \right\rangle + 2\pi i \left\langle n - \frac{\mu}{2}, \epsilon' - \epsilon \right\rangle \right\} \left. \right] \vartheta \left(2\xi, \frac{1}{2}, \frac{\mu}{2} \mid 2\tau \right) \\ &= \sum_{\mu} C_i(\mu) \vartheta \left(2\xi, \frac{1}{2}, \frac{\mu}{2} \mid 2\tau \right), \quad \mu = 0, 1, \end{aligned}$$

where

$$\begin{aligned}
 C_1(\mu) &= \sum_{n \in \mathbb{Z}} \left[-16\pi^2 \left(n - \frac{\mu}{2} \right)^2 k^2 - \lambda \right] \\
 &\quad \times \exp \left\{ -2\pi \tau \left(n - \frac{\mu}{2} \right)^2 + \pi i \left(n - \frac{\mu}{2} \right) \right\}, \\
 C_2(\mu) &= \sum_{n \in \mathbb{Z}} \left[4\pi i \left(n - \frac{\mu}{2} \right) m - 64\pi^3 i \left(n - \frac{\mu}{2} \right)^3 k^2 l + 4\pi i (3\lambda + \alpha) \left(n - \frac{\mu}{2} \right) l \right] \\
 &\quad \times \exp \left\{ -2\pi \tau \left(n - \frac{\mu}{2} \right)^2 + \pi i \left(n - \frac{\mu}{2} \right) \right\}.
 \end{aligned} \tag{3.2}$$

According to Proposition 2, since $H_1(D_x, D_y, D_t)$ is an even function, we have $C_1(\mu = 1) = 0$. Similarly, we can obtain that $C_2(\mu = 0) = 0$. Suppose that the following equations are satisfied:

$$\begin{aligned}
 C_1(\mu = 0) &= \sum_{n \in \mathbb{Z}} (-16\pi^2 n^2 k^2 - \lambda) \exp \{ -2\pi \tau n^2 + \pi i n \} = 0, \\
 C_2(\mu = 1) &= \sum_{n \in \mathbb{Z}} \left[4\pi i \left(n - \frac{1}{2} \right) m - 64\pi^3 i \left(n - \frac{1}{2} \right)^3 k^2 l + 4\pi i (3\lambda + \alpha) \left(n - \frac{1}{2} \right) l \right] \\
 &\quad \times \exp \left\{ -2\pi \tau \left(n - \frac{1}{2} \right)^2 + \pi i \left(n - \frac{1}{2} \right) \right\} = 0.
 \end{aligned} \tag{3.3}$$

Then the Riemann theta functions (3.1) are exact solutions of Eq. (2.10). By introducing the notation

$$\begin{aligned}
 \rho &= e^{-\frac{\pi \tau}{2}}, \\
 \vartheta_1(\xi, \rho) &= \vartheta \left(2\xi, \frac{1}{2}, -\frac{1}{2} \mid 2\tau \right) \\
 &= \sum_{n \in \mathbb{Z}} \rho^{(2n-1)^2} \exp \left[4\pi i \left(n - \frac{1}{2} \right) \left(\xi + \frac{1}{4} \right) \right], \\
 \vartheta_2(\xi, \rho) &= \vartheta \left(2\xi, \frac{1}{2}, 0 \mid 2\tau \right) \\
 &= \sum_{n \in \mathbb{Z}} \rho^{4n^2} \exp \left[4\pi i n \left(\xi + \frac{1}{4} \right) \right],
 \end{aligned} \tag{3.4}$$

we can change Eq. (3.3) into a linear system about the frequency m and the constant λ

$$\begin{aligned}
 k^2 \vartheta_2'' - \lambda \vartheta_2 &= 0, \\
 m \vartheta_1' + k^2 l \vartheta_1''' + (3\lambda + \alpha) l \vartheta_1' &= 0,
 \end{aligned} \tag{3.5}$$

where

$$\vartheta_j^{(p)} = \vartheta_j^{(p)}(0, \rho) = \left. \frac{d^p \vartheta_j}{d\xi^p} \right|_{\xi=0}, \quad j = 1, 2; p = 0, 1, 2, 3.$$

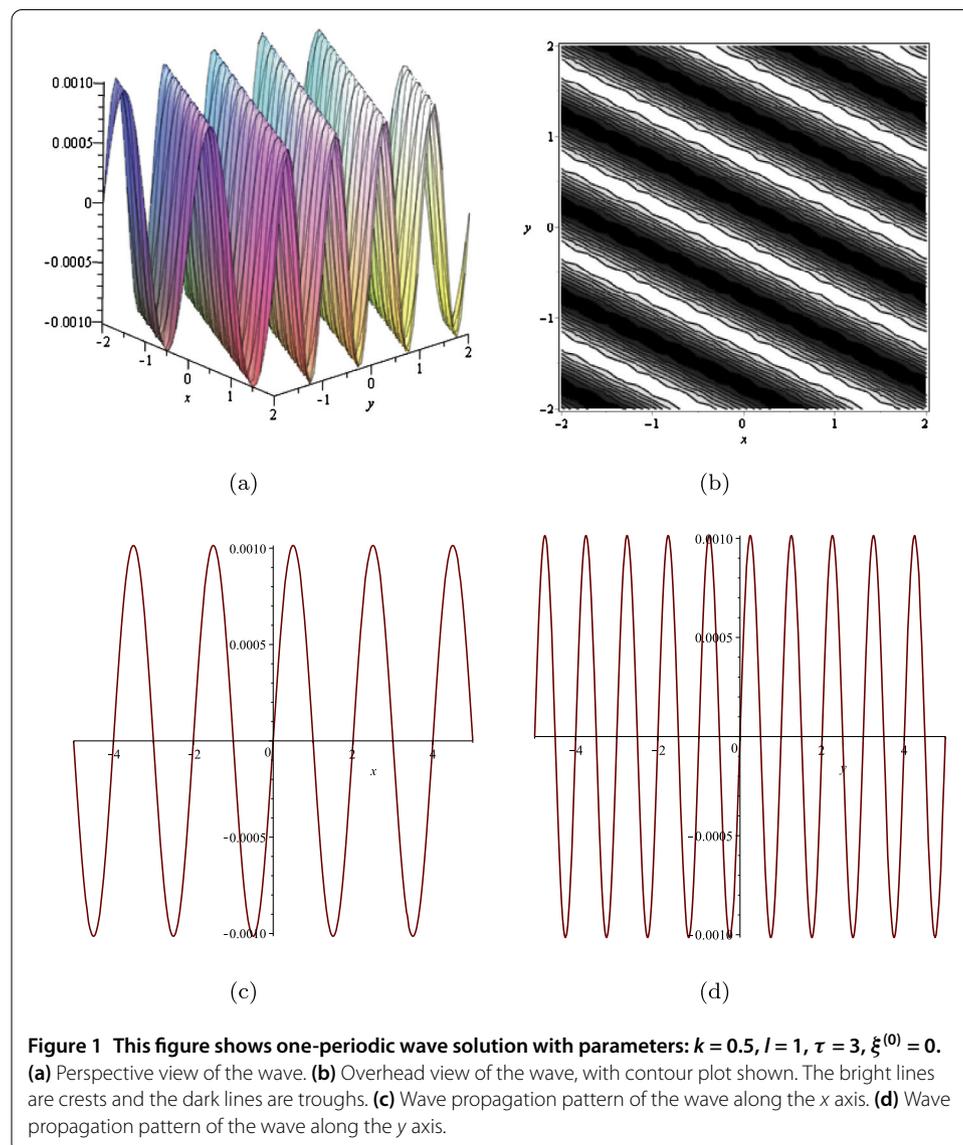
Solving Eq. (3.5), we get

$$\lambda = \frac{k^2 \vartheta_2''}{\vartheta_2}, \quad m = -\frac{k^2 l \vartheta_1''' \vartheta_2 + 3k^2 \vartheta_2'' \vartheta_1' + \alpha l \vartheta_1' \vartheta_2}{\vartheta_1' \vartheta_2}. \tag{3.6}$$

Now we get one-periodic wave solution of Eq. (1.1)

$$u = -2(\ln \vartheta(\xi, 0, 0|\tau))_x, \tag{3.7}$$

where $\xi = kx + ly + mt + \xi^{(0)}$, m are given by Eq. (3.6), and other parameters $k, l, \tau, \xi^{(0)}$ are free. Figure 1 shows the one-periodic wave solution of Eq. (1.1), which is one-dimensional and has two fundamental periods 1 and $i\tau$ in ξ . It can be regarded as a parallel superposition of overlapping one-solitary waves, placed one period apart.



3.2 Asymptotic property of one-periodic waves

Now we proceed to consider the asymptotic properties of the one-periodic wave solution. It is shown that the soliton solution (2.12) can be obtained as a limit of the one-periodic wave solution (3.7). The relation between these two solutions can be established as the following theorem.

Theorem 1 *Suppose that m are determined by Eq. (3.6), let*

$$\xi^{(0)} = \frac{\eta_1^{(0)}}{2\pi i} + \frac{\tau}{2i}, \quad k = \frac{\tilde{k}_1}{2\pi i}, \quad l = \frac{\tilde{l}_1}{2\pi i}, \tag{3.8}$$

where \tilde{k}_1, \tilde{l}_1 , and $\eta_1^{(0)}$ are the same as those in Eq. (2.12). Then the one-periodic solution (3.7) tends to the one-soliton solution (2.12) under a small amplitude limit, that is,

$$u \rightarrow u_1 \quad \text{as } \rho \rightarrow 0.$$

Proof By using Eq. (3.4) we write the coefficients of system (3.5) into power series of ρ :

$$\begin{aligned} \vartheta_1' &= -4\pi\rho + o(\rho^8), & \vartheta_1'' &= -8\pi^2 i\rho + o(\rho^8), \\ \vartheta_1''' &= 16\pi^3 \rho + o(\rho^8), & \vartheta_2 &= 1 - 2\rho^4 + o(\rho^8), \\ \vartheta_2'' &= 32\pi^2 \rho^4 + o(\rho^8). \end{aligned} \tag{3.9}$$

Assume that the solution of system (3.5) has the following form:

$$\begin{aligned} \lambda &= \lambda_0 + \lambda_1\rho + \lambda_2\rho^2 + \dots = \lambda_0 + o(\rho), \\ m &= m_0 + m_1\rho + m_2\rho^2 + \dots = m_0 + o(\rho). \end{aligned} \tag{3.10}$$

Substituting Eqs. (3.9) and (3.10) into Eq. (3.6) and letting $\rho \rightarrow 0$, we obtain

$$\lambda_0 = 0, \quad m_0 = 4\pi^2 k^2 l - \alpha l. \tag{3.11}$$

Combining (3.8) and (3.11), we then obtain

$$\begin{aligned} 2\pi im &\rightarrow 8\pi^3 ik^2 l - 2\pi i\alpha l \\ &= -(\tilde{k}_1^2 + \alpha)\tilde{l}_1 \quad \text{as } \rho \rightarrow 0. \end{aligned} \tag{3.12}$$

In order to show that the one-periodic wave (3.7) degenerates to the one-soliton solution (2.12) under the limit $\rho \rightarrow 0$, we expand the function f in the form

$$f = 1 + \rho^2 (e^{2\pi i\xi} + e^{-2\pi i\xi}) + \rho^8 (e^{4\pi i\xi} + e^{-4\pi i\xi}) + \dots$$

By (3.8) it follows that

$$\begin{aligned} f &= 1 + e^{\xi'} + \rho^2 (e^{-\xi'} + e^{2\xi'}) + \rho^{12} (e^{-2\xi'} + e^{3\xi'}) + \dots \\ &\rightarrow 1 + e^{\xi'} \quad \text{as } \rho \rightarrow 0, \end{aligned} \tag{3.13}$$

where

$$\xi' = 2\pi i\xi - \pi\tau = \tilde{k}_1x + \tilde{l}_1y + 2\pi imt + \eta_1^{(0)}. \tag{3.14}$$

According to Eq. (3.12), we easily get that

$$\xi' \rightarrow \tilde{k}_1x + \tilde{l}_1y - (\tilde{k}_1^2 + \alpha)\tilde{l}_1t + \eta_1^{(0)} = \eta_1. \tag{3.15}$$

Thus, we conclude that the one-periodic solution (3.7) just degenerates to one-soliton solution (2.12) as the amplitude $\rho \rightarrow 0$. □

4 Two-periodic waves and asymptotic properties

We now turn to construct two-periodic wave solutions for Eq. (1.1). In the case $N = 2$, we take $f(x, y, t)$ and $g(x, y, t)$ as

$$\begin{aligned} f(x, y, t) &= \vartheta(\xi, \mathbf{0}, \mathbf{0}|\tau) = \sum_{n \in \mathbb{Z}^2} e^{-\pi \langle \tau n, n \rangle + 2\pi i \langle \xi, n \rangle}, \\ g(x, y, t) &= \vartheta(\xi, \epsilon, \mathbf{0}|\tau) = \sum_{n \in \mathbb{Z}^2} e^{-\pi \langle \tau n, n \rangle + 2\pi i \langle \xi + \epsilon, n \rangle}, \end{aligned} \tag{4.1}$$

where $\epsilon = (\frac{1}{2}, \frac{1}{2})^T$, $n = (n_1, n_2)^T \in \mathbb{Z}^2$, $\xi = (\xi_1, \xi_2)^T \in \mathbb{C}^2$, $\xi_j = k_jx + l_jy + m_jt + \xi_j^{(0)}$, $j = 1, 2$, $k = (k_1, k_2)^T$, $l = (l_1, l_2)^T$, and τ is a positive definite and real-valued symmetric 2×2 matrix, which can be taken of the form

$$\tau = (\tau_{pq})_{2 \times 2}, \quad \tau_{11} > 0, \quad \tau_{22} > 0, \quad \tau_{11}\tau_{22} - \tau_{12}^2 > 0.$$

4.1 Construction of two-periodic waves

By using Proposition 2 we can readily obtain that the constraint equation in (2.13) of $H_1(D_x, D_y, D_t)$ automatically vanishes for $(\mu_1, \mu_2) = (1, 0), (0, 1)$. Similarly, since $H_2(D_x, D_y, D_t)$ is an odd function, its corresponding constraint equation vanishes for $(\mu_1, \mu_2) = (0, 0), (1, 1)$. Therefore, the Riemann theta function (4.1) is a solution of Eq. (2.10) if the following equations are satisfied:

$$\begin{aligned} &\sum_{n \in \mathbb{Z}^2} \left[-16\pi^2 \left\langle n - \frac{\mu}{2}, k \right\rangle^2 - \lambda \right] \\ &\quad \times \exp \left\{ -2\pi \left\langle \tau \left(n - \frac{\mu}{2} \right), n - \frac{\mu}{2} \right\rangle + \pi i(n_1 + n_2) \right\} \Big|_{(\mu_1, \mu_2) = (0, 0), (1, 1)} = 0, \\ &\sum_{n \in \mathbb{Z}^2} \left[4\pi i \left\langle n - \frac{\mu}{2}, m \right\rangle - 64\pi^3 i \left\langle n - \frac{\mu}{2}, k \right\rangle^2 \left\langle n - \frac{\mu}{2}, l \right\rangle + 4\pi i(3\lambda + \alpha) \left\langle n - \frac{\mu}{2}, l \right\rangle \right] \\ &\quad \times \exp \left\{ -2\pi \left\langle \tau \left(n - \frac{\mu}{2} \right), n - \frac{\mu}{2} \right\rangle + \pi i(n_1 + n_2) \right\} \Big|_{(\mu_1, \mu_2) = (0, 1), (1, 0)} = 0. \end{aligned} \tag{4.2}$$

Denote

$$\begin{aligned} \rho_{pq} &= e^{-\frac{\pi \tau pq}{2}}, \quad p, q = 1, 2, \\ \rho &= (\rho_{11}, \rho_{12}, \rho_{22}), \\ \vartheta_r(\xi, \rho) &= \vartheta\left(2\xi, \epsilon, -\frac{s_r}{2} | 2\tau\right) \\ &= \sum_{n \in \mathbb{Z}^2} \exp\left[-\pi\left\langle \tau\left(n - \frac{s_r}{2}\right), n - \frac{s_r}{2}\right\rangle + 2\pi i\left\langle 2\xi + \epsilon, n - \frac{s_r}{2}\right\rangle\right] \\ &= \sum_{n \in \mathbb{Z}^2} \exp\left[4\pi i\left\langle \xi + \frac{\epsilon}{2}, n - \frac{s_r}{2}\right\rangle\right] \prod_{p,q=1}^2 \rho_{pq}^{(2n_p - s_{r,p})(2n_q - s_{r,q})}, \\ s_r &= (s_{r1}, s_{r2}), \quad r = 1, 2, 3, 4, \\ s_1 &= (0, 0), \quad s_2 = (1, 1), \quad s_3 = (0, 1), \quad s_4 = (1, 0). \end{aligned} \tag{4.3}$$

Equation (4.2) can be written as a linear system with respect to λ and m :

$$\begin{aligned} (k \cdot \nabla)^2 \vartheta_r - \lambda \vartheta_r |_{(\xi_1, \xi_2)=(0,0)} &= 0, \quad r = 1, 2, \\ (m \cdot \nabla) \vartheta_r + (k \cdot \nabla)^2 (l \cdot \nabla) \vartheta_r + (3\lambda + \alpha)(l \cdot \nabla) \vartheta_r |_{(\xi_1, \xi_2)=(0,0)} &= 0, \quad r = 3, 4, \end{aligned} \tag{4.4}$$

where $\nabla = (\partial \xi_1, \partial \xi_2)$ and $k \cdot \nabla = k_1 \partial \xi_1 + k_2 \partial \xi_2$. Solving Eq. (4.4), we get a two-periodic wave solution of Eq. (1.1)

$$u = -2(\ln \vartheta(\xi, \mathbf{0}, \mathbf{0} | \tau))_x, \tag{4.5}$$

with $\vartheta(\xi, \mathbf{0}, \mathbf{0} | \tau)$ and m_1, m_2 given by Eqs. (4.1) and (4.4), respectively, whereas the other parameters k_1, k_2, l_1, l_2, τ are free.

4.2 Asymptotic property of two-periodic waves

In a similar way to Theorem 1, we can establish the following relation between the two-periodic solution (4.5) and the two-soliton solution (2.15).

Theorem 2 Assume that $m = (m_1, m_2)^T$ are determined by the linear system (4.4) and take

$$\begin{aligned} k_j &= \frac{\tilde{k}_j}{2\pi i}, \quad l_j = \frac{\tilde{l}_j}{2\pi i}, \quad \xi_j^{(0)} = \frac{\eta_j^{(0)}}{2\pi i} + \frac{\tau_{jj}}{2i}, \quad j = 1, 2, \\ \tau_{12} &= -\frac{A_{12}}{2\pi i}, \end{aligned} \tag{4.6}$$

with $\tilde{k}_j, \tilde{l}_j, \eta_j^{(0)}, j = 1, 2$, and A_{12} as given in Eq. (2.15). Then the two-periodic solution (4.5) tends to the two-soliton solution (2.15) under a small amplitude limit, that is,

$$u \rightarrow u_2 \quad \text{as } \rho_{11}, \rho_{22} \rightarrow 0.$$

Proof We expand the periodic function $f = \vartheta(\xi, \mathbf{0}, \mathbf{0}|\tau)$ in the form

$$f = 1 + (e^{2\pi i\xi_1} + e^{-2\pi i\xi_1})e^{-\pi\tau_{11}} + (e^{2\pi i\xi_2} + e^{-2\pi i\xi_2})e^{-\pi\tau_{22}} + (e^{2\pi i(\xi_1+\xi_2)} + e^{-2\pi i(\xi_1+\xi_2)})e^{-\pi(\tau_{11}+2\tau_{12}+\tau_{22})} + \dots$$

According to Eq. (4.6), we get

$$f = 1 + e^{\xi'_1} + e^{\xi'_2} + e^{\xi'_1+\xi'_2-2\pi\tau_{12}} + e^{-\xi'_1}\rho_{11}^4 + e^{-\xi'_2}\rho_{22}^4 + e^{-\xi'_1-\xi'_2-2\pi\tau_{12}}\rho_{11}^4\rho_{22}^4 + \dots \rightarrow 1 + e^{\xi'_1} + e^{\xi'_2} + e^{\xi'_1+\xi'_2+A_{12}} \text{ as } \rho_{11}, \rho_{22} \rightarrow 0,$$

where $\xi'_j = 2\pi i\xi_j - \pi\tau_{jj} = \tilde{k}'_j x + \tilde{l}'_j y + 2\pi im_j t + \eta_j^{(0)}$, $j = 1, 2$. Then we only need to prove that

$$2\pi im_j \rightarrow -(\tilde{k}_j^2 + \alpha)\tilde{l}_j, \quad j = 1, 2, \text{ as } \rho_{11}, \rho_{22} \rightarrow 0. \tag{4.7}$$

The proof of (4.7) is quite similar to that of formula (3.12) and so is omitted. Thus, the proof is completed. □

5 Conclusions

In this paper, by using the Riemann theta functions, the one-periodic and two-periodic wave solutions for the extended dimensional shallow water wave equation are constructed. Furthermore, the relation between the periodic wave solutions and soliton solutions is investigated, and the asymptotic properties of the quasi-periodic wave solutions are proved. We think that the results can be extended to the case $N > 2$. It should be noted that the solvability of system (2.7), (2.8) is the key to construct a multiperiodic wave solution. Nevertheless, the number of unknown parameters is less than the number of equations when $N > 2$. Therefore, in the case $N > 2$, we cannot get multiperiodic wave solutions directly. How to increase the number of unknown parameters or decrease the number of equations? Such a question will be investigated in the future.

Competing interests

The author declares that there is no conflict of interests in the submission of this manuscript.

Author's contributions

WR obtained the main result and completed all the parts of this manuscript. WR has given approval to the final version of the manuscript.

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References

1. Braum, M: *Differential Equations and Their Applications*. Springer, Berlin (2013)
2. Marin, M: An evolutionary equation in thermoelasticity of dipolar bodies. *J. Math. Phys.* **40**(3), 1391-1399 (1999)
3. Marin, M: On the minimum principle for dipolar materials with stretch. *Nonlinear Anal., Real World Appl.* **10**(3), 1572-1578 (2009)
4. Marin, MI: Nonsimple material problems addressed by the Lagrange's identity. *Bound. Value Probl.* **2013**(1), 135 (2013)
5. El-Sayed, AMA, Gaber, M: The Adomian decomposition method for solving partial differential equations of fractal order in finite domains. *Phys. Lett. A* **359**, 175-182 (2006)

6. Bhrawy, AH, Doha, EH, Baleanu, D, Ezz-eldein, SS: A spectral tau algorithm based on Jacobi operational matrix for numerical solution of time fractional diffusion-wave equations. *J. Comput. Phys.* **293**, 142-156 (2015)
7. Bhrawy, AH, Zaky, MA, Baleanu, D: New numerical approximations for space-time fractional burgers' equations via a Legendre spectral-collocation method. *Rom. Rep. Phys.* **67**, 340-349 (2015)
8. Abdelkawy, MA, Zaky, MA, Bhrawy, AH, Baleanu, D: Numerical simulation of time variable fractional order mobile-immobile advection-dispersion model. *Rom. Rep. Phys.* **67**, 773-791 (2015)
9. Bhrawy, AH, Zaky, MA, Baleanu, D, Abdelkawy, MA: A novel spectral approximation for the two-dimensional fractional sub-diffusion problems. *Rom. J. Phys.* **60**, 344-359 (2015)
10. Bhrawy, AH, Abdelkawy, MA, Alzahrani, AA, Baleanu, D, Alzahrani, EO: A Chebyshev-Laguerre Gauss-Radau collocation scheme for solving time fractional sub-diffusion equation on a semi-infinite domain. *Proc. Rom. Acad., Ser. A : Math. Phys. Tech. Sci. Inf. Sci.* **16**, 490-498 (2015)
11. Bhrawy, AH, Baleanu, D: A spectral Legendre-Gauss-Lobatto collocation method for a space-fractional advection diffusion equations with variable coefficients. *Rep. Math. Phys.* **72**, 219-233 (2013)
12. Ablowitz, MJ, Clarkson, PA: *Soliton, Nonlinear Evolution Equations and Inverse Scattering*. Cambridge University Press, New York (1991)
13. Matveev, VB, Salle, MA: *Darboux Transformation and Solitons*. Springer, Berlin (1991)
14. Gu, CH, Hu, HS, Zhou, ZX: *Darboux Transformation in Soliton Theory and Its Geometric Applications*. Science and Technology Press, Shanghai (1999)
15. Hirota, R: *The Direct Method in Soliton Theory*. Cambridge University Press, New York (2004)
16. Belokolos, ED, Bobenko, AI, Enolskij, VZ, Its, AR, Matveev, VB (eds.): *Algebro-Geometric Approach to Nonlinear Integrable Equations*. Springer, Berlin (1994)
17. Nakamura, A: A direct method of calculating periodic wave solutions to nonlinear evolution equations. I. Exact two-periodic wave solution. *J. Phys. Soc. Jpn.* **47**, 1701-1705 (1979)
18. Nakamura, A: A direct method of calculating periodic wave solutions to nonlinear evolution equations. II. Exact one- and two-periodic wave solution of the coupled bilinear equations. *J. Phys. Soc. Jpn.* **47**, 1365-1370 (1980)
19. Fan, EG, Hon, YC: Quasi-periodic waves and asymptotic behavior for Bogoyavlenskii's breaking soliton equation in $(2 + 1)$ dimensions. *Phys. Rev. E* **78**, 036607 (2008)
20. Hon, YC, Fan, EG, Qin, ZY: A kind of explicit quasi-periodic solution and its limit for the Toda lattice equation. *Mod. Phys. Lett. B* **22**, 547-553 (2008)
21. Fan, EG: Quasi-periodic waves and an asymptotic property for the asymmetrical Nizhnik-Novikov-Veselov equation. *J. Phys. A, Math. Theor.* **42**, 095206 (2009)
22. Ma, WX, Zhou, RG: Exact one-periodic and two-periodic wave solutions to Hirota bilinear equations in $(2 + 1)$ -dimensions. *Mod. Phys. Lett. A* **24**(21), 1677-1688 (2009)
23. Wang, YH, Chen, Y: Integrability of an extended $(2 + 1)$ -dimensional shallow water wave equation with Bell polynomials. *Chin. Phys. B* **22**, 050509 (2013)
24. Ablowitz, MJ, Kaup, DJ, Newell, AC, Segur, H: The inverse scattering transform - Fourier analysis for nonlinear problems. *Stud. Appl. Math.* **53**, 249-315 (1974)
25. Hirota, R, Satsuma, J: N-soliton solutions of model equations for shallow water waves. *J. Phys. Soc. Jpn.* **40**(2), 611-612 (1976)
26. Narain, R, Kara, AH: On the redefinition of the variational and 'partial' variational conservation laws in a class of nonlinear PDEs with mixed derivatives. *Math. Comput. Appl.* **15**, 732-741 (2010)
27. Wazwaz, AM: Multiple-soliton solutions for extended shallow water wave equations. *Stud. Math. Sci.* **1**, 21-29 (2010)
28. Farkas, HM, Kra, I: *Riemann Surfaces*. Springer, New York (1992)
29. Fan, EG, Chow, KW: On the periodic solutions for both nonlinear differential and difference equations: a unified approach. *Phys. Lett. A* **374**, 3629-3634 (2010)

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