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Alternating segment explicit-implicit and implicit-explicit parallel difference method for the nonlinear Leland equation

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Abstract
The nonlinear Leland equation is a Black-Scholes option pricing model with transaction costs and the research of its numerical methods has theoretical significance and practical application value. This paper constructs a kind of difference scheme with intrinsic parallelism-alternating segment explicit-implicit (ASE-I) scheme and alternating segment implicit-explicit (ASI-E) scheme based on the improved Saul'yev asymmetric scheme, explicit-implicit (E-I) scheme, and implicit-explicit (I-E) scheme. Theoretical analysis demonstrates that this kind of scheme is unconditional stable parallel difference scheme. Numerical experiments show that the computational accuracy of this kind of scheme is very close to the classical Crank-Nicolson (C-N) scheme and the alternating segment Crank-Nicolson (ASC-N) scheme. But the computational time of this kind of scheme can save nearly 81% for the classical C-N scheme and save nearly 40% for the ASC-N scheme. Numerical experiments confirm the theoretical analysis, showing the higher efficiency of this kind of scheme given by this paper for solving a nonlinear Leland equation.

MSC: 65M06; 65Y05

Keywords: nonlinear Leland equation; alternating segment explicit-implicit (ASE-I) scheme; alternating segment implicit-explicit (ASI-E) scheme; parallel computing; numerical experiments

1 Introduction

The Black-Scholes (B-S) option pricing model can be accepted by practice fields and theory fields, not only because it has abundant financial implications, but also it is linear and is a simple model. The B-S model can be transformed into a heat conduction equation with a more mature theory in mathematics and can get the analytical solution of the European call option and put option pricing. However, there exist certain differences between the assumptions of the B-S model and the real financial market, such as there being no transaction costs and the fixed volatility hypothesis. In order to meet the needs of the actual financial market, we need to broaden the idealized assumptions and improve the standard B-S model. That has been the focus of academic research; see [1–3]. In the real financial market, because of transaction costs which one needs to pay in securities trading, using the continuous trading strategy is not realistic. So studying the option pricing model with transaction costs (nonlinear Leland model) has great financial practical significance.

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The nonlinear Leland equation is one of the nonlinear B-S option pricing models which need to consider transaction costs and received extensive attention of economists and applied mathematicians in the past 20 years [4–6]. Because one is unable to export the accurate analytical expression of the European option and American option pricing in the case of considering the transaction costs, many researches focus on the study of numerical solutions. In the numerical solution, in order to make the numerical scheme have good computing stability and precision, we often design implicit or half implicit difference scheme. In recent years, Ankudinova and Ehrhardt proposed the Crank-Nicolson (C-N) scheme for solving a nonlinear B-S equation (the Leland model, the Barles-Soner model, and the risk adjustment pricing model) [7]. Wu and Yang put forward the explicit-implicit (E-I) and implicit-explicit (I-E) difference schemes for solving the payment of dividend B-S equation [8]. However, most of the schemes are calculated in a serial way and the efficiency is low.

In order to make full use of the computer advantages of multi-core processors, a parallel algorithm and a parallel program design have become a necessary means to improve the computing efficiency [9]. The implicit scheme generally has good stability, but it is unfavorable for parallel computing. Inspired by the grouping explicit method [10], Zhang *et al.* put forward the thought that using the Saul'yev asymmetric scheme to construct a segment implicit scheme, and one properly used the alternating technology to establish a variety of explicit-implicit and pure implicit alternating parallel schemes (such as an alternating segment explicit-implicit (ASE-I) scheme, an alternating segment Crank-Nicolson (ASC-N) scheme), then one got some numerical results which contained stability and parallelism [11]. Yang *et al.* constructed a new kind of parallel difference scheme-the alternating band Crank-Nicolson (ABdC-N) scheme for solving the quanto option pricing model and proved that it is close to second-order accuracy and unconditionally stable [12]. Yuan *et al.* had put forward a parallel difference scheme with second-order accuracy and unconditional stability for a nonlinear parabolic equation [13]. Wang also gave a kind of alternating segment difference scheme with intrinsic parallelism for the KdV equation and proved that the scheme is linearly absolute stable [14]. Zhang showed the alternating segment explicit-implicit parallel difference scheme for a class of nonlinear evolution equations and got the result that the method has unconditional stability and parallelism [15].

For the research of the parallel difference method for solving the nonlinear Leland equation, Wu *et al.* presented a difference method with intrinsic parallelism-the ASC-N parallel difference scheme [16]. Because of the high timeliness of the option, constructing a difference scheme with good stability and intrinsic parallelism has important practical application value. We apply the E-I and I-E schemes at segment interior points, and the improved asymmetric difference scheme at interior boundary points, and we get a kind of difference numerical difference scheme with intrinsic parallelism-the alternating segment explicit-implicit (ASE-I) scheme and the alternating segment implicit-explicit (ASI-E) scheme.

The plan of this paper is as follows. In Section 2, we construct the ASE-I difference scheme for the nonlinear Leland equation. In Section 3, by using three lemmas, the unique solvability of the difference solution is discussed. We analyze the stability of the ASE-I scheme in Section 4 and the accuracy in Section 5. In Section 6, the ASI-E scheme is put forward by simulating the ASE-I scheme and a theorem is given. Numerical examples are provided to show the effectiveness of the ASE-I and ASI-E schemes in Section 7. Some concluding remarks are included finally.

2 ASE-I parallel difference method

2.1 Nonlinear Leland equation

Assuming that the underlying asset is the transaction cost-paying stock, by the Δ-hedging principle, we can get the following nonlinear Leland equation [1–3], which we will consider for the European options:

$$\frac{\partial V}{\partial t}+\frac{1}{2}\hat{\sigma}^2 S^2\frac{\partial^2 V}{\partial S^2}+rS\frac{\partial V}{\partial S}-rV=0, \tag{1}$$

here V is the price of a European call option (dollar), S is the price of the underlying asset, r is risk-free interest rate, $\hat{\sigma}$ is the revised volatility, $\hat{\sigma}^2=\sigma^2(1+Le\,\mathrm{sign}(V_{SS}))$. In the revised volatility, $Le=2\frac{k}{\sigma}\sqrt{\frac{2}{\pi\delta t}}$ is the Leland number, σ is the volatility, k is a volume of transaction cost, δt is the time difference between the two transactions, t is the time.

The Leland equation is a definite solution problem of nonlinear partial differential equations. When $k>\sigma\sqrt{\pi\delta t/8}$, equation (1) will become a terminal value problem of a positive parabolic equation which is an ill-posed problem [1, 3]. In order to transform the problem (1) into a well-posed problem, we can assume that $k<\sigma\sqrt{\pi\delta t/8}$, and that the transaction cost should be smaller or the process of hedging risk cannot be too often.

In order to solve the equation of the European call option pricing with transaction costs by using numerical methods, equation (1) is to be satisfied on the following boundary conditions [5, 7]:

(1) The value of the option is the pay-off function *i.e.* $V(S,T)=(S-K)^+$.
(2) $\lim_{S\to\infty}\frac{V(S,t)}{S}=1$. When S is sufficiently great, the option price is approximately $S-K$.
(3) If $S(t_0)=0$, then $V(S,t)=0$ for $t>t_0$.

Hence, for the European call option, we need to solve the following equation on the domain $\Sigma=\{0\le S<\infty, 0\le t\le T\}$:

$$\begin{cases}\frac{\partial V}{\partial t}+\frac{1}{2}\tilde{\sigma}^2 S^2\frac{\partial^2 V}{\partial S^2}+rS\frac{\partial V}{\partial S}-rV=0,\\ V(S,T)=\max\{S-K,0\}.\end{cases} \tag{2}$$

In order to be able to solve equation (2), we can substitute its variables as follows [1–3]:

$$S=Ke^x;\qquad 2\tau=\sigma^2(T-t);\qquad V(S,t)=Ke^xU(x,\tau).$$

Then equation (2) will be transformed into the initial-boundary value problem of a partial differential equation with constant coefficients:

$$\begin{cases}\frac{\partial U}{\partial\tau}-D\frac{\partial^2 U}{\partial x^2}-(D+L)\frac{\partial U}{\partial x}=0,\\ U(x,0)=\max\{1-e^{-x},0\},\quad x\in R,\end{cases} \tag{3}$$

here $D=\frac{\hat{\sigma}^2}{\sigma^2}$, $L=\frac{2r}{\sigma^2}$, $x\in R$, $0\le\tau\le\tilde{T}=\frac{\sigma^2 T}{2}$.

Meanwhile, the initial and boundary conditions will be translated into

$$U(x,0)=\max\{1-e^{-x},0\},\quad x\in R,$$

$$\lim_{x\to+\infty} U(x,\tau) = 1 - e^{-x}, \qquad \lim_{x\to-\infty} U(x,\tau) = 0.$$

In the specific calculation, we can select a large enough M^+ and a small enough M^- making the solving area and the boundary conditions

$$\Sigma_0 = \{M^- \le x \le M^+, 0 \le \tau \le \tilde{T}\},$$
$$U(M^+,\tau) = 1 - e^{-M^+}, \qquad U(M^-,\tau) = 0.$$

2.2 Construction of the ASE-I scheme

Let us make a mesh partition on the area Σ_0 and consider the function $U(x,\tau)$ at the discrete set of points

$$x_i = M^- + (i-1)h, \quad i = 1,2,\dots,m,m+1; h = \frac{M^+ - M^-}{m};$$
$$\tau_j = (j-1)p, \quad j = 1,2,\dots,n,n+1; p = \frac{\tilde{T}}{n}.$$

Here h is the space step, p is the time step, and m, n are the number of grid points in the x direction and τ direction, respectively. We use U_i^j to denote the solution of (3) at a finite difference point (x_i,τ_j). In order to construct the ASE-I scheme, we give some difference schemes of equation (3). Let $a = \frac{p(D+L)}{2h}$, $b = \frac{pD}{h^2}$.

First, the classical explicit scheme is

$$\frac{U_i^{j+1} - U_i^j}{p} = D\frac{U_{i+1}^j - 2U_i^j + U_{i-1}^j}{h^2} + (D+L)\frac{U_{i+1}^j - U_{i-1}^j}{2h}.$$

The above scheme can be written as

$$U_i^{j+1} = (b-a)U_{i-1}^j + (1-2b)U_i^j + (a+b)U_{i+1}^j. \tag{4}$$

Second, the classical implicit scheme is

$$\frac{U_i^{j+1} - U_i^j}{p} = D\frac{U_{i+1}^{j+1} - 2U_i^{j+1} + U_{i-1}^{j+1}}{h^2} + (D+L)\frac{U_{i+1}^{j+1} - U_{i-1}^{j+1}}{2h}.$$

The above scheme can be written as

$$-(b-a)U_{i-1}^{j+1} + (1+2b)U_i^{j+1} - (a+b)U_{i+1}^{j+1} = U_i^j. \tag{5}$$

At last, we present the two improved Saul'yev asymmetric schemes,

$$\frac{U_i^{j+1} - U_i^j}{p} = D\frac{U_{i+1}^{j+1} - U_i^{j+1} - U_i^j + U_{i-1}^j}{h^2} + (D+L)\frac{U_{i+1}^{j+1} - U_{i-1}^j}{2h},$$
$$\frac{U_i^{j+1} - U_i^j}{p} = D\frac{U_{i+1}^j - U_i^j - U_i^{j+1} + U_{i-1}^{j+1}}{h^2} + (D+L)\frac{U_{i+1}^j - U_{i-1}^{j+1}}{2h}.$$

The above schemes can be written as

$$(1+b)U_i^{j+1} - (a+b)U_{i+1}^{j+1} = (b-a)U_{i-1}^j + (1-b)U_i^j, \tag{6}$$

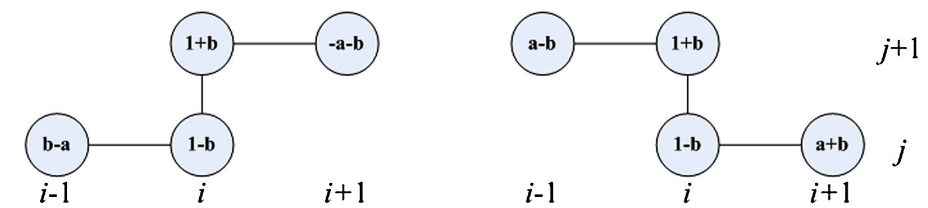


Figure 1 The diagram of schemes (6) and (7).

$$(a-b)U_{i-1}^{j+1} + (1+b)U_i^{j+1} = (1-b)U_i^j + (a+b)U_{i+1}^j. \tag{7}$$

Among the schemes mentioned above, the classic explicit scheme (4) has the property of parallelism and is very suitable for parallel computing, but it is conditionally stable. The classic implicit scheme (5) is unconditionally stable, but it needs to solve an algebraic equation which cannot be implemented on a parallel computer [11]. The improved Saul'yev asymmetric schemes (6), (7) are convenient to parallel computing, but they are conditionally stable (see Figure 1).

The ASE-I scheme which we constructed is combined with the advantages of the above schemes and the design is as follows:

Let $m-1 = Nl$, here N is a positive odd number, l is a positive integer (N, $l \geq 3$) and we divide the points on each time level into N sections. And on the odd level, we arrange the computation according to the rule of 'the explicit segment - the implicit segment - the explicit segment'. When it turns to the even level, the rule changes into 'the implicit segment - the explicit segment - the implicit segment' thus making the implicit segment and the explicit segment doing alternatively at different time levels.

For realizing the parallel computing of the ASE-I scheme, for $i_0 \geq 0$, we consider the calculation of the implicit segment point $(i_0 + i, j+1)$, $i = 1, 2, \ldots, l$. The left endpoint $(i_0 + 1, j+1)$ of the implicit segment is calculated with the improved Saul'yev scheme (6), the right endpoint $(i_0 + l, j+1)$ is calculated with the improved Saul'yev scheme (7), and the 'interior points' $(i_0 + i, j+1)$, $i = 2, 3, \ldots, l-1$, are calculated with the classical implicit scheme (5), leading to the following implicit segment (see Figure 2).

$$\begin{pmatrix} 1+b & -(a+b) & & & \\ a-b & 1+2b & -(a+b) & & \\ & \ddots & \ddots & \ddots & \\ & & a-b & 1+2b & -(a+b) \\ & & & a-b & 1+b \end{pmatrix} \begin{pmatrix} U_{i_0+1}^{j+1} \\ U_{i_0+2}^{j+1} \\ \vdots \\ U_{i_0+l-1}^{j+1} \\ U_{i_0+l}^{j+1} \end{pmatrix} = \begin{pmatrix} (1-b)U_{i_0+1}^j + (b-a)U_{i_0}^j \\ U_{i_0+2}^j \\ \vdots \\ U_{i_0+l-1}^j \\ (1-b)U_{i_0+l}^j + (a+b)U_{i_0+l+1}^j \end{pmatrix}, \tag{8}$$

here, $i_0 = l, 2l, \ldots, (N-2)l$.

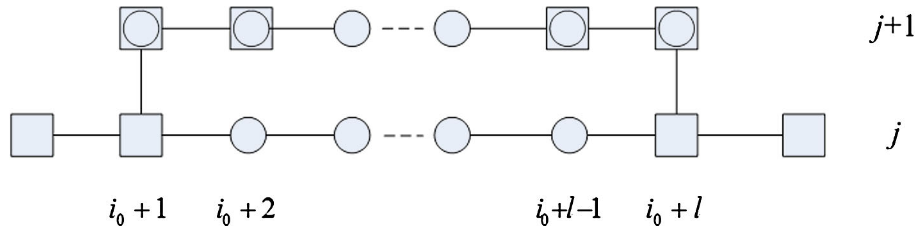


Figure 2 Schematic of implicit segment.

In order to improve the calculation accuracy, the implicit segment will be translated into

$$\begin{pmatrix} 1+2b & -(a+b) & & & \\ a-b & 1+2b & -(a+b) & & \\ & \ddots & \ddots & \ddots & \\ & & a-b & 1+2b & -(a+b) \\ & & & a-b & 1+b \end{pmatrix}\begin{pmatrix} U_1^{j+1} \\ U_2^{j+1} \\ \vdots \\ U_{l-1}^{j+1} \\ U_l^{j+1} \end{pmatrix} = \begin{pmatrix} U_1^j + (b-a)U_0^{j+1} \\ U_2^j \\ \vdots \\ U_{l-1}^j \\ (1-b)U_l^j + (a+b)U_{l+1}^j \end{pmatrix} \tag{9}$$

when $i_0 = 0$ and

$$\begin{aligned} &\begin{pmatrix} 1+b & -(a+b) & & & \\ a-b & 1+2b & -(a+b) & & \\ & \ddots & \ddots & \ddots & \\ & & a-b & 1+2b & -(a+b) \\ & & & a-b & 1+2b \end{pmatrix}\begin{pmatrix} U_{i_0+1}^{j+1} \\ U_{i_0+2}^{j+1} \\ \vdots \\ U_{i_0+l-1}^{j+1} \\ U_{i_0+l}^{j+1} \end{pmatrix} \\ &\quad = \begin{pmatrix} (1-b)U_{i_0+1}^j + (b-a)U_{i_0}^j \\ U_{i_0+2}^j \\ \vdots \\ U_{i_0+l-1}^j \\ U_{i_0+l}^j + (a+b)U_{i_0+l+1}^{j+1} \end{pmatrix} \end{aligned} \tag{10}$$

when $i_0 = (N-1)l$.

The explicit segment is

$$\begin{aligned} \begin{pmatrix} U_{i_0+1}^{j+1} \\ U_{i_0+2}^{j+1} \\ \vdots \\ U_{i_0+l-1}^{j+1} \\ U_{i_0+l}^{j+1} \end{pmatrix} &= \begin{pmatrix} 1-2b & (a+b) & & & \\ b-a & 1-2b & (a+b) & & \\ & \ddots & \ddots & \ddots & \\ & & b-a & 1-2b & (a+b) \\ & & & b-a & 1-2b \end{pmatrix}\begin{pmatrix} U_{i_0+1}^j \\ U_{i_0+2}^j \\ \vdots \\ U_{i_0+l-1}^j \\ U_{i_0+l}^j \end{pmatrix} \\ &\quad + \begin{pmatrix} (b-a)U_{i_0}^j \\ 0 \\ \vdots \\ 0 \\ (b+a)U_{i_0+l+1}^j \end{pmatrix}. \end{aligned} \tag{11}$$

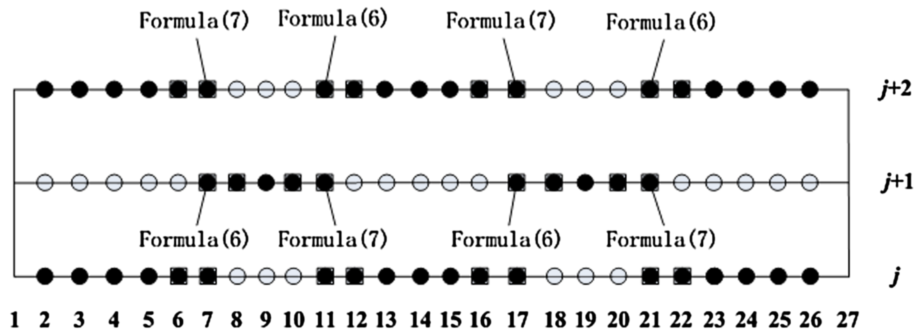


Figure 3 Schematic with segment nodes of the ASE-I scheme.

We use ○ to denote the classical explicit scheme, ● to denote the classical implicit scheme, ▣ to denote the two improved Saul'yev asymmetric schemes. Let $m = 26$, $l = 5$, $N = 5$ and let the schematic of the ASE-I scheme be as given (see Figure 3).

A complete calculation step of the ASE-I scheme is as follows. For odd level:

```
for i = 1 : N
  if mod(i, 2) == 1
    if i == 1
        Solve equation (9) to get U_1^{j+1};
    else if i == N
        Solve equation (10) to get U_N^{j+1};
      else
        Solve equation (8) to get U_i^{j+1};
      end
    end
  else
      Solve equation (11) to get U_i^{j+1};
  end
end for
```

For even level, we just switch the segment implicit scheme and the segment explicit scheme of the odd level to calculate U_i^{j+1}.

The ASE-I scheme can also be expressed as

$$\begin{cases} (I + G_1)U^{j+1} = (I - G_2)U^j + b_1^j, \\ (I + G_2)U^{j+2} = (I - G_1)U^{j+1} + b_1^{j+2}, \end{cases} \quad j = 1, 3, 5, \ldots, \tag{12}$$

where $U^j = (U_2^j, U_3^j, \ldots, U_{m-1}^j, U_m^j)^T$, $b_1^j = ((b-a)U_1^j, 0, \ldots, 0, (b+a)U_{m+1}^j)^T$, $j = 1, 2, 3, \ldots, n+1$,

$$G_1 = \begin{pmatrix} Q_l & & & & & & & \\ & G_l^{(1)} & & & & & & \\ & & Q_l & & & & & \\ & & & G_l^{(2)} & & & & \\ & & & & \ddots & & & \\ & & & & & Q_l & & \\ & & & & & & G_l^{(\frac{N-1}{2})} & \\ & & & & & & & Q_l \end{pmatrix},$$

$$G_2 = \begin{pmatrix} \bar{G}_{l+1}^{(1)} & & & & & & & \\ & Q_{l-2} & & & & & & \\ & & G_{l+2}^{(2)} & & & & & \\ & & & Q_{l-2} & & & & \\ & & & & \ddots & & & \\ & & & & & G_{l+2}^{(\frac{N-1}{2})} & & \\ & & & & & & Q_{l-2} & \\ & & & & & & & \bar{G}_{l+1}^{(\frac{N+1}{2})} \end{pmatrix},$$

in which

$$\bar{G}_{l+1}^{(1)} = \begin{pmatrix} 2b & -(a+b) & & & \\ a-b & 2b & -(a+b) & & \\ & \ddots & \ddots & \ddots & \\ & & a-b & 2b & -(a+b) \\ & & & a-b & b \end{pmatrix}_{(l+1)\times(l+1)},$$

$$\bar{G}_{l+1}^{(\frac{N+1}{2})} = \begin{pmatrix} b & -(a+b) & & & \\ a-b & 2b & -(a+b) & & \\ & \ddots & \ddots & \ddots & \\ & & a-b & 2b & -(a+b) \\ & & & a-b & 2b \end{pmatrix}_{(l+1)\times(l+1)},$$

$$G_{(l')}^{(i)} = \begin{pmatrix} b & -(a+b) & & & \\ a-b & 2b & -(a+b) & & \\ & \ddots & \ddots & \ddots & \\ & & a-b & 2b & -(a+b) \\ & & & a-b & b \end{pmatrix}_{l'\times l'},$$

$$l' = l \text{ or } l+2, i = 1, 2, \ldots, \frac{N-1}{2},$$

and $Q_{l'}$ $(l' = l, l-2)$ is a $l' \times l'$ zero matrix.

3 Existence and uniqueness of the ASE-I scheme solution

In order to discuss the existence and the uniqueness of the ASE-I scheme solution, we need to introduce the following three lemmas.

Lemma 1 (Kellogg [11, 17]) *If $\rho > 0$ and $(C + C^T)$ is a non-negative (or positive) definite, then $(\rho I + C)^{-1}$ exists, and*

$$\left\| (\rho I + C)^{-1} \right\|_2 \leq \rho^{-1}.$$

Lemma 2 (Kellogg [11, 17]) *If $(C + C^T)$ is a non-negative (or positive) definite, for any $\rho \geq 0$, then*

$$\left\| (I - \rho C)(I + \rho C)^{-1} \right\|_2 \leq 1.$$

Lemma 3 *G_1 and G_2 in the ASE-I scheme* (12) *for solving the nonlinear Leland equation are non-negative matrices.*

Proof We only need to prove $G_1 + G_1^T$ and $G_2 + G_2^T$ are non-negative matrices. Because of

$$G_{l'}^{(i)} + \left(G_{l'}^{(i)}\right)^T = \begin{pmatrix} 2b & -2b & & & \\ -2b & 4b & -2b & & \\ & \ddots & \ddots & \ddots & \\ & & -2b & 4b & -2b \\ & & & -2b & 2b \end{pmatrix}_{l' \times l'}, \quad l' = l \text{ or } l + 2,$$

we know that $G_{l'}^{(i)} + (G_{l'}^{(i)})^T$ is a diagonally dominant matrix and the diagonal elements of $G_{l'}^{(i)} + (G_{l'}^{(i)})^T$ are non-negative real numbers. Therefore, $G_{l'}^{(i)} + (G_{l'}^{(i)})^T$ is a non-negative matrix. In the same way, $\bar{G}_{l+1}^{(1)} + (\bar{G}_{l+1}^{(1)})^T$ and $\bar{G}_{l+1}^{(\frac{N+1}{2})} + (\bar{G}_{l+1}^{(\frac{N+1}{2})})^T$ are also non-negative matrices. Therefore, G_1 and G_2 are non-negative matrices. □

From the initial conditions and the boundary conditions of the nonlinear Leland equation, we know the difference solution of the first time layer. Assuming the value U_i^{2j} of the $(2j)$th time layer is known, the value U_i^{2j+1} of the $(2j + 1)$th time layer waits for calculating. From the ASE-I scheme (12), the matrix equation for calculating the value of the $(2j + 1)$th time layer is

$$(I + G_1)U^{2j+1} = (I - G_2)U^{2j} + b_1^{2j}. \tag{13}$$

Apparently the right of equation (13) is known and $(I + G_1)^{-1}$ exists by Lemma 3 and Lemma 1. Then equation (13) has a unique solution.

In the same way, applying the ASE-I scheme to calculate the value of the $(2j + 2)$th time layer, the matrix equation is

$$(I + G_2)U^{2j+2} = (I - G_1)U^{2j+1} + b_1^{2j+2}. \tag{14}$$

We could also prove that the matrix equation (14) has a unique solution. Then we could get the following.

Theorem 1 *The solution of the ASE-I scheme* (12) *for solving a nonlinear Leland equation exists and is unique*.

4 Stability of the ASE-I scheme

By eliminating U^{j+1} from equation (12), we obtain

$$U^{j+2} = YU^j + b',$$

here Y is the growth matrix of the ASE-I scheme. The growth matrix of the ASE-I scheme is

$$Y = (I + G_2)^{-1}(I - G_1)(I + G_1)^{-1}(I - G_2).$$

From Lemmas 1-3, we can get the following inequality easily:

$$\begin{aligned} &\bigl\|(I + G_2)^{-1}\bigr\|_2 \le 1, \\ &\bigl\|(I - G_i)(I + G_i)^{-1}\bigr\|_2 \le 1, \quad i = 1, 2. \end{aligned}$$

So

$$\begin{aligned} \bigl\|Y^n\bigr\|_2 &\le \bigl\|(I + G_2)^{-1}\bigr\|_2 \cdot \bigl\|(I - G_1)(I + G_1)^{-1}\bigr\|_2^n \bigl\|(I - G_2)(I + G_2)^{-1}\bigr\|_2^{n-1} \cdot \bigl\|(I - G_2)\bigr\|_2 \\ &\le \bigl\|(I - G_2)\bigr\|_2 \le 1 + 4b + 2a. \end{aligned}$$

Therefore we have the following theorem.

Theorem 2 *The ASE-I scheme* (12) *for solving the nonlinear Leland equation is absolutely stable*.

5 Accuracy of the ASE-I scheme

We take the inside points without interior boundary points as 'interior points'. From the segment construction of the ASE-I scheme, we know that the ASE-I scheme uses the classic E-I scheme at an 'interior point' of odd and even levels, and it uses the two improved Saul'yev asymmetric schemes at the 'interior boundary points'. The truncation error of the classic E-I scheme is of second order in time and space [8]. The ASE-I scheme just has a finite number of 'interior boundary points', so the overall accuracy of the ASE-I scheme is close to that of the C-N scheme.

The truncation error of the ASE-I scheme at the 'interior boundary points' will be given in the following. We denote the truncation error as $T_1(p,h)$ when we use (6), we denote the truncation error as $T_2(p,h)$ when we use (7), and we let each point of (6) and (7) be expanded as the Taylor series at the point (x_{i-1},τ_j), (x_{i+1},τ_j). Then we get

$$\begin{aligned} T_1(p,h) = {}&\left(\frac{\partial U}{\partial \tau} + h\frac{\partial^2 U}{\partial \tau\,\partial x} + \frac{h^2}{2}\frac{\partial^3 U}{\partial \tau\,\partial x^2} + \frac{p}{2}\frac{\partial^2 U}{\partial \tau^2} + \frac{ph}{2}\frac{\partial^3 U}{\partial \tau^2\,\partial x} + \frac{p^2}{6}\frac{\partial^3 U}{\partial \tau^3}\right) \\ &- D\left(\frac{\partial^2 U}{\partial x^2} + h\frac{\partial^3 U}{\partial x^3} + \frac{p}{h}\frac{\partial^2 U}{\partial x\,\partial \tau} + \frac{3p}{2}\frac{\partial^3 U}{\partial x^2\partial \tau} + \frac{p^2}{2h}\frac{\partial^3 U}{\partial x\,\partial \tau^2}\right) \end{aligned}$$

$$
\begin{aligned}
&- (D+L)\biggl(\frac{\partial U}{\partial x} + h\frac{\partial^2 U}{\partial x^2} + \frac{2h^2}{3}\frac{\partial^3 U}{\partial x^3} + \frac{p}{2h}\frac{\partial U}{\partial \tau} + p\frac{\partial^2 U}{\partial x\,\partial \tau} \\
&+ hp\frac{\partial^3 U}{\partial \tau\,\partial x^2} + \frac{p^2}{4h}\frac{\partial^2 U}{\partial \tau^2} + \frac{p^2}{2}\frac{\partial^3 U}{\partial x\,\partial \tau^2} + \frac{p^3}{12h}\frac{\partial^3 U}{\partial \tau^3}\biggr) + O\bigl(p^\alpha h^\beta\bigr), \\
T_2(p,h) = {}& \biggl(\frac{\partial U}{\partial \tau} - h\frac{\partial^2 U}{\partial \tau\,\partial x} + \frac{h^2}{2}\frac{\partial^3 U}{\partial \tau\,\partial x^2} + \frac{p}{2}\frac{\partial^2 U}{\partial \tau^2} - \frac{ph}{2}\frac{\partial^3 U}{\partial \tau^2\,\partial x} + \frac{p^2}{6}\frac{\partial^3 U}{\partial \tau^3}\biggr) \\
&- D\biggl(\frac{\partial^2 U}{\partial x^2} - h\frac{\partial^3 U}{\partial x^3} - \frac{p}{h}\frac{\partial^2 U}{\partial x\,\partial \tau} + \frac{3p}{2}\frac{\partial^3 U}{\partial x^2\partial \tau} - \frac{p^2}{2h}\frac{\partial^3 U}{\partial x\,\partial \tau^2}\biggr) \\
&- (D+L)\biggl(\frac{\partial U}{\partial x} - h\frac{\partial^2 U}{\partial x^2} + \frac{2h^2}{3}\frac{\partial^3 U}{\partial x^3} - \frac{p}{2h}\frac{\partial U}{\partial \tau} + p\frac{\partial^2 U}{\partial x\,\partial \tau} \\
&- hp\frac{\partial^3 U}{\partial \tau\,\partial x^2} - \frac{p^2}{4h}\frac{\partial^2 U}{\partial \tau^2} + \frac{p^2}{2}\frac{\partial^3 U}{\partial x\,\partial \tau^2} - \frac{p^3}{12h}\frac{\partial^3 U}{\partial \tau^3}\biggr) + O\bigl(p^\alpha h^\beta\bigr),
\end{aligned}
$$

where $\alpha + \beta = 4$. Because of

$$
\begin{aligned}
&\frac{\partial U}{\partial \tau} - D\frac{\partial^2 U}{\partial x^2} - (D+L)\frac{\partial U}{\partial x} = 0, \\
&\frac{\partial}{\partial x}\biggl(\frac{\partial U}{\partial \tau} - D\frac{\partial^2 U}{\partial x^2} - (D+L)\frac{\partial U}{\partial x}\biggr) = 0, \\
&\frac{\partial}{\partial \tau}\biggl(\frac{\partial U}{\partial \tau} - D\frac{\partial^2 U}{\partial x^2} - (D+L)\frac{\partial U}{\partial x}\biggr) = 0,
\end{aligned}
$$

we can get

$$
\begin{aligned}
T_1(p,h) = {}& \biggl(\frac{h^2}{2}\frac{\partial^3 U}{\partial \tau\,\partial x^2} + \frac{ph}{2}\frac{\partial^3 U}{\partial \tau^2\,\partial x} + \frac{p^2}{6}\frac{\partial^3 U}{\partial \tau^3}\biggr) \\
&- D\biggl(\frac{p}{h}\frac{\partial^2 U}{\partial x\,\partial \tau} + p\frac{\partial^3 U}{\partial x^2\,\partial \tau} + \frac{p^2}{2h}\frac{\partial^3 U}{\partial x\,\partial \tau^2}\biggr) \\
&- (D+L)\biggl(\frac{2h^2}{3}\frac{\partial^3 U}{\partial x^3} + \frac{p}{2h}\frac{\partial U}{\partial \tau} + \frac{p}{2}\frac{\partial^2 U}{\partial x\,\partial \tau} + hp\frac{\partial^3 U}{\partial \tau\,\partial x^2} + \frac{p^2}{4h}\frac{\partial^2 U}{\partial \tau^2} \\
&+ \frac{p^2}{2}\frac{\partial^3 U}{\partial x\,\partial \tau^2} + \frac{p^3}{12h}\frac{\partial^3 U}{\partial \tau^3}\biggr) + O\bigl(p^\alpha h^\beta\bigr), \\
T_2(p,h) = {}& \biggl(\frac{h^2}{2}\frac{\partial^3 U}{\partial \tau\,\partial x^2} - \frac{ph}{2}\frac{\partial^3 U}{\partial \tau^2\,\partial x} + \frac{p^2}{6}\frac{\partial^3 U}{\partial \tau^3}\biggr) \\
&- D\biggl(-\frac{p}{h}\frac{\partial^2 U}{\partial x\,\partial \tau} + p\frac{\partial^3 U}{\partial x^2\,\partial \tau} - \frac{p^2}{2h}\frac{\partial^3 U}{\partial x\,\partial \tau^2}\biggr) \\
&- (D+L)\biggl(\frac{2h^2}{3}\frac{\partial^3 U}{\partial x^3} - \frac{p}{2h}\frac{\partial U}{\partial \tau} + \frac{p}{2}\frac{\partial^2 U}{\partial x\,\partial \tau} - hp\frac{\partial^3 U}{\partial \tau\,\partial x^2} - \frac{p^2}{4h}\frac{\partial^2 U}{\partial \tau^2} \\
&+ \frac{p^2}{2}\frac{\partial^3 U}{\partial x\,\partial \tau^2} - \frac{p^3}{12h}\frac{\partial^3 U}{\partial \tau^3}\biggr) + O\bigl(p^\alpha h^\beta\bigr).
\end{aligned}
$$

Noticing that $T_1(p,h)$ and $T_2(p,h)$ contain the same form as regards the expression of the function, respectively, but we have the reversed symbol. For these items we have the following:

$$-D\frac{p}{h}\frac{\partial^2 U}{\partial x\,\partial\tau}\bigg|_{i-1}^{j} + D\frac{p}{h}\frac{\partial^2 U}{\partial x\,\partial\tau}\bigg|_{i+1}^{j} = 2pD\frac{\partial^3 U}{\partial x^2\,\partial\tau}\bigg|_{\xi}^{j}, \quad i-1 \le \xi \le i+1.$$

This part of the 'interior boundary point' can be offset when the ASE-I scheme alternatively uses (6) and (7) at different times. Ultimately, we can get the following theorem.

Theorem 3 *The truncation error of the ASE-I scheme* (12) *for solving a nonlinear Leland equation at interior points is* $O(p^2 + h^2)$, *and at the improved Saul'yev asymmetric schemes* (6), (7) *of interior boundary points it is* $O(p + h^2)$.

Hence the error of the points which are near the interior boundary point is bigger than that of the other interior points. The result will be proved in the following numerical experiments.

6 ASI-E parallel difference method

Imitating the method constructed in the ASE-I scheme, we give the ASI-E scheme for solving the nonlinear Leland equation.

On the odd level, we arrange the computation according to the rule of 'the implicit segment-the explicit segment-the implicit segment'. When it turns to the even level, the rule changes into 'the explicit segment-the implicit segment-the explicit segment'. Getting the ASI-E difference scheme for solving the nonlinear Leland equation, we have

$$\begin{cases} (I + G_2)U^{j+1} = (I - G_1)U^j + b_1^{j+1}, \\ (I + G_1)U^{j+2} = (I - G_2)U^{j+1} + b_1^{j+1}, \end{cases} \tag{15}$$

here $j = 1, 3, 5, \ldots$; G_1, G_2 and b_1 are as in the above definition.

Imitating the analytical and proved method of the ASE-I scheme (12), we have the following theorem.

Theorem 4 *The ASI-E scheme* (15) *for solving a nonlinear Leland equation is uniquely solvable, absolutely stable, its truncation error is* $O(p^2 + h^2)$ *at the interior points, and it is* $O(p + h^2)$ *at the interior boundary points.*

7 Numerical experiments

Numerical experiments will be done in Matlab 2008a, based on the Intel Core i5-4200 CPU@1.60GHz. We use the ASE-I scheme (12) and the ASI-E scheme (15) of this paper, the ASC-N scheme in [16] and the classic C-N scheme to calculate European call option prices with transaction costs. For the nonlinear Leland equation (1) it is very difficult to obtain an analytical solution [3, 4]. Therefore, we will let the numerical solution of the C-N scheme approximately substitute the exact solution of a European call option pricing problem with transaction costs and compare these various difference schemes.

Example We consider a European call option on stocks with transaction cost. Assuming the initial price of the underlying stock is 70 dollars, the strike price of an option is 50 dollars, the risk-free interest rate is 0.01 per year, the deadline of the option is 6 months, the volatility is 0.2 per year, the ratio of the transaction cost is 0.02, δt is 1/12.

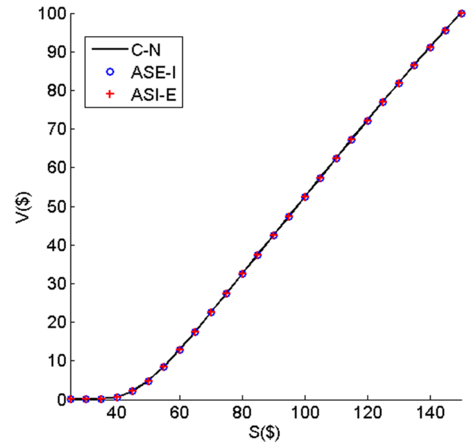


Figure 4 Numerical solutions' comparison of ASE-I, ASI-E, and C-N schemes.

Table 1 Comparison of several schemes' numerical results

Schemes	S ($)					Time (s)
	55	65	75	85	95	
C-N	8.4734	17.5879	27.4546	37.4391	47.4317	5.2358
ASC-N [16]	8.4734	17.5879	27.4546	37.4391	47.4317	1.2225
ASE-I	8.4734	17.5879	27.4546	37.4391	47.4317	0.5195
ASI-E	8.4734	17.5879	27.4546	37.4391	47.4317	0.4995

Solution We use the following symbols:

$$S = 70, \quad K = 50, \quad T = 6, \quad r = 0.1, \quad \sigma = 0.2, \quad k = 0.02, \quad \delta t = \frac{1}{12}.$$

Let

$$M^{+} = \ln 3.0, \quad M^{-} = -\ln 3.0, \quad m = 1{,}001, \quad n = 1{,}000, \quad l = 200, \quad N = \frac{m-1}{l} = 5.$$

First of all, we give the numerical solutions of the ASE-I and ASI-E schemes.

From Figure 4 and Table 1 we can see that the numerical solutions of the ASE-I and ASI-E schemes are very close to those of the C-N and ASC-N schemes.

Second, we regard the solution U_i^j of the classical C-N scheme as the control solution and the solution $\bar{U}_i^j$ of the other schemes as perturbation solutions. Let the grid ratio be $r1 = \frac{p}{h^2}$ and give the absolute error (AE) under the different $r1$. The definition of AE is as follows:

$$AE = \left|U_i^n - \bar{U}_i^n\right|.$$

Observing Figures 5, 6 and Tables 2, 3, we see that the AE of the numerical solutions between he ASE-I, ASI-E schemes, and C-N scheme has the same magnitudes as that of the ASC-N scheme, showing that the accuracy of the ASE-I and the ASI-E schemes is close to that of the ASC-N scheme. Because grid points correspond with the stock price (40$ or 100$) near the interior boundary and the AEs of these points are bigger than that of other points, this accords with the theory (see the details in Section 3 and Theorem 3). In addition, when $r1$ is increasing, the ASE-I and ASI-E schemes still have a good accuracy.

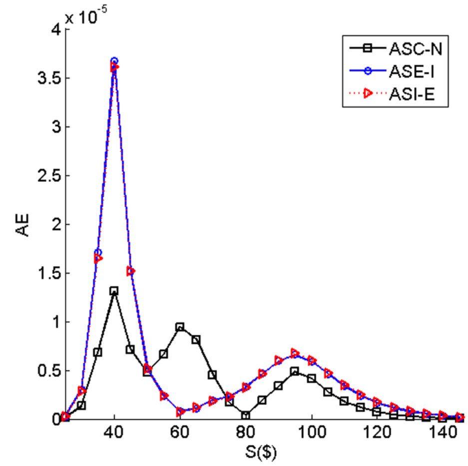


Figure 5 AE of numerical solution when $r1 \approx 2.066$.

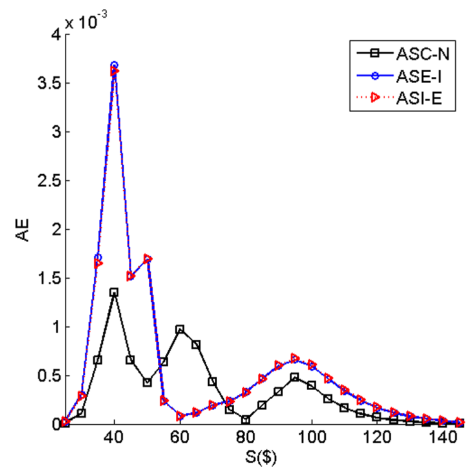


Figure 6 AE of numerical solution when $r1 \approx 20.66$.

Table 2 AE of numerical solution when $r1 \approx 2.066$ ($m = 1{,}001$, $n = 1{,}000$)

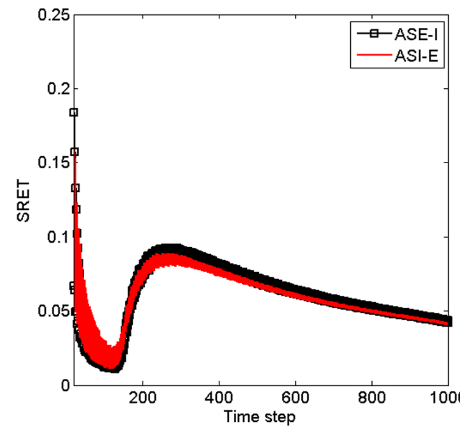
Schemes	S ($)				
	55	65	75	85	95
ASC-N [16]	6.581×10^{-6}	8.112×10^{-6}	1.677×10^{-6}	1.925×10^{-6}	4.838×10^{-6}
ASE-I	2.367×10^{-6}	1.106×10^{-6}	2.314×10^{-6}	4.638×10^{-6}	6.585×10^{-6}
ASI-E	2.323×10^{-6}	1.146×10^{-6}	2.232×10^{-6}	4.609×10^{-6}	6.693×10^{-6}

Table 3 AE of numerical solution when $r1 \approx 20.66$ ($m = 1{,}001$, $n = 100$)

Schemes	S ($)				
	55	65	75	85	95
ASC-N [16]	6.314×10^{-4}	8.119×10^{-4}	1.471×10^{-4}	1.922×10^{-4}	4.748×10^{-4}
ASE-I	2.380×10^{-4}	1.140×10^{-4}	2.319×10^{-4}	4.649×10^{-4}	6.615×10^{-4}
ASI-E	2.335×10^{-4}	1.180×10^{-4}	2.236×10^{-4}	4.620×10^{-4}	6.723×10^{-4}

Thirdly, we will give the proof of stability and the convergence order of the ASE-I and the ASI-E schemes. We analyze the sum of the relative error at every time level (SRET) and the convergence order in the temporal direction (COT) and the spatial direction (COS).

Figure 7 Curves of SRET at time layer.



The definitions of SRET, COT, and COS are as follows:

$$SRET(j) = \sum_{i=1}^{m} \frac{|U_i^j - \bar{U}_i^j|}{U_i^j},$$

$$COT = \frac{\log(L^2_{\Delta\tau_1}/L^2_{\Delta\tau_2})}{\log(\Delta\tau_1/\Delta\tau_2)},$$

$$COS = \frac{\log(L^2_{\Delta x_1}/L^2_{\Delta x_2})}{\log(\Delta x_1/\Delta x_2)}.$$

The error of the L^2 measurement norm is defined as follows:

$$L^2_{\Delta\tau,m} = \left\| U_m^j - \bar{U}_m^j \right\|^2 = \left\{ \sum_{j=1}^{n} \left(U_m^j - \bar{U}_m^j\right)^2 p \right\}^{\frac{1}{2}},$$

$$L^2_{n,\Delta x} = \left\| U_i^n - \bar{U}_i^n \right\|^2 = \left\{ \sum_{i=1}^{m} \left(U_i^n - \bar{U}_i^n\right)^2 h \right\}^{\frac{1}{2}}.$$

From Figure 7 we can see that the SRET of the ASE-I and ASI-E schemes is larger in the beginning and decreasing along with the movement of the time step; and it is bounded. This shows that the ASE-I and ASI-E schemes have better stability.

Table 4 and Table 5 show that the convergence order of the ASE-I and ASI-E schemes in the temporal direction is approaching $O(p^2)$ and in the spatial direction it is $O(h^2)$.

Next, observing Table 1, the computing times of the ASE-I and ASI-E schemes (0.5195s, 0.4995s) are less than that of the C-N and ASC-N schemes (5.2358s, 1.2225s). In order to better compare the computing efficiency of the several difference schemes, we choose different points at the space grid and let $m = 101, 301, 501, 701, 901, 1{,}001$, $n = 1{,}000$. Because the calculated amount of the ASI-E scheme is the same as that of the ASI-E scheme, we just need to compare the ASE-I scheme, the ASC-N scheme, and the C-N scheme, and the results are in Figure 8 and Table 6.

From Figure 8 and Table 6 we see that when the number of grid points we need calculated is greater than a certain range, the ASE-I and ASI-E schemes of this paper show a clear superiority in computation time. With the increase of the grid number, the computing time of the difference schemes rises for the nonlinear Leland equation. But the

Table 4 COT of ASC-N, ASE-I, and ASI-E schemes when $m = 1{,}001$

Time grid (n)	ASC-N [16]		ASE-I		ASI-E	
	L^2-error	COT	L^2-error	COT	L^2-error	COT
100	1.3959×10^{-7}	–	1.4273×10^{-7}	–	1.4249×10^{-7}	–
400	1.0276×10^{-8}	1.8818	1.0318×10^{-8}	1.8949	1.0315×10^{-8}	1.8940
700	3.3784×10^{-9}	1.9123	3.3862×10^{-9}	1.9226	3.3856×10^{-9}	1.9218
1,000	1.6579×10^{-9}	1.9252	1.6606×10^{-9}	1.9342	1.6604×10^{-9}	1.9335

Table 5 COS of ASC-N, ASE-I, and ASI-E schemes when $n = 1{,}000$

Space grid (m)	ASC-N [16]		ASE-I		ASI-E	
	L^2-error	COS	L^2-error	COS	L^2-error	COS
301	2.1853×10^{-7}	–	4.6645×10^{-7}	–	4.3609×10^{-7}	–
501	7.2702×10^{-7}	2.3592	1.4714×10^{-6}	2.2754	1.3071×10^{-6}	2.1544
701	1.3532×10^{-6}	2.1567	2.6557×10^{-6}	2.3109	2.8593×10^{-6}	2.2244
901	1.8025×10^{-6}	1.9545	3.7376×10^{-6}	2.0935	3.5504×10^{-6}	1.9926

Figure 8 Comparison of the three schemes' calculation time.

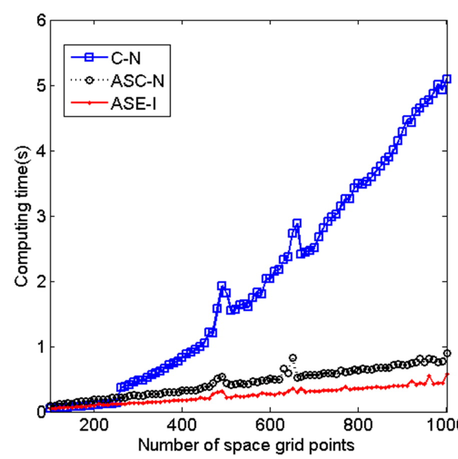


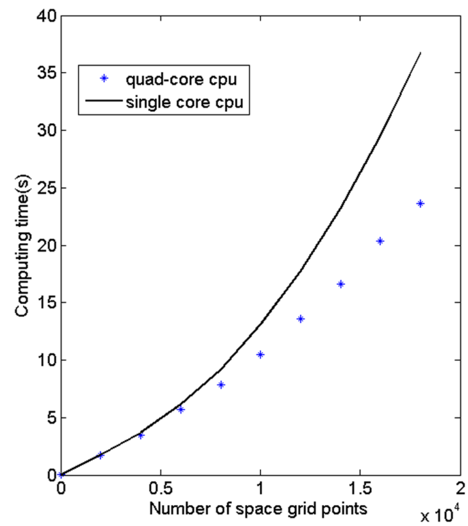
Table 6 Comparison of the three schemes' calculation time at $n = 1{,}000$

Space grid (m)	Time (s)		
	C-N	ASC-N [16]	ASE-I
101	0.0526	0.0880	0.0563
301	0.4738	0.2396	0.1371
501	1.8090	0.4338	0.2221
701	2.5048	0.5632	0.3075
901	4.2826	0.7152	0.3999
1,001	5.0936	0.8897	0.5744

increased amplitude of the computing time of the C-N scheme is greater than that of the ASE-I, ASC-N schemes. The computing time of the ASE-I scheme saves nearly 81% for the C-N scheme by calculating and saves nearly 40% for the ASC-N scheme, showing the computing efficiency of the ASE-I scheme is best.

As is well known, the parallel scheme has superiority in computing time. But when the amount of calculation data is small, the impact of the data communication on the cycle can reduce the computing efficiency. For programming of the ASE-I scheme in our example, we, respectively, adopt the serial for loop and the parallel parfor loop. For the serial

Figure 9 Comparison of the serial and parallel calculation time.



for loop, numerical array and the loop body are performed in the same Matlab process, so there are no data communication problems. But, for a parallel parfor loop, numerical arrays are created in the Matlab client, while parallel computing of the parfor loop body is finished under the Matlab worker, so numerical arrays need to be transmitted from the Matlab client to the Matlab worker. Because of taking up time and processor resources in data communication, we need to consider the data communication problem in parallel programming [18].

Last, we give the computation time of the ASE-I scheme in the case of the single-core cpu and quad-core cpu. The result is in Figure 9.

Figure 9 and the first line of Table 6 show that when the number of grid points is less than a certain range, the serial scheme is more effective than the parallel scheme, meaning that the data communication problems have an effect on the execution efficiency of the programming in the case of small data quantity (grid points). And when we have a larger amount of data (grid points), the influence of the loop body execution is greater than that of the data communication, meaning that using the parallel computing is more effective.

In the practical application, in order to make the numerical results more precise, we tend to a dense mesh and the number of space points becomes higher. The ASE-I parallel method has obvious localization characteristics in computing and communications and is very suitable for large-scale parallel computing in a distributed storage system on the application.

8 Conclusion

For the nonlinear Leland equation, this paper constructs the ASE-I and ASI-E parallel difference schemes with unconditional stability and high accuracy characteristics. Theoretical analysis gets the result that the numerical solutions of the ASE-I and ASI-E schemes are very close to that of the C-N and ASC-N schemes. Under the same computing accuracy, the ASE-I and ASI-E schemes are greatly improved as regards the computing efficiency. Numerical experiment demonstrates that the computing time of the ASE-I and ASI-E schemes save nearly 81% for the C-N scheme and saved nearly 40% for the ASC-N scheme, showing the practicability of this kind of parallel difference schemes for solving a nonlinear Leland equation.

The ASE-I and ASI-E schemes given by this paper can be extended to solve other nonlinear B-S models with transaction costs, such as the Barles-Soner model and the risk adjustment pricing model, and they can better solve the timeliness problem of the option pricing.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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