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Positive asymptotically almost periodic solutions for hematopoiesis model

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Abstract

By using the fixed point theory and Lyapunov functional, we establish the existence and stability of asymptotically almost periodic solution to hematopoiesis of the form $x'(t) = -a(t)x(t) + \sum_{i=1}^{k} \frac{b_i(t)}{1+x^n(t-\tau_i(t))}$, $t \in \mathbb{R}$. Unlike many previous related results, we do not assume the condition $\inf_{t\in\mathbb{R}} a(t) > 0$, which is a key assumption in their proofs.

MSC: 34K14

Keywords: asymptotically almost periodic; almost periodic; hematopoiesis

1 Introduction

In this paper, we consider the following hematopoiesis model:

$$x'(t) = -a(t)x(t) + \sum_{i=1}^{k} \frac{b_i(t)}{1 + x^n(t - \tau_i(t))}, \quad t \in \mathbb{R}, \tag{1.1}$$

where $n > 0$, k is a positive integer, $a : \mathbb{R} \to \mathbb{R}$ is continuous, and $b_i, \tau_i : \mathbb{R} \to \mathbb{R}^+$ are all continuous functions for $i = 1, 2, \ldots, k$.

The above model originates from the work of Mackey and Glass [1], where they proposed the following nonlinear delay differential equation:

$$h'(t) = -\alpha h(t) + \frac{\beta}{1 + h^n(t - \tau)} \tag{1.2}$$

as an appropriate model of hematopoiesis, which describes the process of production of all types of blood cells generated by a remarkable self-regulated system that is responsive to the demands put upon it. In medical terms, $h(t)$ denotes the density of mature cells in the blood circulation at time t and τ is the time delay between the production of immature cells in the bone marrow and their maturation for release in circulating bloodstream. It is assumed that the cells are lost from the circulation at a rate α, and the flux of the cells into the circulation from the stem cell compartment depends on the density of mature cells at the previous time $t - \tau$.

Recently, the existence of periodic solutions and almost periodic solutions for equation (1.1) and its various forms have attracted much attention (see, *e.g.*, [2–12] and references therein). Stimulated by these works, we aim to make further study of this topic. As one will see, there are two differences of our work from many earlier works on almost periodic solutions to equation (1.1). The first difference is that we do not assume that $\inf_{t\in\mathbb{R}} a(t) > 0$ (even do not assume that a is nonnegative). The second difference is that we investigate the existence and stability of asymptotically almost periodic solution to equation (1.1). In fact, to the best of our knowledge, it seems that until now there is no results concerning asymptotically almost periodic solution to equation (1.1). Recall that in 1940s, Fréchet introduced the notion of asymptotically almost periodicity, which turns out to be one of the most interesting and important generalizations of almost periodicity. So, we think it will be of interest for some colleagues to investigate the existence and stability of asymptotically almost periodic solution to equation (1.1). That is the main motivation of this paper.

Throughout the rest of this paper, we denote by $\mathbb{R}$ the set of real numbers, by $\mathbb{R}^+$ the set of nonnegative real numbers, and by $\mathbb{N}$ the set of positive integers. Moreover, for each bounded function $f : \mathbb{R} \to \mathbb{R}$, we denote

$$f^+ = \sup_{t\in\mathbb{R}} f(t), \qquad f^- = \inf_{t\in\mathbb{R}} f(t).$$

Next, let us recall some definition and basic properties for almost periodic function and asymptotically almost periodic functions. For more details, we refer the reader to [13–15].

Definition 1.1 A set $E \subset \mathbb{R}$ is called relatively dense if there exists a number $l > 0$ such that

$$[a, a + l] \cap E \neq \emptyset$$

for every $a \in \mathbb{R}$.

Definition 1.2 A continuous function $f : \mathbb{R} \to \mathbb{R}$ is called almost periodic if for every $\varepsilon > 0$, the set

$$P_\varepsilon := \Big\{ \tau \in \mathbb{R} : \sup_{t\in\mathbb{R}} \big| f(t + \tau) - f(t) \big| < \varepsilon \Big\}$$

is relatively dense. We denote the set of all such functions by $\mathrm{AP}(\mathbb{R}, \mathbb{R})$.

Recall that, for every $f \in \mathrm{AP}(\mathbb{R}, \mathbb{R})$, the limit

$$\lim_{T\to+\infty} \frac{1}{T} \int_0^T f(t)\,dt$$

exists. Throughout the rest of this paper, we denote

$$M(f) = \lim_{T\to+\infty} \frac{1}{T} \int_0^T f(t)\,dt, \quad f \in \mathrm{AP}(\mathbb{R}, \mathbb{R}).$$

Also, we denote by $C_0(\mathbb{R}, \mathbb{R})$ be the set of all continuous functions $f : \mathbb{R} \to \mathbb{R}$ with $\lim_{t\to\infty} f(t) = 0$.

Definition 1.3 A continuous function $f:\mathbb{R}\to\mathbb{R}$ is called asymptotically almost periodic if there exist $g\in \mathrm{AP}(\mathbb{R},\mathbb{R})$ and $h\in C_0(\mathbb{R},\mathbb{R})$ such that

$$f(t)=g(t)+h(t),\quad t\in\mathbb{R}.$$

We denote the set of all such functions by $\mathrm{AAP}(\mathbb{R},\mathbb{R})$. Moreover, we denote by $\mathrm{AAP}(\mathbb{R},\mathbb{R}^+)$ the set of all nonnegative asymptotically almost periodic functions from $\mathbb{R}$ to $\mathbb{R}$.

Lemma 1.4 [15] *Let $f,g\in \mathrm{AAP}(\mathbb{R},\mathbb{R})$. Then the following assertions hold*:

(a) $\mathrm{AAP}(\mathbb{R},\mathbb{R})$ *is a Banach space under the norm* $\|f\|=\sup_{t\in\mathbb{R}}|f(t)|$.
(b) $f+g\in \mathrm{AAP}(\mathbb{R},\mathbb{R})$ *and* $f\cdot g\in \mathrm{AAP}(\mathbb{R},\mathbb{R})$.
(c) $f/g\in \mathrm{AAP}(\mathbb{R},\mathbb{R})$ *provided that* $\inf_{t\in\mathbb{R}}|g(t)|>0$.

2 Preliminary results

In this section, we present some essential lemmas which are needed in proving the main results.

Lemma 2.1 *Let $a\in \mathrm{AAP}(\mathbb{R},\mathbb{R})$ with $M(a)>0$. Then for every $\alpha\in(0,M(a))$, there exists $T_0>0$ such that, for all $s,t\in\mathbb{R}$ with $s\le t$, we have*

$$\int_t^s a(u)\,du\le\alpha(T_0+s-t).$$

Proof Since $a\in \mathrm{AAP}(\mathbb{R},\mathbb{R})$ and $M(a)>\alpha>0$, it follows from [15], p.208, Lemma 1.5, that there exists $T>0$ such that

$$\int_t^s a(u)\,du<\alpha(s-t)$$

for all $s,t\in\mathbb{R}$ with $s-t<-T$. On the other hand, we have

$$\int_t^s a(u)\,du\le\|a\|\cdot T\le\|a\|\cdot T+\alpha(T+s-t)=\alpha\left[T\cdot\left(\frac{\|a\|}{\alpha}+1\right)+s-t\right]$$

for all $s,t\in\mathbb{R}$ with $-T\le s-t\le 0$. Then, taking $T_0=T\cdot(\frac{\|a\|}{\alpha}+1)$, the conclusion follows. □

Lemma 2.2 *Let $f,a\in \mathrm{AAP}(\mathbb{R},\mathbb{R})$ with $M(a)>0$. Then the equation*

$$x'(t)=-a(t)x(t)+f(t)\tag{2.1}$$

has a unique asymptotically almost periodic solution.

Proof By [15], p.209, Theorem 1.6, equation (2.1) has a unique bounded solution $x(t)$ given by

$$x(t)=\int_{-\infty}^t e^{-\int_s^t a(r)\,dr}f(s)\,ds.\tag{2.2}$$

Next, let us show that the unique bounded solution given by (2.2) is asymptotically almost periodic.

Let $f = g + h$, where $g \in \mathrm{AP}(\mathbb{R},\mathbb{R})$ and $h \in C_0(\mathbb{R},\mathbb{R})$. Denote

$$G(t) = \int_{-\infty}^{t} e^{-\int_s^t a(r)\,dr} g(s)\,ds, \qquad H(t) = \int_{-\infty}^{t} e^{-\int_s^t a(r)\,dr} h(s)\,ds, \quad t \in \mathbb{R}.$$

Fix $\alpha \in (0, M(a))$. By Lemma 2.1, there exists $T_0 > 0$ such that

$$\int_t^s a(u)\,du \le \alpha(T_0 + s - t) \tag{2.3}$$

for all $s, t \in \mathbb{R}$ with $s \le t$. Now, we divide the remaining proof by three steps.

Step 1. $H \in C_0(\mathbb{R},\mathbb{R})$.

For every $\varepsilon > 0$, there exists $T_1 > 0$ such that $\int_{T_1}^{+\infty} e^{-\alpha s}\,ds < \varepsilon$. Also, since $h \in C_0(\mathbb{R},\mathbb{R})$, there exists $T_2 > T_1$ such that

$$\sup_{s\in[0,T_1]} \bigl|h(t-s)\bigr| < \varepsilon, \quad t \ge T_2.$$

Then, by (2.3), we conclude that, for $t \ge T_2$,

$$\begin{aligned} \bigl|H(t)\bigr| &= \left|\int_{-\infty}^{t} e^{-\int_s^t a(r)\,dr} h(s)\,ds\right| \\ &= \left|\int_0^{+\infty} e^{-\int_{t-s}^t a(r)\,dr} h(t-s)\,ds\right| \\ &\le \left|\int_0^{T_1} e^{-\int_{t-s}^t a(r)\,dr} h(t-s)\,ds\right| + \left|\int_{T_1}^{+\infty} e^{-\int_{t-s}^t a(r)\,dr} h(t-s)\,ds\right| \\ &\le e^{\alpha T_0} \int_0^{T_1} e^{-\alpha s}\,ds \cdot \sup_{s\in[0,T_1]} \bigl|h(t-s)\bigr| + e^{\alpha T_0} \int_{T_1}^{+\infty} e^{-\alpha s}\,ds \cdot \|h\| \\ &< e^{\alpha T_0}\left(\frac{1}{\alpha} + \|h\|\right)\varepsilon, \end{aligned}$$

which means that $H \in C_0(\mathbb{R},\mathbb{R})$.

Step 2. For every $n \in \mathbb{N}$, $G_n \in \mathrm{AAP}(\mathbb{R},\mathbb{R})$, where $G_n(t) = \int_0^n e^{-\int_{t-s}^t a(r)\,dr} g(t-s)\,ds$ for all $t \in \mathbb{R}$.

Let $a = b + c$, where $b \in \mathrm{AP}(\mathbb{R},\mathbb{R})$, $c \in C_0(\mathbb{R},\mathbb{R})$. Then

$$G_n(t) = \int_0^n e^{-\int_{t-s}^t b(r)\,dr} g(t-s)\,ds + \int_0^n \bigl[e^{-\int_{t-s}^t a(r)\,dr} - e^{-\int_{t-s}^t b(r)\,dr}\bigr] g(t-s)\,ds = I_n(t) + J_n(t),$$

where

$$I_n(t) = \int_0^n e^{-\int_{t-s}^t b(r)\,dr} g(t-s)\,ds, J_n(t) = \int_0^n \bigl[e^{-\int_{t-s}^t a(r)\,dr} - e^{-\int_{t-s}^t b(r)\,dr}\bigr] g(t-s)\,ds.$$

We first show that $I_n \in \mathrm{AP}(\mathbb{R},\mathbb{R})$ for every $n \in \mathbb{N}$. For every $\varepsilon > 0$, there exists $\delta \in (0,\varepsilon)$ such that, for all $x_1, x_2 \in [-n\|a\|, n\|a\|]$ with $|x_1 - x_2| \le \delta$, we have

$$\bigl|e^{x_1} - e^{x_2}\bigr| < \varepsilon. \tag{2.4}$$

For the above $\delta > 0$, there exists a relatively dense set $P_\delta \subset \mathbb{R}$ such that, for all $\tau \in P_\delta$,

$$\sup_{t\in\mathbb{R}}\bigl|g(t+\tau)-g(t)\bigr| < \delta, \qquad \sup_{t\in\mathbb{R}}\bigl|b(t+\tau)-b(t)\bigr| < \frac{\delta}{n}. \tag{2.5}$$

Combining (2.4) and (2.5), we conclude that, for all $\tau \in P_\delta$,

$$\begin{aligned}
&\bigl|I_n(t+\tau)-I_n(t)\bigr| \\
&\quad\le \left|\int_0^n e^{-\int_{t+\tau-s}^{t+\tau} b(r)\,dr} g(t+\tau-s)\,ds - \int_0^n e^{-\int_{t-s}^{t} b(r)\,dr} g(t+\tau-s)\,ds\right| \\
&\qquad + \left|\int_0^n e^{-\int_{t-s}^{t} b(r)\,dr} g(t+\tau-s)\,ds - \int_0^n e^{-\int_{t-s}^{t} b(r)\,dr} g(t-s)\,ds\right| \\
&\quad\le \int_0^n \bigl|e^{-\int_{t+\tau-s}^{t+\tau} b(r)\,dr} - e^{-\int_{t-s}^{t} b(r)\,dr}\bigr|\,ds\cdot\|g\| + \int_0^n e^{-\int_{t-s}^{t} b(r)\,dr}\,ds\cdot\delta \\
&\quad\le n\varepsilon\|g\| + n\delta e^{n\|b\|} < n\bigl(\|g\| + e^{n\|b\|}\bigr)\varepsilon,
\end{aligned}$$

where we have used the fact that, for all $s \in [0,n]$,

$$\begin{aligned}
\left|\int_{t+\tau-s}^{t+\tau} b(r)\,dr - \int_{t-s}^{t} b(r)\,dr\right| &= \left|\int_{-s}^{0} b(r+t+\tau)\,dr - \int_{-s}^{0} b(r+t)\,dr\right| \\
&\le \int_{-s}^{0}\bigl|b(r+t+\tau)-b(r+t)\bigr|\,dr \le s\cdot\frac{\delta}{n} \le \delta.
\end{aligned}$$

Moreover, we claim that $J_n \in C_0(\mathbb{R},\mathbb{R})$ for every $n \in \mathbb{N}$. In fact, since $c \in C_0(\mathbb{R},\mathbb{R})$, for the above $\delta > 0$, there exists sufficiently large $M > 0$ such that, for all $t \ge M$ and $s \in [0,n]$, we have

$$\left|\int_{t-s}^{t} a(r)\,dr - \int_{t-s}^{t} b(r)\,dr\right| \le \int_{t-s}^{t}\bigl|c(r)\bigr|\,dr < \delta.$$

Combining this with (2.4), we have, for all $t \ge M$,

$$\bigl|J_n(t)\bigr| \le \int_0^n \bigl|e^{-\int_{t-s}^{t} a(r)\,dr} - e^{-\int_{t-s}^{t} b(r)\,dr}\bigr|\,ds\cdot\|g\| \le n\varepsilon\|g\|.$$

This completes the proof of Step 2.

Step 3. $G \in \mathrm{AAP}(\mathbb{R},\mathbb{R})$ and $x \in \mathrm{AAP}(\mathbb{R},\mathbb{R})$.

By (2.3), we have

$$\begin{aligned}
\sup_{t\in\mathbb{R}}\bigl|G(t)-G_n(t)\bigr| &= \sup_{t\in\mathbb{R}}\left|\int_n^{+\infty} e^{-\int_{t-s}^{t} a(r)\,dr} g(t-s)\,ds\right| \\
&\le e^{\alpha T_0}\|g\|\int_n^{+\infty} e^{-\alpha s}\,ds = \frac{e^{\alpha T_0}\|g\|}{\alpha}e^{-\alpha n} \to 0, \quad n\to\infty.
\end{aligned}$$

Therefore, by Step 2, we conclude that $G \in \mathrm{AAP}(\mathbb{R},\mathbb{R})$ and thus $x \in \mathrm{AAP}(\mathbb{R},\mathbb{R})$. This completes the proof. □

Lemma 2.3 *Let $x, \tau \in \mathrm{AAP}(\mathbb{R},\mathbb{R})$. Then $x(\cdot - \tau(\cdot)) \in \mathrm{AAP}(\mathbb{R},\mathbb{R})$.*

Proof Let

$$x = y + z, \qquad \tau = \tau_1 + \tau_2,$$

where $y, \tau_1 \in \mathrm{AP}(\mathbb{R},\mathbb{R})$ and $z, \tau_2 \in C_0(\mathbb{R},\mathbb{R})$. It is not difficult to see that

$$y\big(\cdot - \tau_1(\cdot)\big) \in \mathrm{AP}(\mathbb{R},\mathbb{R}).$$

In view of the boundedness of τ and the uniform continuity of y, we have

$$z\big(\cdot - \tau(\cdot)\big) \in C_0(\mathbb{R},\mathbb{R})$$

and

$$y\big(\cdot - \tau(\cdot)\big) - y\big(\cdot - \tau_1(\cdot)\big) \in C_0(\mathbb{R},\mathbb{R}),$$

respectively. Observing that

$$x\big(t - \tau(t)\big) = y\big(t - \tau_1(t)\big) + z\big(t - \tau(t)\big) + y\big(t - \tau(t)\big) - y\big(t - \tau_1(t)\big), \quad t \in \mathbb{R},$$

we obtain $x(\cdot - \tau(\cdot)) \in \mathrm{AAP}(\mathbb{R},\mathbb{R})$. □

3 Main results

In order to obtain our existence theorem, we make the following assumptions:

(H0) $a \in \mathrm{AAP}(\mathbb{R},\mathbb{R})$ with $M(a) > 0$, and $b_i, \tau_i \in \mathrm{AAP}(\mathbb{R},\mathbb{R}^+)$ with $b_i^- > 0$ for all $i = 1, 2, \ldots, k$.

(H1) There exists $\alpha \in (0, M(a))$ such that

$$M_2 := \frac{\sum_{i=1}^k \frac{b_i^-}{1+M_1^n}}{a^+} \le \frac{e^{\alpha T_0}\sum_{i=1}^k b_i^+}{\alpha} := M_1,$$

where T_0 is defined in Lemma 2.1.

(H2) For the case of $n \in (0,1]$, we have

$$\frac{e^{\alpha T_0}\sum_{i=1}^k b_i^+}{\alpha} < \frac{(1+M_2^n)^2 M_2^{1-n}}{n};$$

for the case of $n > 1$, we have

$$\frac{e^{\alpha T_0}\sum_{i=1}^k b_i^+}{\alpha} < \frac{4n}{n^2-1}\sqrt[n]{\frac{n-1}{n+1}}.$$

Theorem 3.1 *Under the assumptions* (H0)-(H2), *equation* (1.1) *has a unique asymptotically almost periodic solution in*

$$\Omega = \big\{\varphi \in \mathrm{AAP}(\mathbb{R},\mathbb{R}) : M_2 \le \varphi(t) \le M_1, \forall t \in \mathbb{R}\big\}.$$

Proof Fix $\varphi \in \Omega$. Let us consider the following differential equation:

$$x'(t) = -a(t)x(t) + \sum_{i=1}^{k} \frac{b_i(t)}{1 + \varphi^n(t - \tau_i(t))}. \tag{3.1}$$

It follows from Lemma 2.3 that $\varphi(\cdot - \tau_i(\cdot)) \in \mathrm{AAP}(\mathbb{R}, \mathbb{R})$. Then, by Lemma 1.4, we can obtain

$$\sum_{i=1}^{k} \frac{b_i(\cdot)}{1 + \varphi^n(\cdot - \tau_i(\cdot))} \in \mathrm{AAP}(\mathbb{R}, \mathbb{R}).$$

By Lemma 2.2, equation (3.1) has a unique asymptotically almost periodic solution given by

$$x^{\varphi}(t) = \int_{-\infty}^{t} e^{-\int_s^t a(r)\,dr} \sum_{i=1}^{k} \frac{b_i(s)}{1 + \varphi^n(s - \tau_i(s))}\, ds, \quad t \in \mathbb{R}. \tag{3.2}$$

Now we define a mapping T on Ω by

$$(T\varphi)(t) = x^{\varphi}(t), \quad \varphi \in \Omega, t \in \mathbb{R}.$$

Next, we show that $T(\Omega) \subset \Omega$. It suffices to prove that

$$M_2 \leq (T\varphi)(t) \leq M_1$$

for all $t \in \mathbb{R}$ and $\varphi \in \Omega$. For every $t \in \mathbb{R}$ and $\varphi \in \Omega$, by Lemma 2.1, we have

$$\begin{aligned}
(T\varphi)(t) &= x^{\varphi}(t) \\
&= \int_{-\infty}^{t} e^{-\int_s^t a(r)\,dr} \sum_{i=1}^{k} \frac{b_i(s)}{1 + \varphi^n(s - \tau_i(s))}\, ds \\
&\leq \int_{-\infty}^{t} e^{-\int_s^t a(r)\,dr} \sum_{i=1}^{k} b_i(s)\, ds \\
&\leq \sum_{i=1}^{k} b_i^+ \int_{-\infty}^{t} e^{-\int_s^t a(r)\,dr}\, ds \\
&= \sum_{i=1}^{k} b_i^+ \int_{0}^{+\infty} e^{-\int_{t-s}^t a(r)\,dr}\, ds \\
&\leq \sum_{i=1}^{k} b_i^+ \int_{0}^{+\infty} e^{\alpha T_0} e^{-\alpha s}\, ds \\
&= \frac{e^{\alpha T_0} \sum_{i=1}^{k} b_i^+}{\alpha} = M_1.
\end{aligned}$$

Moreover, for every $t \in \mathbb{R}$ and $\varphi \in \Omega$, we have

$$\begin{aligned}
(T\varphi)(t) &= x^{\varphi}(t) \\
&= \int_{-\infty}^{t} e^{-\int_s^t a(r)\,dr} \sum_{i=1}^{k} \frac{b_i(s)}{1 + \varphi^n(s - \tau_i(s))}\, ds
\end{aligned}$$

$$
\begin{aligned}
&\geq \int_{-\infty}^{t} e^{-\int_s^t a(r)\,dr} \sum_{i=1}^{k} \frac{b_i(s)}{1+M_1^n}\,ds \\
&\geq \sum_{i=1}^{k} \frac{b_i^-}{1+M_1^n} \int_0^{+\infty} e^{-\int_{t-s}^t a(r)\,dr}\,ds \\
&\geq \sum_{i=1}^{k} \frac{b_i^-}{1+M_1^n} \int_0^{+\infty} e^{-a^+ s}\,ds \\
&= \frac{\sum_{i=1}^{k} \frac{b_i^-}{1+M_1^n}}{a^+} = M_2.
\end{aligned}
$$

Thus, T is a self-mapping from Ω to Ω.

Next, let us show that T is a contraction mapping. We consider two cases.

Case I. $n \in (0,1]$.

By mean value theorem and direct calculations, one can obtain

$$
\left| \frac{1}{1+x^n} - \frac{1}{1+y^n} \right| \leq \frac{n}{(1+M_2^n)^2 M_2^{1-n}} \cdot |x-y| \tag{3.3}
$$

for all $x, y \geq M_2$. By using (3.3) and Lemma 2.1, we conclude, for every $\varphi, \psi \in \Omega$,

$$
\begin{aligned}
&\|T\varphi - T\psi\| \\
&\quad = \sup_{t\in\mathbb{R}} \left|(T\varphi)(t) - (T\psi)(t)\right| \\
&\quad = \sup_{t\in\mathbb{R}} \left| \int_{-\infty}^{t} e^{-\int_s^t a(r)\,dr} \sum_{i=1}^{k} b_i(s) \left[\frac{1}{1+\varphi^n(s-\tau_i(s))} - \frac{1}{1+\psi^n(s-\tau_i(s))} \right] ds \right| \\
&\quad \leq \sup_{t\in\mathbb{R}} \int_{-\infty}^{t} e^{-\int_s^t a(r)\,dr} \sum_{i=1}^{k} b_i(s) \cdot \frac{n}{(1+M_2^n)^2 M_2^{1-n}} \cdot \left|\varphi\bigl(s-\tau_i(s)\bigr) - \psi\bigl(s-\tau_i(s)\bigr)\right| ds \\
&\quad \leq \frac{n \sum_{i=1}^{k} b_i^+}{(1+M_2^n)^2 M_2^{1-n}} \cdot \|\varphi - \psi\| \cdot \sup_{t\in\mathbb{R}} \int_{-\infty}^{t} e^{-\int_s^t a(r)\,dr}\,ds \\
&\quad \leq \frac{n e^{\alpha T_0} \sum_{i=1}^{k} b_i^+}{\alpha (1+M_2^n)^2 M_2^{1-n}} \cdot \|\varphi - \psi\|.
\end{aligned}
$$

Case II. $n > 1$.

By the mean value theorem and direct calculations, one can obtain

$$
\left| \frac{1}{1+x^n} - \frac{1}{1+y^n} \right| \leq \frac{n^2-1}{4n} \sqrt[n]{\frac{n+1}{n-1}} \cdot |x-y| \tag{3.4}
$$

for all $x, y \geq 0$. By using (3.4) and Lemma 2.1, we conclude that, for every $\varphi, \psi \in \Omega$,

$$
\begin{aligned}
&\|T\varphi - T\psi\| \\
&\quad = \sup_{t\in\mathbb{R}} \left| \int_{-\infty}^{t} e^{-\int_s^t a(r)\,dr} \sum_{i=1}^{k} b_i(s) \left[\frac{1}{1+\varphi^n(s-\tau_i(s))} - \frac{1}{1+\psi^n(s-\tau_i(s))} \right] ds \right|
\end{aligned}
$$

$$\le \sup_{t\in\mathbb{R}} \int_{-\infty}^{t} e^{-\int_s^t a(r)\,dr} \sum_{i=1}^{k} b_i(s) \cdot \frac{n^2-1}{4n} \sqrt[n]{\frac{n+1}{n-1}} \cdot \left|\varphi\left(s-\tau_i(s)\right) - \psi\left(s-\tau_i(s)\right)\right| ds$$
$$\le \frac{e^{\alpha T_0} \sum_{i=1}^{k} b_i^+}{\alpha} \cdot \frac{n^2-1}{4n} \sqrt[n]{\frac{n+1}{n-1}} \cdot \|\varphi - \psi\|.$$

In all cases, by (H2), T is a contraction. Thus, T has a unique fixed point in Ω, *i.e.*, equation (1.1) has a unique asymptotically almost periodic solution in Ω. □

Remark 3.2 Compared with most earlier results concerning almost periodic solutions to equation (1.1), in Theorem 3.1, we do not assume that $a^- > 0$ (see also Remark 3.6).

Next, let us study exponential stability of asymptotically almost periodic solution of (1.1). For convenience, we only discuss the case of $n > 1$.

Theorem 3.3 *Let* $n > 1$. *Suppose that* (H0)-(H2) *are satisfied*, $x(t)$ *is the unique asymptotically almost periodic solution of* (1.1) *in* Ω, *and* $y(t)$ *is an arbitrary nonnegative global solution of* (1.1). *Then there exists a constant* $\lambda > 0$ *such that*

$$\left|x(t) - y(t)\right| \le M e^{\alpha T_0} e^{-\lambda t} \tag{3.5}$$

for all $t \in [-\tau, +\infty)$, *where* $\tau = \max_{1\le i\le k} \tau_i^+$ *and* $M = \sup_{t\in[-\tau,0]} |x(t) - y(t)|$.

Proof Let $N = \frac{n^2-1}{4n} \sqrt[n]{\frac{n+1}{n-1}}$. By (H2), we have

$$\frac{e^{\alpha T_0} \sum_{i=1}^{k} b_i^+}{\alpha} < \frac{1}{N}.$$

Thus, there exists $\lambda \in (0, \alpha)$ such that

$$\lambda - \alpha + e^{\alpha T_0} \sum_{i=1}^{k} b_i^+ \cdot N e^{\lambda\tau} < 0. \tag{3.6}$$

Now, setting $z(t) = x(t) - y(t)$, it is not difficult to see that

$$z(t) = e^{-\int_0^t a(u)\,du} z(0) + \int_0^t e^{-\int_s^t a(u)\,du} \sum_{i=1}^{k} b_i(s) \cdot \left[\frac{1}{1+x^n(s-\tau_i(s))} - \frac{1}{1+y^n(s-\tau_i(s))}\right] ds.$$

We claim that, for every $\varepsilon > 0$, we have

$$\left|z(t)\right| = \left|x(t) - y(t)\right| \le (M+\varepsilon) e^{\alpha T_0} e^{-\lambda t}, \quad t \in [-\tau, +\infty). \tag{3.7}$$

Otherwise, for some $\varepsilon > 0$,

$$\left\{t > 0 : \left|z(t)\right| > (M+\varepsilon) e^{\alpha T_0} e^{-\lambda t}\right\} \ne \emptyset.$$

Let

$$t_0 = \inf\left\{t > 0 : \left|z(t)\right| > (M+\varepsilon) e^{\alpha T_0} e^{-\lambda t}\right\}.$$

Then $t_0 > 0$ and

$$\left|z(t_0)\right| = (M+\varepsilon)e^{\alpha T_0}e^{-\lambda t_0}, \qquad \left|z(t)\right| \le (M+\varepsilon)e^{\alpha T_0}e^{-\lambda t}, \quad t \in [-\tau, t_0).$$

Combining this with (3.4), (3.6), and Lemma 2.1, we conclude

$$\begin{aligned}
\left|z(t_0)\right| &= \left| e^{-\int_0^{t_0} a(u)\,du} z(0) + \int_0^{t_0} e^{-\int_s^{t_0} a(u)\,du} \cdot \sum_{i=1}^k b_i(s) \right.\\
&\quad \left. \cdot \left[\frac{1}{1+x^n(s-\tau_i(s))} - \frac{1}{1+y^n(s-\tau_i(s))}\right] ds \right| \\
&\le e^{\alpha T_0}e^{-\alpha t_0}\left|z(0)\right| + \int_0^{t_0} e^{\alpha T_0}e^{-\alpha(t_0-s)} \cdot \sum_{i=1}^k b_i^+ N \left|z\bigl(s-\tau_i(s)\bigr)\right| ds \\
&\le e^{\alpha T_0}e^{-\alpha t_0}(M+\varepsilon) + Ne^{\alpha T_0}\int_0^{t_0} e^{-\alpha(t_0-s)}\sum_{i=1}^k b_i^+ \cdot (M+\varepsilon)e^{\alpha T_0}e^{-\lambda(s-\tau_i(s))}\,ds \\
&\le e^{\alpha T_0}e^{-\alpha t_0}(M+\varepsilon) + Ne^{\alpha T_0}e^{\lambda\tau}\int_0^{t_0} e^{-\alpha(t_0-s)}\sum_{i=1}^k b_i^+ \cdot (M+\varepsilon)e^{\alpha T_0}e^{-\lambda s}\,ds \\
&= e^{\alpha T_0}e^{-\alpha t_0}(M+\varepsilon) + e^{\alpha T_0}e^{-\alpha t_0}(M+\varepsilon)\int_0^{t_0} Ne^{\lambda\tau}e^{\alpha T_0}\sum_{i=1}^k b_i^+ \cdot e^{(\alpha-\lambda)s}\,ds \\
&< e^{\alpha T_0}e^{-\alpha t_0}(M+\varepsilon) + e^{\alpha T_0}e^{-\alpha t_0}(M+\varepsilon)\int_0^{t_0} (\alpha-\lambda)e^{(\alpha-\lambda)s}\,ds \\
&= e^{\alpha T_0}e^{-\alpha t_0}(M+\varepsilon)e^{(\alpha-\lambda)t_0} = (M+\varepsilon)e^{\alpha T_0}e^{-\lambda t_0},
\end{aligned}$$

which is a contradiction. Thus, for every $\varepsilon > 0$, (3.7) holds. By the arbitrariness of ε, we conclude that

$$\left|z(t)\right| \le Me^{\alpha T_0}e^{-\lambda t}, \quad t \in [-\tau, +\infty).$$

This completes the proof. □

Next, we give two examples to illustrate our main results.

Example 3.4 Let $n = k = 1$, $a(t) = 1 + \ln 2 \cdot (\sin t + \sin \pi t) - \frac{1}{2}e^{-t^2}$, and

$$b_1(t) = \frac{1 + \sin^2 t + \sin^2 \sqrt{2}t}{48}, \qquad \tau_1(t) = \cos^2 t + \cos^2 \sqrt{2}t + \frac{1}{1+t^2}.$$

It is easy to see that (H0) holds. For all $t, s \in \mathbb{R}$ with $s \le t$, we have

$$\int_t^s a(u)\,du \le (s-t) + \ln 2 \cdot \left(2 + \frac{2}{\pi}\right) + \frac{1}{2}(t-s) \le \frac{1}{2}\bigl[(s-t) + 6\ln 2\bigr].$$

So, we can choose $\alpha = \frac{1}{2}$ and $T_0 = 6\ln 2$. By a direct calculation, we can obtain

$$e^{\alpha T_0} = 8, \qquad b_1^+ = \frac{1}{16}, \qquad b_1^- = \frac{1}{48}, \qquad a^+ = 1 + \ln 4.$$

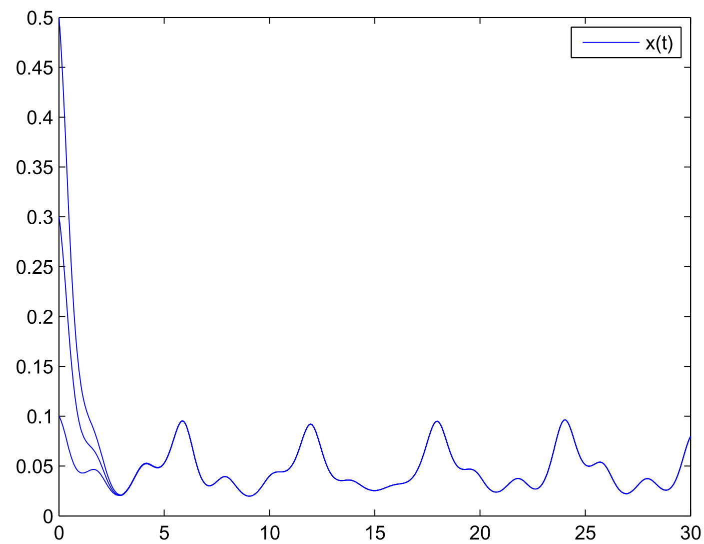

Figure 1 A numerical solution of Example 3.4 with initial value $x(t) \equiv 0.1, 0.3, 0.5$, $t \in [-3, 0]$.

Thus, we have

$$M_1 = \frac{e^{\alpha T_0}\sum_{i=1}^{k} b_i^+}{\alpha} = 1, \qquad M_2 = \frac{\sum_{i=1}^{k} \frac{b_i^-}{1+M_1^n}}{a^+} = \frac{1}{96(1+\ln 4)} \le M_1,$$

which shows that (H1) holds. Moreover,

$$\frac{e^{\alpha T_0}\sum_{i=1}^{k} b_i^+}{\alpha} = 1 < (1+M_2)^2 = \frac{(1+M_2^n)^2 M_2^{1-n}}{n},$$

which means that (H2) holds. Thus, by Theorem 3.1, equation (1.1) has a unique asymptotically almost periodic solution in Ω. A numerical simulation is given in Figure 1.

Example 3.5 Let $n = k = 2$, $a(t) = 10 + 5(\sin \frac{15}{\ln 2} t + \sin \frac{15\pi}{\ln 2} t) + e^{-t^2}$,

$$b_1(t) = 1 + \sin^2 t + \sin^2 \sqrt{2}t, \qquad b_2(t) = 1 + \cos^2 t + \cos^2 \sqrt{2}t,$$

and

$$\tau_1(t) = \cos^2 t + \cos^2 \sqrt{2}t + \frac{1}{1+t^2}, \qquad \tau_2(t) = \sin^2 t + \sin^2 \sqrt{2}t + \frac{1}{1+t^4}.$$

It is easy to see that (H0) holds. For all $t, s \in \mathbb{R}$ with $s \le t$, we have

$$\int_t^s a(u)\,du \le 10(s-t) + \frac{5\ln 2}{15}\left(2 + \frac{2}{\pi}\right) \le 10(s-t) + \ln 2 \le 9\left[(s-t) + \frac{1}{9}\ln 2\right].$$

So, we can choose $\alpha = 9$ and $T_0 = \frac{1}{9}\ln 2$. By a direct calculation, we can obtain

$$e^{\alpha T_0} = 2, \qquad b_1^+ = b_2^+ = 3, \qquad b_1^- = b_2^- = 1, \qquad a^+ \in [20, 21].$$

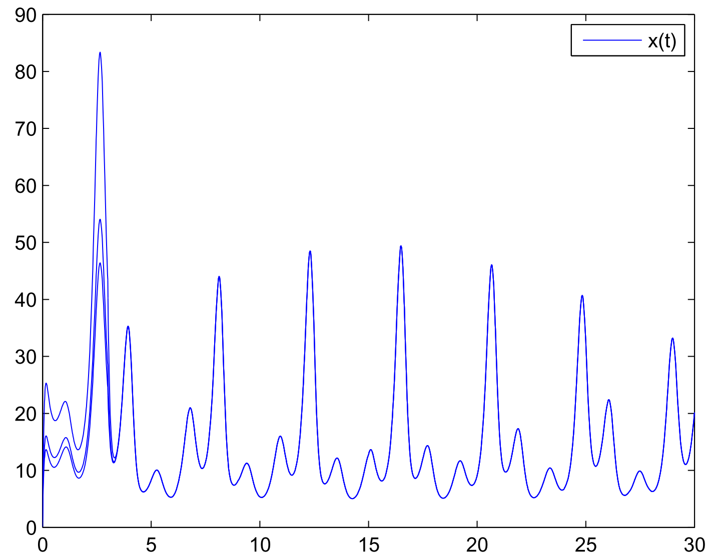

Figure 2 A numerical solution of Example 3.5 with initial value $x(t) \equiv 0.1, 1, 1.3$, $t \in [-3, 0]$.

Thus, we have

$$M_1 = \frac{e^{\alpha T_0} \sum_{i=1}^k b_i^+}{\alpha} = \frac{4}{3}, \qquad M_2 = \frac{\sum_{i=1}^k \frac{b_i^-}{1+M_1^n}}{a^+} \le \frac{\sum_{i=1}^k \frac{b_i^-}{1+M_1^n}}{20} = \frac{9}{250} \le M_1,$$

which shows that (H1) holds. Moreover,

$$\frac{e^{\alpha T_0} \sum_{i=1}^k b_i^+}{\alpha} = \frac{4}{3} < \frac{8}{3\sqrt{3}} = \frac{4n}{n^2-1} \sqrt[n]{\frac{n-1}{n+1}},$$

which means that (H2) holds. Thus, by Theorem 3.1 and Theorem 3.3, equation (1.1) has a unique asymptotically almost periodic solution $x_0(t)$ in Ω, and every nonnegative global solution of (1.1) converges exponentially to $x_0(t)$ as $t \to +\infty$. A numerical simulation is given in Figure 2.

Remark 3.6 It is easy to see that $a^- < 0$ in Example 3.4 and $a^- = 0$ in Example 3.5. So, many earlier results, which requires $a^- > 0$, cannot be applied to the above two examples.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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