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Powers under umbral composition and degeneration for Sheffer sequences

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Abstract
In this paper we study the powers under umbral composition and degeneration for Sheffer sequences, where we presented several applications related to Bernoulli polynomials, Frobenius-Euler polynomials, falling factorial polynomials and Bell polynomials and their degeneration cases.

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1 Introduction

The aim of this paper is to use umbral calculus and to study powers under umbral composition and degeneration for Sheffer sequences. Umbral calculus (see [1, 2]) has been used in numerous problems of applied mathematics, theoretical physics, approximation theory, and several diverse areas of mathematics. In this paper, umbral calculus is considered for some special Sheffer polynomials (to be defined in next section) such as Bell polynomials, Bernoulli polynomials, Frobenius-Euler polynomials, Korobov polynomials, degenerate Bernoulli polynomials, and falling factorial polynomials.

The *order* $O(f(t))$ of the non-zero power series $f(t)$ is the smallest integer k for which the coefficient of t^k does not vanish. A series $g(t)$ with $O(g(t)) = 0$ is called an *invertible series* and a series $f(t)$ with $O(f(t)) = 1$ is called a *delta series*.

Let $g(t)$ be an invertible series and let $f(t)$ be a delta series. Then there exists a unique sequence $s_n(x)$ of polynomials such that $\langle g(t)f(t)^k | s_n(x)\rangle = n!\delta_{n,k}$, for $n, k \geq 0$ (see [2]). The sequence $s_n(x)$ is called the *Sheffer sequence* for the Sheffer pair $(g(t), f(t))$, which is indicated by $s_n(x) \sim (g(t), f(t))$. It is well known that $s_n(x) \sim (g(t), f(t))$ if and only if

$$\frac{1}{g(\bar{f}(t))} e^{x\bar{f}(t)} = \sum_{n\geq 0} s_n(x) \frac{t^n}{n!},$$

where $\bar{f}(t)$ is the compositional inverse of $f(t)$ determined by $f(\bar{f}(t)) = \bar{f}(f(t)) = t$.

For each nonnegative integer m, the mth power of an invertible series $g(t)$ will be indicated by $(g(t))^m$, while the compositional powers of a delta series $f(t)$ will be denoted by $f^m(t) = f \circ f \circ \cdots \circ f(t)$. For $p_n(x)$ and $q_n(x) = \sum_{k=0}^{n} q_{n,k} t^k$, the umbral composition of $q_n(x)$

with $p_n(x)$, denoted by $q_n \circ p_n(x)$, is defined by $q_n \circ p_n(x) = \sum_{k=0}^{n} q_{n,k} p_k(x)$. The main goal of this paper is to study the powers under umbral composition and degeneration for Sheffer sequences. Moreover, we present several applications related to Bernoulli polynomials, Frobenius-Euler polynomials, falling factorial polynomials and Bell polynomials and their degeneration cases (for definitions, see below). For instance, see Corollaries 3.2, 3.4, 3.6, and 3.8.

2 Preliminaries: powers under umbral composition and degeneration for Sheffer sequences

We start by stating the following theorem, which is given as Theorem 3.5.5 in Roman's book [2].

Theorem 2.1 *The set of Sheffer sequences is a group under operation of umbral composition. If $s_n(x) \sim (g(t), f(t))$ and $r_n(x) \sim (h(t), \ell(t))$, then $r_n(x) \circ s_n(x) \sim (g(t)h(f(t)), \ell(f(t)))$. The identity under umbral composition is $x^n \sim (1, t)$, and the inverse of the sequence $s_n(x) \sim (g(t), f(t))$ is the Sheffer sequence for $(g(\bar{f}(t))^{-1}, \bar{f}(t))$.*

As a corollary, we see that, if $s_n(x) \sim (g(t), f(t))$ and $r_n(x) \sim (1, \ell(t))$, then the generating function for $r_n \circ s_n(x)$ is obtained from that for $s_n(x)$ by substituting $\bar{\ell}(t)$ for t. As $r_n \circ s_n(x) \sim (g(t), \ell(f(t)))$, and the compositional inverse of $\ell(f(t))$ is $\bar{f}(\bar{\ell}(t))$,

$$g\big(\bar{f}\big(\bar{\ell}(t)\big)\big)^{-1} e^{x\bar{f}(\bar{\ell}(t))} = \sum_{n\geq 0} r_n \circ s_n(x) \frac{t^n}{n!}. \tag{1}$$

From the definition of umbral composition, we see that the mth power under umbral composition of $r_n(x) \sim (h(t), \ell(t))$ is given by

$$r_n^{(m)}(x) \sim \left(\prod_{i=1}^{m-1} h\big(\ell^i(t)\big), \ell^m(t) \right), \tag{2}$$

for $m \in \mathbb{Z}_{>0}$. In particular, for the Appel sequence $r_n(x) \sim (h(t), t)$, we have $r_n^{(m)}(x) \sim ((h(t))^m, t)$; for the associated sequence $r_n(x) \sim (1, \ell(t))$, we have $r_n^{(m)}(x) \sim (1, \ell^m(t))$.

For $n \geq 0$, we write $r_n(x) = \sum_{k=0}^{n} r_{n,k} x^k = \sum_{k\geq 0} r_{n,k} x^k$, where we agree that $r_{i,j} = 0$ for all $i < j$. In general, we write

$$r_n^{(m)}(x) = \sum_{k-0}^{n} r_{n,k}^{(m)} x^k = \sum_{k\geq 0} r_{n,k}^{(m)} x^k,$$

for all $m \in \mathbb{Z}_{>0}$. Then we see that

$$r_{n,k}^{(m)} = \sum_{\ell_1,\ldots,\ell_{m-1}=0}^{n} r_{n,\ell_1} r_{\ell_1,\ell_2} \cdots r_{\ell_{m-1},k}, \quad m \geq 2, \tag{3}$$

$$r_{n,k}^{(1)} = r_{n,k}$$

(from now on, the sum in (3) is understood as $r_{n,k}$ for $m = 1$). From (2), the generating function for $r_n^{(m)}(x)$ is given by

$$\sum_{n\geq 0} r_n^{(m)}(x)\frac{t^n}{n!} = \frac{1}{\prod_{i=0}^{m-1} h(\ell^i(\bar{\ell}^m(t)))} e^{x\bar{\ell}^m(t)} = \frac{1}{\prod_{i=0}^{m-1} h(\bar{\ell}^{m-i}(t))} e^{x\bar{\ell}^m(t)}.$$

Let $R = R_{h(t),\ell(t)}$ be the lower triangular matrix of infinite size where rows and columns are indexed by nonnegative integers and the nth row consists of the coefficients of $r_n(x)$, namely $r_{n,0},\ldots,r_{n,n},0,0,\ldots$. Then, as was noted in [3], from (3) we see that $r_{n,k}^{(m)}$ is the (n,k)-entry of R^m.

Let $r_n(x) = \sum_{k=0}^n r_{n,k}x^k \sim (h(t),t)$. Thus, by (2), we have

$$r_n^{(m)}(x) = \sum_{k=0}^n r_{n,k}^{(m)} x^k \sim \big(\big(h(t)\big)^m, t\big).$$

Observe here that the notation $r_n^{(m)}(x)$ for the mth power of $r_n(x)$ under umbral composition agrees with that for mth order polynomial of $r_n(x)$.

Now let us give two examples. At first, let $\alpha > 0$. Let us first consider the Bernoulli polynomials $B_n^{(\alpha)}(x)$ of order α (see [4]). So, if

$$r_n(x) = B_n^{(\alpha)}(x) = \sum_{k=0}^n \binom{n}{k} B_{n-k}^{(\alpha)} x^k \sim \left(\left(\frac{e^t-1}{t}\right)^\alpha, t\right),$$

then

$$r_n^{(m)}(x) = B_n^{(\alpha m)}(x) = \sum_{k=0}^n \binom{n}{k} B_{n-k}^{(\alpha m)} x^k \sim \left(\left(\frac{e^t-1}{t}\right)^{\alpha m}, t\right).$$

Thus, from (3), we get the following result.

Theorem 2.2 *For all $n \geq k \geq 0$ and $\alpha, m \in \mathbb{Z}_{>0}$,*

$$\binom{n}{k} B_{n-k}^{(\alpha m)} = \sum_{\ell_1,\ldots,\ell_{m-1}=0}^n \binom{n}{\ell_1}\binom{\ell_1}{\ell_2}\cdots\binom{\ell_{m-1}}{k} B_{n-\ell_1}^{(\alpha)} B_{\ell_1-\ell_2}^{(\alpha)}\cdots B_{\ell_{m-1}-k}^{(\alpha)}.$$

As a second example, let us take the Frobenius-Euler polynomials $H_n^{(\alpha)}(x|\lambda)$ of order α, $1 \neq \lambda \in \mathbb{C}$ (see [5–7]). So, if

$$r_n(x) = H_n^{(\alpha)}(x|\lambda) = \sum_{k=0}^n \binom{n}{k} H_{n,k}^{(\alpha)}(\lambda) x^k \sim \left(\left(\frac{e^t-\lambda}{1-\lambda}\right)^\alpha, t\right),$$

then

$$r_n^{(m)}(x) = H_n^{(\alpha m)}(x|\lambda) = \sum_{k=0}^n \binom{n}{k} H_{n,k}^{(\alpha m)}(\lambda) x^k \sim \left(\left(\frac{e^t-\lambda}{1-\lambda}\right)^{\alpha m}, t\right).$$

Thus, again from (3), we obtain the following result.

Theorem 2.3 *For all $n \geq k \geq 0$ and $\alpha, m \in \mathbb{Z}_{>0}$,*

$$\binom{n}{k} H_{n-k}^{(\alpha m)} = \sum_{\ell_1,\dots,\ell_{m-1}=0}^{n} \binom{n}{\ell_1}\binom{\ell_1}{\ell_2}\cdots\binom{\ell_{m-1}}{k} H_{n-\ell_1}^{(\alpha)}(\lambda) H_{\ell_1-\ell_2}^{(\alpha)}(\lambda)\cdots H_{\ell_{m-1}-k}^{(\alpha)}(\lambda).$$

3 Powers under umbral composition applied to associated sequences

In this section, we study the powers under umbral composition applied to associated sequences. Throughout this section, let $r_n(x) = \sum_{k=0}^{n} r_{n,k} x^k \sim (1, f(t))$ and $r_n^{(m)}(x) = \sum_{k=0}^{n} r_{n,k}^{(m)} x^k \sim (1, f^m(t))$.

3.1 Generalized falling factorial polynomials

As a first interesting case, let us consider the generalized falling factorial polynomials $(x|\lambda)_n = x(x-\lambda)\cdots(x-(n-1)\lambda)$, for $n \geq 1$, and $(x|\lambda)_0 = 1$ (see [4]). So

$$r_n(x) = (x|\lambda)_n = \sum_{k=0}^{n} \lambda^{n-k} S_1(n,k) x^k \sim \bigl(1, f(t)\bigr)$$

with $f(t) = \frac{e^{\lambda t}-1}{\lambda}$. Then, by (3), we obtain

$$\begin{aligned} r_{n,k}^{(m)} &= \sum_{\ell_1,\dots,\ell_{m-1}=0}^{n} \lambda^{n-\ell_1} S_1(n,\ell_1) \lambda^{\ell_1-\ell_2} S_1(\ell_1,\ell_2)\cdots\lambda^{\ell_{m-1}-k} S_1(\ell_{m-1},k) \\ &= \lambda^{n-k} \sum_{\ell_1,\dots,\ell_{m-1}=0}^{n} S_1(n,\ell_1) S_1(\ell_1,\ell_2)\cdots S_1(\ell_{m-1},k). \end{aligned} \tag{4}$$

To proceed, we recall the transfer formula (see [2]): for $p_n(x) \sim (1, f(t))$ and $q_n(x) \sim (1, \ell(t))$, we have $q_n(x) = x(f(t)/\ell(t))^n x^{-1} p_n(x)$, for all $n \geq 1$. As $x^n \sim (1,t)$, we have, for $n \geq 1$,

$$\begin{aligned} r_n(x) &= x\left(\frac{\lambda t}{e^{\lambda t}-1}\right)^n x^{-1} x^n = x \sum_{k\geq 0} \beta_k^{(n)} \frac{\lambda^k}{k!} t^k x^{n-1} \\ &= x \sum_{k=0}^{n-1} \binom{n-1}{k} \lambda^k \beta_k^{(n)} x^{n-1-k} = x \sum_{k=0}^{n-1} \binom{n-1}{k} \lambda^{n-1-k} \beta_{n-1-k}^{(n)} x^k \\ &= \sum_{k=1}^{n} \binom{n-1}{k-1} \lambda^{n-k} \beta_{n-k}^{(n)} x^k. \end{aligned}$$

Thus, for $n \geq 1$, $0 \leq m \leq n-1$, we observe that

$$\bigl(f(t)\bigr)^m x^{-1} r_n(x) = \bigl(f(t)\bigr)^m \bigl(t/f(t)\bigr)^n x^{n-1} = \bigl(t/f(t)\bigr)^{n-m} t^m x^{n-1}.$$

Therefore,

$$\bigl(f(t)\bigr)^m x^{-1} r_n(x) = \sum_{\ell=0}^{n-1-m} \frac{(n-1)_{\ell+m}}{\ell!} \lambda^\ell B_\ell^{(n-m)} x^{n-1-\ell-m}.$$

Now, we get, for $n \geq 1$, $r_n^{(2)}(x) = x(f(t)/f^2(t))^n x^{-1} r_n(x)$, which gives

$$\begin{aligned} r_n^{(2)}(x) &= x \sum_{k_2=0}^{n-1} B_{k_2}^{(n)} \frac{\lambda^{k_2}}{k_2!} \bigl(f(t)\bigr)^{k_2} x^{-1} r_n(x) \\ &= \sum_{k_2=0}^{n-1} \sum_{k_1=0}^{n-1-k_2} \binom{n-1}{k_1, k_2, n-1-k_1-k_2} \lambda^{k_1+k_2} B_{k_2}^{(n)} B_{k_1}^{(n-k_2)} x^{n-k_1-k_2} \\ &= \sum_{k=1}^{n} \left(\sum_{k_1+k_2=n-k} \binom{n-1}{k_1, k_2, k-1} \lambda^{n-k} B_{k_2}^{(n)} B_{k_1}^{(n-k_1)} \right) x^k. \end{aligned} \tag{5}$$

By induction on m, we obtain the following result.

Theorem 3.1 *For all $m, n \geq 1$,*

$$r_n^{(m)}(x) = \sum_{k=1}^{n} \left(\sum_{k_1+\cdots+k_m=n-k} \binom{n-1}{k_1, \ldots, k_m, k-1} \lambda^{n-k} \prod_{j=1}^{m} B_{k_j}^{(n-\sum_{i=j+1}^{m} k_i)} \right) x^k.$$

Note that by combining the two expressions (see (4) and Theorem 3.1) for $r_n^{(m)}(x)$, we obtain the same result as obtained in [8], Theorem 4:

Corollary 3.2 *For all $1 \leq k \leq n$ and $m \geq 1$,*

$$\begin{aligned} &\sum_{\ell_1,\ldots,\ell_{m-1}=0}^{n} S_1(n, \ell_1) S_1(\ell_1, \ell_2) \cdots S_1(\ell_{m-1}, k) \\ &= \sum_{k_1+\cdots+k_m=n-k} \binom{n-1}{k_1, \ldots, k_m, k-1} B_{k_m}^{(n)} B_{k_{m-1}}^{(n-k_m)} \cdots B_{k_1}^{(n-k_m-\cdots-k_2)}. \end{aligned}$$

Note that for $m = 1$, the above corollary reduces to $S_1(n,k) = \binom{n-1}{k-1} B_{n-k}^{(n)}$.

3.2 Degenerate Bell polynomials

Now, let us take the associated sequence $r_n(x)$ to $f(t) = \frac{(1+t)^\lambda - 1}{\lambda}$. So $r_n(x) = \sum_{k=0}^n S_2(n, k|\lambda) x^k \sim (1, f(t))$ and $r_n^{(m)}(x) = \sum_{k=0}^n r_{n,k}^{(m)} x^k \sim (1, f^m(t))$. Here, $S_2(n,k|\lambda)$ are the degenerate Stirling numbers of the second kind obtained by $\frac{1}{k!}((1+\lambda t)^{1/\lambda} - 1)^k = \sum_{n \geq k} S_2(n,k|\lambda) \frac{t^n}{n!}$ (see [9]). Indeed, as $\bar{f}(t) = (1+\lambda t)^{1/\lambda} - 1$, we get

$$\begin{aligned} \sum_{n \geq 0} r_n(x) \frac{t^n}{n!} = e^{x((1+\lambda t)^{1/\lambda} - 1)} &= \sum_{k \geq 0} \bigl((1+\lambda t)^{1/\lambda} - 1\bigr)^k \frac{x^k}{k!} \\ &= \sum_{n \geq 0} \left(\sum_{k=0}^{n} S_2(n, k|\lambda) x^k \right) \frac{t^n}{n!}. \end{aligned}$$

Thus, $r_n(x) = \sum_{k=0}^n S_2(n,k|\lambda) x^k$. As $\lambda \to 0$, $r_n(x)$ tends to the Bell polynomial $Bel_n(x) = \sum_{k=0}^n S_2(n,k) x^k$. Hence they may be called the degenerate Bell polynomials. From (3), we obtain

$$r_{n,k}^{(m)} = \sum_{\ell_1,\ldots,\ell_{m-1}=0}^{n} S_2(n, \ell_1|\lambda) S_2(\ell_1, \ell_2|\lambda) \cdots S_2(\ell_{m-1}, k|\lambda). \tag{6}$$

On the other hand, from the transfer formula, we get, for $n \geq 1$, $r_n(x) = x(\lambda t/((1+t)^\lambda - 1))^n x^{-1} x^n$.

Recall that the *Korobov polynomials* $K_{n,(r)}(\lambda, x)$ of order r (see [10]) are given by generating function $(\lambda t/((1+t)^\lambda - 1))^r (1+t)^n = \sum_{n\geq 0} K_{n,(r)}(\lambda, x)\frac{t^n}{n!}$. For $x = 0$, $K_{n,(r)}(\lambda) = K_{n,(r)}(\lambda, 0)$ are called the *Korobov numbers* of order r. Note that $K_{n,(r)}(\lambda, x)$ should be distinguished from $K_n^{(r)}(\lambda, x)$, which denotes the rth power under umbral composition of $K_n(\lambda, x)$. Thus,

$$r_n(x) = x \sum_{k\geq 0} K_{k,(n)}(\lambda)\frac{t^k}{k!} x^{n-1} = \sum_{k=1}^{n} \binom{n-1}{k-1} K_{n-k,(n)}(\lambda) x^k.$$

To proceed further, we observe the following: for $0 \leq m \leq n-1$,

$$\begin{aligned} f(t)^m x^{-1} r_n(x) &= f(t)^m \big(t/f(t)\big)^n x^{n-1} = \big(t/f(t)\big)^{n-m} t^m x^{n-1} \\ &= (n-1)_m \big(\lambda t/\big((1+t)^\lambda - 1\big)\big)^{n-m} x^{n-1-m} \\ &= (n-1)_m \sum_{\ell\geq 0} K_{\ell,(n-m)}(\lambda)\frac{t^\ell}{\ell!} x^{n-1-m} \\ &= \sum_{\ell=0}^{n-1-m} \frac{(n-1)_{\ell+m}}{\ell!} K_{\ell,(n-m)}(\lambda) x^{n-1-\ell-m}. \end{aligned}$$

Thus, by induction on m and (3), we can state the following formula.

Theorem 3.3 *For all* $m, n \geq 1$,

$$r_n^{(m)}(x) = \sum_{k=1}^{n} \left(\sum_{k_1+\cdots+k_m = n-k} \binom{n-1}{k_1, \ldots, k_m, k-1} \prod_{j=1}^{m} K_{k_j,(n-\sum_{i=j+1}^{m} k_i)}(\lambda) \right) x^k.$$

Combining the two expressions for $r_n^{(m)}(x)$ (see (6) and Theorem 3.3), we obtain the following corollary.

Corollary 3.4 *For all* $1 \leq k \leq n$ *and* $m \geq 1$,

$$\begin{aligned} &\sum_{\ell_1,\ldots,\ell_{m-1}=0}^{n} S_2(n, \ell_1|\lambda) S_2(\ell_1, \ell_2|\lambda) \cdots S_2(\ell_{m-1}, k|\lambda) \\ &= \sum_{k_1+\cdots+k_m=n-k} \binom{n-1}{k_1, \ldots, k_m, k-1} \prod_{j=1}^{m} K_{k_j,(n-\sum_{i=j+1}^{m} k_i)}(\lambda). \end{aligned}$$

Note that the above corollary with $m = 1$ shows that $S_2(n, \ell_1|\lambda) = \binom{n-1}{k-1} K_{n-k,(n)}(\lambda)$.

3.3 Degenerate falling factorial polynomials

As for third example, let us consider the associated sequence $r_n(x)$ to $f(t) = (1+\lambda t)^{1/\lambda} - 1$. So, $r_n(x) = \sum_{k=0}^{n} S_1(n, k|\lambda) x^k \sim (1, f(t))$ and $r_n^{(m)}(x) = \sum_{k=0}^{n} r_{n,k}^{(m)} x^k \sim (1, f^m(t))$. Here, $S_1(n, k|\lambda)$ are the *degenerate Stirling numbers of the first kind* (see [4, 9]) given by $\frac{1}{k!}((1+t)^\lambda - 1)^k/\lambda^k = \sum_{n\geq k} S_1(n, k|\lambda)\frac{t^n}{n!}$. $r_n(x)$ may be called the degenerate falling factorial polynomials, since, as $\lambda \to 0$, $r_n(x)$ tends to the falling factorial polynomial $(x)_n = \sum_{k=0}^{n} S_1(n, k) x^k$.

From (3), we obtain

$$r_{n,k}^{(m)} = \sum_{\ell_1,\dots,\ell_{m-1}=0}^{n} S_1(n,\ell_1|\lambda)S_1(\ell_1,\ell_2|\lambda)\cdots S_1(\ell_{m-1},k|\lambda). \tag{7}$$

We recall that the degenerate Bernoulli polynomials $\beta_{n,(r)}(\lambda,x)$ of order r are defined by the generating function

$$t^r/\big((1+\lambda t)^{1/\lambda}-1\big)^r(1+\lambda t)^{x/\lambda} = \sum_{n\geq 0}\beta_{n,(r)}(\lambda,x)\frac{t^n}{n!}.$$

For $x=0$, $\beta_{n,(r)}(\lambda)=\beta_{n,(r)}(\lambda,0)$ are called the degenerate Bernoulli numbers of order r. Here, $\beta_{n,(r)}(\lambda,x)$ should not be confused with $\beta_n^{(r)}(\lambda,x)$, which denotes the rth power under umbral composition of $\beta_n(\lambda,x)$. So, by these definitions, for $n\geq 1$, we have

$$\begin{aligned} r_n(x) &= x\frac{t^n}{((1+\lambda t)^{1/\lambda}-1)^n}x^{n-1} = x\sum_{k\geq 0}\beta_{k,(n)}(\lambda)\frac{t^k}{k!}x^{n-1} \\ &= \sum_{k=1}^{n}\binom{n-1}{k-1}\beta_{n-k,(n)}(\lambda)x^k. \end{aligned}$$

Thus, for $0\leq m\leq n-1$, we have

$$\begin{aligned} f(t)^m x^{-1} r_n(x) &= \big(t/f(t)\big)^{n-m}t^m x^{n-1} \\ &= (n-1)_m\frac{t^{n-m}}{((1+\lambda t)^{1/\lambda}-1)^{n-m}}x^{n-1-m} \\ &= \sum_{\ell=0}^{n-1-m}\frac{(n-1)_{\ell+m}}{\ell!}\beta_{\ell,(n-m)}(\lambda)x^{n-1-\ell-m}. \end{aligned}$$

By using similar arguments to (5), we obtain

$$\begin{aligned} r_n^{(2)}(x) &= x\big(f(t)/f^2(t)\big)^n x^{-1}r_n(x) \\ &= \sum_{k=1}^{n}\left(\sum_{k_+k_2=n-k}\binom{n-1}{k_1,k_2,k-1}\beta_{k_2,(n)}(\lambda)\beta_{k_1,(n-k_2)}(\lambda)\right)x^k. \end{aligned}$$

Hence, by induction on m, we derive the following result.

Theorem 3.5 *For all $m,n\geq 1$,*

$$r_n^{(m)}(x) = \sum_{k=1}^{n}\left(\sum_{k_1+\cdots+k_m=n-k}\binom{n-1}{k_1,\dots,k_m,k-1}\prod_{j=1}^{m}\beta_{k_j,(n-\sum_{i=j+1}^{m}k_i)}(\lambda)\right)x^k.$$

Combining the two expressions for $r_n^{(m)}(x)$ (see (7) and Theorem 3.5), we obtain the following corollary.

Corollary 3.6 *For all* $1 \leq k \leq n$ *and* $m \geq 1$,

$$\sum_{\ell_1,\ldots,\ell_{m-1}=0}^{n} S_1(n,\ell_1|\lambda)S_1(\ell_1,\ell_2|\lambda)\cdots S_1(\ell_{m-1},k|\lambda)$$
$$= \sum_{k_1+\cdots+k_m=n-k} \binom{n-1}{k_1,\ldots,k_m,k-1} \prod_{j=1}^{m} \beta_{k_j,(n-\sum_{i=j+1}^{m} k_i)}(\lambda).$$

Note that the above corollary with $m = 1$ shows that $S_1(n,k|\lambda) = \binom{n-1}{k-1}\beta_{n-k,(n)}(\lambda)$.

3.4 Generalized Bell polynomials

One more example is Bell polynomials, also called the exponential polynomials, which are given by the generating function $e^{x(e^t-1)} = \sum_{n\geq 0} Bel_n(x)\frac{t^n}{n!}$ (see [4]). Let $r_n(x) = \lambda^n Bel_n(x/\lambda) = \sum_{k=0}^{n} \lambda^{n-k} S_n(n,k)x^k \sim (1, f(t))$ with $f(t) = \frac{1}{\lambda}\log(1+\lambda t)$. Thus, $\bar{f}(t) = \frac{e^{\lambda t}-1}{\lambda}$, which implies

$$\sum_{n\geq 0} r_n(x)\frac{t^n}{n!} = e^{\frac{x}{\lambda}(e^{\lambda t}-1)} = \sum_{n\geq 0} \lambda^n Bel_n(x/\lambda)\frac{t^n}{n!}$$
$$= \sum_{n\geq 0}\left(\sum_{k=0}^{n} \lambda^{n-k} S_2(n,k)x^k\right)\frac{t^n}{n!}.$$

Thus, from (3), we get

$$r_{n,k}^{(m)} = \lambda^{n-k} \sum_{\ell_1,\ldots,\ell_{m-1}=0}^{n} S_2(n,\ell_1)S_2(\ell_1,\ell_2)\cdots S_2(\ell_{m-1},k). \tag{8}$$

To proceed we recall that the *Bernoulli polynomials of the second kind* $b_{n,(r)}(x)$ of order r are given by the generating function

$$\bigl(t/\log(1+t)\bigr)^r(1+t)^x = \sum_{n\geq 0} b_{n,(r)}(x)\frac{t^n}{n!}$$

(see [4]). For $x = 0$, $b_{n,(r)} = b_{n,(r)}(0)$ are the Bernoulli numbers of the second kind of order r. Here, $b_{n,(r)}(x)$ should be distinguished from $b_n^{(r)}(x)$, which denotes the rth power under umbral composition of $b_n(x)$. So, for $n \geq 1$,

$$r_n(x) = x\left(\frac{\lambda t}{\log(1+\lambda t)}\right)^n x^{n-1} = x\sum_{k\geq 0} b_{k,(n)}\frac{\lambda^k t^k}{k!}x^{n-1}$$
$$= \sum_{k=1}^{n}\binom{n-1}{k-1}\lambda^{n-k} b_{n-k,(n)}x^k.$$

For $0 \leq m \leq n-1$, we observe that

$$f(t)^m x^{-1} r_n(x) = \bigl(t/f(t)\bigr)^{n-m} t^m x^{n-1} = \sum_{\ell=0}^{n-1-m} \frac{(n-1)_{\ell+m}}{\ell!}\lambda^\ell b_{\ell,(n-m)}x^{n-1-\ell-m}.$$

Thus, by induction on m (similar to Theorem 3.1), one can obtain the following formula.

Theorem 3.7 *For all $m, n \geq 1$,*

$$r_n^{(m)}(x) = \sum_{k=1}^{n}\left(\sum_{k_1+\cdots+k_m=n-k}\lambda^{n-k}\binom{n-1}{k_1,\ldots,k_m,k-1}\prod_{j=1}^{m}b_{k_j,(n-\sum_{i=j+1}^{m}k_i)}\right)x^k.$$

Combining the two expressions for $r_n^{(m)}(x)$ (see (8) and Theorem 3.7), we obtain the following corollary.

Corollary 3.8 *For all $1 \leq k \leq n$ and $m \geq 1$,*

$$\begin{aligned}&\sum_{\ell_1,\ldots,\ell_{m-1}=0}^{n}S_2(n,\ell_1)S_2(\ell_1,\ell_2)\cdots S_2(\ell_{m-1},k)\\&\quad=\sum_{k_1+\cdots+k_m=n-k}\binom{n-1}{k_1,\ldots,k_m,k-1}\prod_{j=1}^{m}b_{k_j,(n-\sum_{i=j+1}^{m}k_i)}.\end{aligned}$$

Note that the above corollary with $m = 1$ shows that $S_2(n,k) = \binom{n-1}{k-1}b_{n-k,(n)}$.

4 Degenerations of Sheffer sequences

Let $s_n(x) \sim (g(t),f(t))$ and $r_n(x) \sim (1,\ell(t))$. Then $g(\bar{f}(t))^{-1}e^{x\bar{f}(t)} = \sum_{n\geq 0}s_n(x)\frac{t^n}{n!}$. In view of Theorem 2.1 and (1), we have

$$g\bigl(\bar{f}\bigl(\bar{\ell}^m(t)\bigr)\bigr)^{-1}e^{x\bar{f}(\bar{\ell}^m(t))} = \sum_{n\geq 0}r_n^{(m)}\circ s_n(x)\frac{t^n}{n!}.$$

In other words, the generating function for $r_n^{(m)}\circ s_n(x)$ is obtained from that of $s_n(x)$ by replacing t by $\bar{\ell}(t)$ exactly m times. In particular, for $\ell(t) = \frac{1}{\lambda}(e^{\lambda t}-1)$ (resp. $\ell(t) = \frac{1}{\lambda}\log(1+\lambda t)$), $r_n^{(m)}\circ s_n(x)$ will be called the mth degeneration of $s_n(x)$ by $\bar{\ell}(t) = \frac{1}{\lambda}\log(1+\lambda t)$ (resp. $\ell(t) = \frac{1}{\lambda}(e^{\lambda t}-1)$).

The research in this section was motivated by the following example (see [8]) $b_n(x) \sim (t/(e^t-1), e^t-1) = (g(t),f(t))$ and $r_n(x) = (x|\lambda)_n \sim (1,(e^{\lambda t}-1)/\lambda) = (1,\ell(t))$. Note here that $b_n(x)$ is the Bernoulli polynomial of the second kind whose generating function is given by

$$\frac{t}{\log(1+t)}(1+t)^x = \sum_{n\geq 0}b_n(x)\frac{t^n}{n!}. \tag{9}$$

From this consideration, we see that the generating function of $r_n^{(m)}\circ b_n(x)$, the mth degeneration of $b_n(x)$ by $\bar{\ell}(t) = \frac{1}{\lambda}\log(1+\lambda t)$, is obtained from $b_n(x)$ in (9) by replacing t by $\frac{1}{\lambda}\log(1+\lambda t)$ exactly m times. The polynomial $r_n^{(m)}\circ b_n(x)$ is denoted by $c_{n,\lambda}^{(m)}(x)$ in [8] and is given by

$$c_{n,\lambda}^{(m)}(x) = \sum_{k=0}^{n}\lambda^{n-k}\left(\sum_{\ell_1,\ldots,\ell_{m-1}=0}^{n}S_1(n,\ell_1)S_1(\ell_1,\ell_2)\cdots S_1(\ell_{m-1},k)\right)b_k(x).$$

This agrees with the result in [8].

In general, the mth degeneration of $s_n(x) \sim (g(t), f(t))$ by $\bar{\ell}(t) = \frac{1}{\lambda}\log(1+\lambda t)$ is given by

$$r_n^{(m)} \circ s_n(x) = \sum_{k=0}^{n} \lambda^{n-k} \left(\sum_{\ell_1,\ldots,\ell_{m-1}=0}^{n} S_1(n,\ell_1) S_1(\ell_1,\ell_2) \cdots S_1(\ell_{m-1},k) \right) s_k(x),$$

where $r_n(x) \sim (1, (e^{\lambda t}-1)/\lambda) = (1, \ell(t))$.

Also, the mth degeneration of $s_n(x) \sim (g(t), f(t))$ by $\bar{\ell}(t) = \frac{1}{\lambda}(e^{\lambda t}-1)$ is given by

$$r_n^{(m)} \circ s_n(x) = \sum_{k=0}^{n} \lambda^{n-k} \left(\sum_{\ell_1,\ldots,\ell_{m-1}=0}^{n} S_2(n,\ell_1) S_2(\ell_1,\ell_2) \cdots S_2(\ell_{m-1},k) \right) s_k(x),$$

where $r_n(x) \sim (1, \log(1+\lambda t)/\lambda) = (1, \ell(t))$. On the other hand,

$$r_n^{(m)} \circ s_n(x) = \sum_{k=0}^{n} \left(\sum_{\ell_1,\ldots,\ell_{m-1}=0}^{n} S_2(n,\ell_1|\lambda) S_2(\ell_1,\ell_2|\lambda) \cdots S_2(\ell_{m-1},k|\lambda) \right) s_k(x)$$

are polynomials whose generating function is obtained from that of $s_n(x) \sim (g(t), f(t))$ by replacing t by $\bar{\ell}(t) = (1+\lambda t)^{1/\lambda} - 1$ exactly m times (here $r_n(x) \sim (1, \ell(t)) = (1, ((1+t)^{\lambda}-1)/\lambda))$.

In addition,

$$r_n^{(m)} \circ s_n(x) = \sum_{k=0}^{n} \left(\sum_{\ell_1,\ldots,\ell_{m-1}=0}^{n} S_1(n,\ell_1|\lambda) S_1(\ell_1,\ell_2|\lambda) \cdots S_1(\ell_{m-1},k|\lambda) \right) s_k(x)$$

are the polynomials whose generating function is obtained from that of $s_n(x) \sim (g(t), f(t))$ by replacing t by $\bar{\ell}(t) = ((1+t)^{\lambda}-1)/\lambda$ exactly m times (here $r_n(x) \sim (1, \ell(t)) = (1, (1+\lambda t)^{1/\lambda} - 1))$.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to this work. All authors read and approved the final manuscript.

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