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Solutions of complex difference and q-difference equations

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Abstract

Using Nevanlinna theory of the value distribution of meromorphic functions, the growth of entire solutions and the form of transcendental meromorphic solutions of some types of systems of higher-order complex difference equations are investigated. Some new results are obtained. We also investigate the problem of the existence of solutions of complex q-difference equations, and we obtain some new results, which are different from analogue differential equations. Improvements and extensions of some results in the literature are presented. Some examples show that our results are, in a sense, the best possible.

MSC: 30D35

Keywords: value distribution; entire solutions; meromorphic solutions; systems of higher-order complex difference equations; complex q-difference equation

1 Introduction and notation

Throughout the paper, we use the standard notations of the Nevanlinna theory of meromorphic functions (see [1]).

Many authors, such as Weissenborn [2], Toda [3], Gao [4, 5] and Malmquist [6] *etc.* have investigated complex differential equation theory, they obtained many results, such as the following.

Theorem A (Malmquist theorem) (see [6]) *Let $a_0(z),\dots,a_p(z)$, $b_0(z),\dots,b_q(z)$ be rational functions. If the differential equation*

$$\frac{dw}{dz}=R\bigl(z,w(z)\bigr)=\frac{P(z,w(z))}{Q(z,w(z))}=\frac{a_0(z)+a_1(z)w(z)+\cdots+a_p(z)w^p(z)}{b_0(z)+b_1(z)w(z)+\cdots+b_q(z)w^q(z)},$$

where $P(z,w(z))$ and $Q(z,w(z))$ do not have any common factors in $w(z)$, admits a transcendental meromorphic solution, then we have

$$q=0,\qquad p\le 2.$$

Theorem B (see [3]) *When $0\le p\le m-1$, the differential equation*

$$\bigl[\Omega_1(z,w)\bigr]^m=\sum_{j=0}^{p}a_j(z)w^j(z),$$

where $\Omega_1(z,w) = \sum_{(i)} a_{(i)}(z)w^{i_0}(w')^{i_1}\cdots(w^{(n)})^{i_n}$, $0 \le p \le m\max\{i_0 + 2i_1 + \cdots + (n+1)i_n\}$, $a_p(z) \neq 0$, *has no admissible meromorphic solutions except of the following form*:

$$\left[\Omega_1(z,w)\right]^m = a_p(z)\left(w + b(z)\right)^p,$$

where $b(z) = \frac{a_{p-1}(z)}{pa_p(z)}$.

Recently, difference Nevanlinna theory has become a subject of great interest. Halburd and Korhonen [7] and Chiang and Feng [8] *etc.* investigated complex difference equation theory, they got a lot of good results, such as the following.

In 2005, Laine *et al.* [9] studied the following difference equations:

$$\sum_{\{J\}} \alpha_J(z)\left(\prod_{j\in J} w(z+c_j)\right) = \frac{P(z,w(z))}{Q(z,w(z))}, \tag{1.1}$$

where $P(z,w(z))$, $Q(z,w(z))$ are relatively prime polynomials in $w(z)$ over the field of rational functions,the coefficient $\alpha_J(z)$ is a rational function for every J and $q := \deg_w^Q > 0$, $p := \deg_w^P$. They obtained the following result.

Theorem C (see [9]) *Assume* $w(z)$ *is a transcendental meromorphic solution of* (1.1). *If* $w(z)$ *has finitely many poles, then it must be of the following form*:

$$w(z) = r(z)e^{g(z)} + s(z),$$

where $r(z)$ *and* $s(z)$ *are rational functions, and* $g(z)$ *is a transcendental entire function satisfying a difference equation of one of the following forms*: *either*

$$\sum_{j\in J} g(z+c_j) = (j_0 - q)g(z) + d,$$

or

$$\sum_{j\in J} g(z+c_j) = \sum_{j\in I} g(z+c_j) + d,$$

here I *and* J *are non-empty disjoint subsets of* $\{1,2,\ldots,n\}$, $j_0 \in \{0,1,\ldots,p\}$, $d \in C$.

Gao *etc.* have studied solutions of some types of complex difference equations and solutions of system of complex difference equations, they also got some results (see [10, 11]).

On the basis of them, we also did some work. In this paper, first of all, we will investigate solutions of systems of the higher-order complex difference equations (1.2)-(1.4),

$$\begin{cases} [w_1(z+c_1)]^{m_1} = \sum_{i=0}^{p} a_i(z)w_2^i, & p \le m_1, \\ [w_2(z+c_2)]^{m_2} = \sum_{j=0}^{q} b_j(z)w_1^j, & q \le m_2, \end{cases} \tag{1.2}$$

where $c_1, c_2 \in C\setminus\{0\}$;

$$\begin{cases} [\Omega_1(z,w_1)]^{m_1} = \sum_{i=0}^{p} a_i(z)w_2^i, \\ [\Omega_2(z,w_2)]^{m_2} = \sum_{j=0}^{q} b_j(z)w_1^j, \end{cases} \tag{1.3}$$

where

$$\Omega_1(z,w_1) = \sum_{i=1}^{k_1} a_{(i)}(z) w_1^{i_0}\bigl(w_1(z+c_1)\bigr)^{i_1}\cdots\bigl(w_1(z+c_n)\bigr)^{i_n},$$

$$\Omega_2(z,w_2) = \sum_{j=1}^{k_2} b_{(j)}(z) w_2^{j_0}\bigl(w_2(z+c_1)\bigr)^{j_1}\cdots\bigl(w_2(z+c_n)\bigr)^{j_n},$$

$pq \le m_1m_2 \sum_{i=1}^{k_1}\lambda_{1i}\sum_{j=1}^{k_2}\lambda_{2j}$, $k_1 \ge 2$, $k_2 \ge 2$, $c_i, c_j \in C\setminus\{0\}$, $i = 0,1,\dots,n$, $j = 0,1,\dots,n$, $\lambda_{1i} = i_0 + i_1 + \cdots + i_n$, $\lambda_{2j} = j_0 + j_1 + \cdots + j_n$, $T(r,a_{(i)}(z)) = o\{T(r,w_1)\}$, $T(r,b_{(j)}(z)) = o\{T(r,w_2)\}$;

$$\begin{cases} \sum_{\{I\}} a_I(z)(\prod_{i\in I} w_1(z+c_i)) = \frac{P_1(z,w_2(z))}{Q_1(z,w_2(z))}, \\ \sum_{\{J\}} b_J(z)(\prod_{j\in J} w_2(z+c_j)) = \frac{P_2(z,w_1(z))}{Q_2(z,w_1(z))}, \end{cases} \tag{1.4}$$

where $\{I\}$, $\{J\}$ are two finite sets of multi-indices, $c_i, c_j \in C$ $(i \in I, j \in J)$, the coefficients $a_I(z)$, $b_J(z)$ are rational functions, $P_1(z,w_2(z))$, $Q_1(z,w_2(z))$ are relatively prime polynomials in w_2. $P_2(z,w_1(z))$, $Q_2(z,w_1(z))$ are relatively prime polynomials in w_1. The coefficients of $P_1(z,w_2(z))$, $P_2(z,w_1(z))$ are, respectively, rational functions, $Q_1(z,w_2(z))$, $Q_2(z,w_1(z))$ are polynomials and their coefficients are, respectively, small functions of w_2, w_1. We have $q_1 := \deg_{w_2}^{Q_1} > 0$, $q_2 := \deg_{w_1}^{Q_2} > 0$, $p_1 := \deg_{w_2}^{P_1}$, $p_2 := \deg_{w_1}^{P_2}$.

Second, we will investigate solutions of higher-order complex q-difference equations.

The standard logarithmic derivative lemma and Wiman-Valiron theory (see [1]) play important roles in the study of growth and value distribution of meromorphic solutions of differential equations. When talking about linear q-difference equations, Barnett *et al.* [12] and Bergweiler *et al.* [13] give analogs of logarithmic derivative lemma and Wiman-Valiron theory, respectively. Using the q-difference analog of the lemma on the logarithmic derivative, Zhang and Korhonen [14] investigated the relation of the characteristic function of zero order meromorphic function f and its shift. They also concluded that $T(r,f(qz)) \sim T(r,f)$, when $q \in C\setminus\{0\}$. One of its applications to q-difference equations is as follows.

Theorem D (see [14]) *Let $q_1,\dots,q_n \in C\setminus\{0\}$, and let $a_0(z),\dots,a_p(z)$, $b_0(z),\dots,b_q(z)$ be rational functions. If the q-difference equation*

$$\sum_{i=1}^{n} w(q_iz) = R\bigl(z,w(z)\bigr) = \frac{P(z,w(z))}{Q(z,w(z))} = \frac{a_0(z)+a_1(z)w(z)+\cdots+a_p(z)w^p(z)}{b_0(z)+b_1(z)w(z)+\cdots+b_q(z)w^q(z)},$$

where $P(z,w(z))$ and $Q(z,w(z))$ do not have any common factors in $w(z)$, admits a transcendental meromorphic solution of zero order, then we have

$$\max\{p,q\} \le n.$$

Let w be a non-constant meromorphic function of zero order, if meromorphic function g satisfies $T(r,g) = o\{T(r,w)\} = S(r,w)$, outside an exceptional set of zero logarithmic density, then g is called a small function of w.

We also will investigate the complex q-difference equations (1.5)-(1.7),

$$\Omega(z,w) = \frac{a_0(z) + a_1(z)w(z) + \cdots + a_p(z)w^p(z)}{b_0(z) + b_1(z)w(z) + \cdots + b_q(z)w^q(z)}, \tag{1.5}$$

where $\Omega(z,w) = \sum_{i=1}^{k} a_{(i)}(z)w^{i_0}(w(q_1z))^{i_1}\cdots(w(q_nz))^{i_n}$, $q_1,\ldots,q_n \in C \setminus \{0\}$, the coefficients $\{a_{(i)}(z)\}$ are small functions of w (outside an exceptional set of zero logarithmic density). $a_0(z),\ldots,a_p(z)$, $b_0(z),\ldots,b_q(z)$ are rational functions, and $T(r,a_i) = o\{T(r,w)\}$, $i = 0,1,\ldots,p$. $T(r,b_j) = o\{T(r,w)\}$, $j = 0,1,\ldots,q$. We have

$$\bigl[w(q_1z)\bigr]^m = \sum_{i=0}^{p} a_i(z)w^i(z), \quad q_1 \in C\backslash\{0\}, 0 \le p \le m, a_p(z) \not\equiv 0; \tag{1.6}$$

$$\bigl[\Omega(z,w)\bigr]^m = \sum_{i=0}^{p} a_i(z)w^i(z), \quad k \ge 2, 0 \le p \le m\sum_{i=1}^{k}\lambda_i, a_p(z) \not\equiv 0, \tag{1.7}$$

where $\Omega(z,w) = \sum_{i=1}^{k} a_{(i)}(z)w^{i_0}(w(q_1z))^{i_1}\cdots(w(q_nz))^{i_n}$, $q_i \in C\backslash\{0\}$, $i = 1,\ldots,n$, $\lambda_i = i_0 + i_1 + \cdots + i_n$, and the coefficients $\{a_{(i)}(z)\}$ are small functions of w (outside an exceptional set of zero logarithmic density).

The remainder of the paper is organized as follows. In Section 2, we study the existence of admissible meromorphic solutions of systems of complex difference equation (1.2), the problem of the order of entire solutions of systems of complex difference equation (1.3), and the form of transcendental meromorphic solutions of systems of complex difference equation (1.4), and obtain three theorems. In Section 3, we study the problem of the existence of solutions of complex q-difference equations (1.5), (1.6) and (1.7), and we obtain three theorems, and then we give some remarks and some examples, which show that the results obtained in Section 3 are, in a sense, the best possible. In Section 4, we give a series of lemmas for the proofs of the theorems. In Section 5, we prove theorems for complex difference equations by a lemma given in Section 3. In Section 6, we prove theorems for complex q-difference equations by a lemma given in Section 3.

2 Results for systems of complex difference equations

Definition 2.1 Let $(w_1(z),w_2(z))$ be a set of meromorphic solutions of (1.2), $S_1(r)$ means the sum of characteristic functions of all coefficients in (1.2). If the meromorphic solutions $(w_1(z),w_2(z))$ of (1.2) satisfy

$$S_1(r) = o\bigl\{T(r,w_i)\bigr\} = S(r,w_i), \quad i = 1,2,$$

outside a possible exceptional set with finite logarithmic measure, then we say $(w_1(z), w_2(z))$ is a set of admissible meromorphic solutions of (1.2).

We obtain the following results as regards complex difference equations.

Theorem 2.1 *Let $0 \le p < m_1$, $0 \le q < m_2$, the system of higher-order complex difference equations* (1.2) *has no admissible meromorphic solutions of finite order.*

Theorem 2.2 *Let $0 \le p < m_1$, $0 \le q < m_2$, $k_1 \ge 2$, $k_2 \ge 2$, the system of higher-order complex difference equations* (1.3) *does not admit any entire solutions $(w_1(z),w_2(z))$ of order*

$\rho(w_1, w_2) = \max\{\rho(w_1), \rho(w_2)\}$ *greater than* ρ *except the following form*:

$$\begin{cases} [\Omega_1(z,w_1)]^{m_1} = a_p(z)(w_2(z)+c(z))^p, \\ [\Omega_2(z,w_2)]^{m_2} = b_q(z)(w_1(z)+d(z))^q, \end{cases} \tag{2.1}$$

$$\begin{cases} [\Omega_1(z,w_1)]^{m_1} = \sum_{i=0}^{p} a_i(z)w_2^i, \\ [\Omega_2(z,w_2)]^{m_2} = b_q(z)(w_1(z)+d(z))^q, \end{cases} \tag{2.2}$$

$$\begin{cases} [\Omega_1(z,w_1)]^{m_1} = a_p(z)(w_2(z)+c(z))^p, \\ [\Omega_2(z,w_2)]^{m_2} = \sum_{j=0}^{q} b_j(z)w_1^j, \end{cases} \tag{2.3}$$

where $c(z)$, $d(z)$ *are meromorphic functions of order at most* ρ, $\rho = \max\{\rho(a_i), i = 0,1,\dots,p, \rho(a_{(i)}), \rho(b_j), j = 0,1,\dots,q, \rho(b_{(j)})\}$.

Theorem 2.3 *Let* $(w_1(z), w_2(z))$ *be a set of transcendental meromorphic solutions of* (1.4). *If* $w_1(z)$, $w_2(z)$ *have finitely many poles, they must be of the following form*:

$$w_1(z) = r_2(z)e^{g_2(z)} + s_1(z), \qquad w_2(z) = r_1(z)e^{g_1(z)} + s_2(z),$$

where $r_i(z)$, $s_i(z)$, $i = 1,2$ *are rational functions and* $g_i(z)$, $i = 1,2$, *are transcendental entire functions satisfying the following cases*:

(i)

$$\sum_{i\in I_2} g_2(z+c_i) - \sum_{i\in I_3} g_2(z+c_i) = d_1, \qquad \sum_{j\in J_2} g_1(z+c_j) - \sum_{j\in J_3} g_1(z+c_j) = d_2,$$

(ii)

$$\sum_{i\in I_4} g_2(z+c_i) + (q_1 - j_1)g_1(z) = d_1, \qquad \sum_{j\in J_4} g_1(z+c_j) + (q_2 - j_2)g_2(z) = d_2,$$

here I_2, I_3, I_4, J_2, J_3, J_4 *are non-empty subsets of* $\{1,2,\dots,n\}$, $I_2 \cap I_3 = \emptyset$, $J_2 \cap J_3 = \emptyset$, $j_1 \in \{0,1,\dots,p_1\}$, $j_2 \in \{0,1,\dots,p_2\}$, $q_1 > 0$, $q_2 > 0$, $d_i \in C$, $i = 1,2$.

Example 2.1 $(w_1(z), w_2(z)) = (\frac{1}{z}e^{\cos z - 1} + \frac{1}{z+2\pi}, \frac{1}{z}e^{\cos z + 2} + \frac{1}{z-2\pi})$ is a set of transcendental meromorphic solution of the following system of higher-order complex difference equations:

$$\begin{cases} w_1(z+\pi)w_1(z-\pi) + w_1(z+2\pi)w_1(z-2\pi) = \frac{P_1(z,w_2)}{Q_1(z,w_2)}, \\ w_2(z+\pi)w_2(z-\pi) + w_2(z+2\pi)w_2(z-2\pi) = \frac{P_2(z,w_1)}{Q_2(z,w_1)}, \end{cases}$$

where

$$\begin{aligned} P_1(z,w_2) &= e^{-8}z^3(z+4\pi)\bigl(z^2-\pi^2\bigr)(z+3\pi)\left(w_2 - \frac{1}{z-2\pi}\right)^4 \\ &\quad + 2e^{-5}z\bigl(z^2-\pi^2\bigr)(z+3\pi)\bigl(z^2+2\pi z-4\pi^2\bigr)\left(w_2 - \frac{1}{z-2\pi}\right)^3 \\ &\quad + e^{-2}\bigl(z^2-4\pi^2\bigr)(z-\pi)\bigl(2z^2+8\pi z+3\pi^2\bigr)\left(w_2 - \frac{1}{z-2\pi}\right)^2 \\ &\quad + \frac{2e^{-1}}{z+\pi}\bigl(z^2-4\pi^2\bigr)(z+4\pi)\bigl(z^2+2\pi z-\pi^2\bigr)\left(w_2 - \frac{1}{z-2\pi}\right) \end{aligned}$$

$$
\begin{aligned}
&\quad + \frac{1}{z}\left(z^2-4\pi^2\right)(z+4\pi)(z+3\pi),\\
Q_1(z,w_2) &= e^{-2}z\left(z^2-4\pi^2\right)(z+4\pi)\left(z^2-\pi^2\right)(z+3\pi)\left(w_2-\frac{1}{z-2\pi}\right)^2,\\
P_2(z,w_1) &= e^4z^3(z-4\pi)\left(z^2-\pi^2\right)(z-3\pi)\left(w_1-\frac{1}{z+2\pi}\right)^4\\
&\quad + 2ez\left(z^2-\pi^2\right)(z-3\pi)\left(z^2-2\pi z+4\pi^2\right)\left(w_1-\frac{1}{z+2\pi}\right)^3\\
&\quad + e^{-2}\left(z^2-4\pi^2\right)(z+\pi)\left(2z^2-8\pi z+3\pi^2\right)\left(w_1-\frac{1}{z+2\pi}\right)^2\\
&\quad + 2e^{-1}\left(z^2-4\pi^2\right)(z-4\pi)\frac{(z^2-2\pi z-\pi^2)}{z-\pi}\left(w_1-\frac{1}{z+2\pi}\right)\\
&\quad + \frac{1}{z}\left(z^2-4\pi^2\right)(z-4\pi)(z-3\pi),\\
Q_2(z,w_1) &= e^{-2}z\left(z^2-4\pi^2\right)(z-4\pi)\left(z^2-\pi^2\right)(z-3\pi)\left(w_1-\frac{1}{z+2\pi}\right)^2.
\end{aligned}
$$

Easily, we find that $w_1(z)$, $w_2(z)$ have finitely many poles, $q_1 = 2$, $q_2 = 2$, and

$$g_1(z) = \cos z + 2, \qquad g_2(z) = \cos z - 1$$

are transcendental entire functions satisfying

$$
\begin{aligned}
g_1(z+\pi) + g_1(z-\pi) &= -2\cos z + 4 = (0-2)g_2(z) + 2,\\
g_2(z+\pi) + g_2(z-\pi) &= -2\cos z - 2 = (0-2)g_1(z) + 2.
\end{aligned}
$$

This example shows that Theorem 2.3 is not vacuous.

3 Results for complex q-difference equations

Definition 3.1 Let $w(z)$ be an entire solution of (1.6) (or (1.7)), $S_2(r)$ means the sum of characteristic functions of all coefficients in (1.6) (or (1.7)). If the entire solution $w(z)$ of (1.6) (or (1.7)) satisfies

$$S_2(r) = o\{T(r,w)\} = S(r,w),$$

outside a possible exceptional set of zero logarithmic density, then we say $w(z)$ is an admissible entire solution of (1.6) (or (1.7)).

We obtain the following results as regards complex q-difference equations.

Theorem 3.1 *Let $q_j \in C\backslash\{0\}$, $j = 1,\ldots,n$. If the q-difference equation* (1.5) *admits a transcendental meromorphic solution of zero order, then we have*

$$\max\{p,q\} \le \sum_{i=1}^{k} \lambda_i,$$

where $\lambda_i = i_0 + i_1 + \cdots + i_n$.

Remark 3.1 The following Example 3.1 shows the upper bound in Theorem 3.1 can be reached.

Example 3.1 The function $f(z) = \sum_{n=1}^{\infty} \frac{z^n}{2^{\frac{1}{2}n(n+1)}}$ is a transcendental entire function of order zero and satisfies the q-difference equation $f(2z) = z + zf(z)$. Then $w(z) = \frac{1}{f(z)+1}$ is a transcendental meromorphic function of order zero and satisfies the following q-difference equation:

$$w(2z) + w(4z) = \frac{z(2z+1)w(z) + 2(z+1)w^2(z)}{2z^3 + z(4z+1)w(z) + (2z+1)w^2(z)}.$$

Easily we find that

$$p = q = 2, \qquad \lambda_1 = \lambda_2 = 1.$$

Thus

$$\max\{p, q\} = 2 = \lambda_1 + \lambda_2.$$

Remark 3.2 It is clear that Theorem B is a special case of Theorem 3.1.

Remark 3.3 It is unlikely that most of the q-difference equations studied in this paper have meromorphic solutions due to the properties of the q-difference operator. The reason for this is the following. Consider as an example one of the simplest possible q-difference equations,

$$f(qz) = f(z).$$

This equation has no meromorphic solutions unless $|q| = 1$ and the argument of q is a rational number. Otherwise, we take any complex value a such that $f(z_0) = a$. Without loss of generality we may assume that $|q| \leq 1$ (otherwise consider $\frac{1}{q}$ instead of q). Now by the equation above, $f(q^n z_0) = a$ for all $n \in \mathbf{N}$. But this implies that a has a finite accumulation point in the complex plane (at $z = 0$ if $|q| < 1$), which is impossible. A similar phenomenon can easily rule out the possibility of the existence of meromorphic solutions in most (but not all) cases of q-difference equations.

Theorem 3.2 *Let* $0 \leq p < m$, *the q-difference equation* (1.6) *has no admissible entire solutions of zero order.*

Remark 3.4 If $p = m$, from Theorem 3.1, we know (1.6) admits a transcendental meromorphic solution of zero order, but it is uncertain whether the form of (1.6) (the excluded q-difference equation as Theorem C),

$$\bigl[w(q_1 z)\bigr]^m = a_p(z)\bigl(w(z) + b(z)\bigr)^p, \quad \text{where } b(z) = \frac{a_{p-1}(z)}{pa_p(z)},$$

has an admissible entire solution.

Example 3.2 $w(z) = z + 1$ is an admissible entire solution of zero order of the following q-difference equation:

$$w(2z) = 2\left[w(z) - \frac{1}{2}\right].$$

Clearly, we get

$$p = 1, \qquad m = 1.$$

Example 3.3 $w(z) = z^2 - 1$ is a non-admissible entire solution of the complex q-difference equation of the form

$$\left[w(-\sqrt{2}z)\right]^2 = \left(\frac{1}{z^4} - \frac{4}{z^2} + 4\right)\left[w(z) + 1\right]^2.$$

In this case

$$p = 2, \qquad m = 2.$$

Theorem 3.3 *Let* $0 \leq p < m$, $k \geq 2$, *the q-difference equation* (1.7) *has no admissible entire solutions of zero order except the following form*:

$$\left[\Omega(z, w)\right]^m = a_p(z)\big(w(z) + b(z)\big)^p,$$

where $b(z) = \frac{a_{p-1}(z)}{pa_p(z)}$.

Remark 3.5 It is uncertain whether the excluded q-difference equation has an admissible solution.

Example 3.4 $w(z) = z - 1$ is an admissible entire solution of zero order of the following q-difference equation:

$$\left[w^2(2z) + \frac{1}{2}w\left(\frac{67}{2}z\right) - 4\frac{1}{2}w\left(-\frac{1}{2}z\right)w(-2z) + w(z)w\left(\frac{1}{2}z\right)\right]^2 = 9.$$

Clearly, we get

$$p = 0, \qquad m = 2.$$

Example 3.5 $w(z) = z^2 + 1$ is a non-admissible entire solution of the complex q-difference equation of the form

$$\begin{aligned}&\left[2w\left(-\frac{1}{2}z\right)w(\sqrt{2}z) - w^2(-2z) - 2w\left(\frac{\sqrt{5}}{2}z\right) - 3w(2z) - w^2(z) + 4w(z)w(2z)\right]^4 \\ &\quad = \left[w(z) - \left(z^2 + 2\right)\right]^2.\end{aligned}$$

In this case

$$p = 2, \qquad m = 4.$$

4 Lemmas for the proof of theorems

We need the following lemmas to prove theorems.

Lemma 4.1 (see [15]) *Let*

$$R(z,w(z)) = \frac{a_0(z)+a_1(z)w(z)+\cdots+a_p(z)w^p(z)}{b_0(z)+b_1(z)w(z)+\cdots+b_q(z)w^q(z)}$$

be an irreducible rational function in $w(z)$ *with the meromorphic coefficients* $\{a_i(z)\}$ *and* $\{b_j(z)\}$. *If* $w(z)$ *is a meromorphic function, then*

$$T(r,R(z,w(z))) = \max\{p,q\}T(r,w(z)) + O\Bigl\{\sum T(r,a_i(z)) + \sum T(r,b_j(z))\Bigr\}.$$

Lemma 4.2 (see [16]) *Let* $T:[0,+\infty)\to[0,+\infty)$ *be a non-decreasing continuous function,* $\delta\in(0,1)$, $s\in(0,+\infty)$. *If* T *is of finite order, i.e.*

$$\lim_{r\to\infty}\frac{\log T(r)}{\log r} = \rho < \infty,$$

then

$$T(r+s) = T(r) + o\left(\frac{T(r)}{r^\delta}\right),$$

outside an exceptional set of finite logarithmic measure.

Lemma 4.3 (see [3]) *Let* $g_0(z)$ *and* $g_1(z)$ *be meromorphic functions in* $|z|<\infty$ *and linearly independent over* C, *and put*

$$g_0(z) + g_1(z) = \Phi.$$

Then

$$T(r,g_0) \le T(r,\Phi) + \overline{N}(r,\Phi) + \overline{N}(r,g_0) + 2\overline{N}(r,g_1) + \overline{N}\left(r,\frac{1}{g_0}\right) + \overline{N}\left(r,\frac{1}{g_1}\right) + S(r),$$

or

$$T(r,g_0) \le m(r,\Phi) + N(r,g_0) + \overline{N}(r,g_1) + \overline{N}(r,g_0) + \overline{N}\left(r,\frac{1}{g_0}\right) + \overline{N}\left(r,\frac{1}{g_1}\right) + S(r),$$

where, when g_0, g_1 *are rational*

$$S(r) = O(1),$$

when g_0, g_1 *are of finite order,*

$$S(r) = O(\log r)$$

in the other cases,

$$S(r) = O\bigl(\log^+ T(r,g_0) + \log^+ T(r,g_1)\bigr) + O(\log r) \quad (r\to\infty, r\notin E).$$

Lemma 4.4 (see [2]) *Let $w(z)$ be a meromorphic function and let Φ be given by*

$$\Phi = w^n + a_{n-1}w^{n-1} + \cdots + a_0,$$
$$T(r,a_j) = S(r,w), \quad j = 0,1,\ldots,n-1.$$

Then either

$$\Phi = \left(w + \frac{a_{n-1}}{n}\right)^n,$$

or

$$T(r,w) \le \overline{N}\left(r,\frac{1}{\Phi}\right) + \overline{N}(r,w) + S(r,w).$$

Lemma 4.5 (see [1]) *Let $a_1, a_2, \ldots, a_n$ ($n \ge 2$) be rational functions and let $g_1, g_2, \ldots, g_n$ be entire functions such that $g_i - g_j$ is not a constant for every pair $i,j \in \{1,2,\ldots,n\}$ such that $i \ne j$. If*

$$\sum_{j=1}^{n} a_j e^{g_j} = 0,$$

then

$$a_1 = a_2 = \cdots = a_n = 0.$$

Lemma 4.6 (see [14]) *Let $w(z)$ be a transcendental meromorphic function of zero order and $q \in C \setminus \{0\}$. Then*

$$N\big(r,w(qz)\big) = \big(1+o(1)\big)N\big(r,w(z)\big),$$
$$T\big(r,w(qz)\big) = \big(1+o(1)\big)T\big(r,w(z)\big),$$

for all r on a set of logarithmic density 1.

By Lemma 4.6, we can easily obtain the following lemma.

Lemma 4.7 *Let $w(z)$ be meromorphic function of zero order, $\{a_{(i)}(z)\}$ be small function of $w(z)$,*

$$\Omega(z,w) = \sum_{i=1}^{k} a_{(i)}(z)w^{i_0}\big(w(q_1z)\big)^{i_1}\cdots\big(w(q_nz)\big)^{i_n}.$$

If

$$\lambda_i = i_0 + i_1 + \cdots + i_n,$$

then

$$T\big(r,\Omega(z,w)\big) \le \sum_{i=1}^{k} \lambda_i T(r,w) + S(r,w).$$

5 Proof of Theorems 2.1-2.3

Proof of Theorem 2.1 Let $(w_1(z), w_2(z))$ be a set of admissible meromorphic solutions of the system of complex difference equations (1.2). It follows from Lemma 4.1 and Lemma 4.2 that

$$pT(r,w_2) + S(r,w_2) = m_1 T(r,w_1) + S(r,w_1), \tag{5.1}$$

$$qT(r,w_1) + S(r,w_1) = m_2 T(r,w_2) + S(r,w_2). \tag{5.2}$$

From (5.1) and (5.2), we obtain

$$p = m_1 \frac{T(r,w_1)}{T(r,w_2)} + \frac{S(r,w_1)}{T(r,w_2)}, \tag{5.3}$$

$$q = m_2 \frac{T(r,w_2)}{T(r,w_1)} + \frac{S(r,w_2)}{T(r,w_1)}. \tag{5.4}$$

By (5.3) and (5.4), we have

$$pq = m_1 m_2.$$

This is a contradiction. Theorem 2.1 is proved. □

Proof of Theorem 2.2 Suppose that (1.3) does not have the form (2.1), (2.2), (2.3), and (1.3) admits an entire solution $(w_1(z), w_2(z))$ of order greater than ρ. We rewrite (1.3) as follows:

$$\begin{cases} [\Omega_1(z,w_1)]^{m_1} = a_p(z)(w_2(z)+c(z))^p + \sum_{l=0}^{t} a_l(z) w_2^l, & 0 \le t \le p-2, \\ [\Omega_2(z,w_2)]^{m_2} = b_q(z)(w_1(z)+d(z))^q + \sum_{s_1=0}^{s} b_{s_1}(z) w_1^{s_1}, & 0 \le s \le q-2, \end{cases} \tag{5.5}$$

where $c(z) = \frac{a_{p-1}}{pa_p}$, $d(z) = \frac{b_{q-1}}{qb_q}$, $a_l(z)$ is a rational function, $b_{s_1}(z)$ is a rational function, there is at least one l such that $a_l(z) \not\equiv 0$, and there is at least one s_1 such that $b_{s_1}(z) \not\equiv 0$.

Let

$$A_0 = -a_p(z)\bigl(w_2(z)+c(z)\bigr)^p, \qquad B_0 = \bigl[\Omega_1(z,w_1)\bigr]^{m_1}, \qquad \Phi_0 = \sum_{l=0}^{t} a_l(z) w_2^l,$$

$$A_1 = -b_q(z)\bigl(w_1(z)+d(z)\bigr)^q, \qquad B_1 = \bigl[\Omega_2(z,w_2)\bigr]^{m_2}, \qquad \Phi_1 = \sum_{s_1=0}^{s} b_{s_1}(z) w_1^{s_1},$$

then $A_0 + B_0 = \Phi_0$, $A_1 + B_1 = \Phi_1$. Noting that the orders of $c(z)$ and $a_l(z)$ are at most ρ, the orders of $d(z)$ and $b_{s_1}(z)$ are at most ρ, $\Phi_0 \not\equiv 0$, $\Phi_1 \not\equiv 0$. As the proof of Theorem 1 in [5], we can prove that A_0 and B_0 are linearly independent, A_1 and B_1 are linearly independent.

It follows from Lemma 4.3 that

$$\begin{aligned} T(r,A_0) &\le m(r,\Phi_0) + N(r,A_0) + \overline{N}(r,A_0) + \overline{N}(r,B_0) \\ &\quad + \overline{N}\left(r,\frac{1}{A_0}\right) + \overline{N}\left(r,\frac{1}{B_0}\right) + S(r). \end{aligned} \tag{5.6}$$

Next we estimate each term of (5.6).

By Lemma 4.1, we obtain

$$T(r,A_0) \geq pT(r,w_2) - pT\bigl(r,c(z)\bigr) - T\bigl(r,a_p(z)\bigr), \tag{5.7}$$

$$\begin{aligned} m(r,\Phi_0) &\leq tm(r,w_2) + \sum_{l=0}^{t} m\bigl(r,a_l(z)\bigr) + O(1) \\ &\leq tT(r,w_2) + \sum_{l=0}^{t} T\bigl(r,a_l(z)\bigr) + O(1). \end{aligned} \tag{5.8}$$

As $w_2(z)$ is an entire function,

$$N(r,A_0) \leq N(r,a_p) + pN\bigl(r,c(z)\bigr), \tag{5.9}$$

$$\overline{N}(r,A_0) \leq \overline{N}(r,a_p) + \overline{N}(r,a_{p-1}) + \overline{N}\left(r,\frac{1}{a_p}\right) \tag{5.10}$$

and (5.9), (5.10) show that the order of $N(r,A_0)$, $\overline{N}(r,A_0)$, is at most ρ, respectively.

We have

$$\begin{aligned} \overline{N}\left(r,\frac{1}{A_0}\right) &\leq \overline{N}\left(r,\frac{1}{a_p}\right) + \overline{N}\left(r,\frac{1}{w_2(z)+c(z)}\right) \\ &\leq T\bigl(r,w_2(z)\bigr) + T\bigl(r,a_p(z)\bigr) + T\bigl(r,c(z)\bigr) + O(1), \end{aligned} \tag{5.11}$$

$$\begin{aligned} \overline{N}\left(r,\frac{1}{B_0}\right) &\leq T\bigl(r,\Omega_1(z,w_1)\bigr) \\ &\leq \frac{p}{m_1}T(r,w_2) + \frac{1}{m_1}\sum_{i=0}^{p} T\bigl(r,a_i(z)\bigr). \end{aligned} \tag{5.12}$$

As $w_1(z)$ is an entire function, then

$$\overline{N}(r,B_0) = \overline{N}\bigl(r,\Omega_1(z,w_1)\bigr) \leq S(r,w_1). \tag{5.13}$$

Equation (5.13) shows that the order of $\overline{N}(r,B_0)$ is at most ρ.

On the other hand

$$S(r) = O\bigl(\log T(r,w_2)\bigr) + O\left(\sum_{i=0}^{p} \log^+ T(r,a_i)\right) + O(\log r), \quad r \notin E.$$

From (5.6) to (5.13), we have

$$\begin{aligned} &\left(p - t - 1 - \frac{p}{m_1} - o(1)\right)T(r,w_2) \\ &\quad \leq N(r,A_0) + \overline{N}(r,A_0) + \overline{N}(r,B_0) + S(r,w_1) \\ &\qquad + K_1\sum_{i=0}^{p} T(r,a_i) + O\left(\sum_{i=0}^{p} \log^+ T(r,a_i)\right) + O(\log r). \end{aligned} \tag{5.14}$$

Similarly to the case above, we get

$$\begin{aligned}&\left(q-s-1-\frac{q}{m_2}-o(1)\right)T(r,w_1)\\&\quad\le N(r,A_1)+\overline{N}(r,A_1)+\overline{N}(r,B_1)+S(r,w_2)\\&\qquad+K_2\sum_{j=0}^{q}T(r,b_j)+O\left(\sum_{j=0}^{q}\log^+T(r,b_j)\right)+O(\log r).\end{aligned}\tag{5.15}$$

Noting that the order of right-hand side of the inequality (5.14) and (5.15) is at most ρ, respectively, and $p-t-1-\frac{p}{m_1}-o(1)>0$, $q-s-1-\frac{q}{m_2}-o(1)>0$, therefore, the order of w_1, w_2 is at most ρ. That is a contradiction.

This completes the proof of Theorem 2.2. □

Proof of Theorem 2.3 Note that $Q_1(z,w_2)$, $Q_2(z,w_1)$ are monic polynomials and their coefficients are, respectively, small functions of w_2, w_1. By Lemma 4.4, we have

$$Q_2(z,w_1)=\bigl(w_1(z)-s_1(z)\bigr)^{q_2},\qquad Q_1(z,w_2)=\bigl(w_2(z)-s_2(z)\bigr)^{q_1}$$

or

$$T(r,w_1)\le\overline{N}\left(r,\frac{1}{Q_2(z,w_1)}\right)+\overline{N}(r,w_1)+S(r,w_1),\tag{5.16}$$

$$T(r,w_2)\le\overline{N}\left(r,\frac{1}{Q_1(z,w_2)}\right)+\overline{N}(r,w_2)+S(r,w_2).\tag{5.17}$$

On the other hand, $P_1(z,w_2)$ and $Q_1(z,w_2)$, $P_2(z,w_1)$, and $Q_2(z,w_1)$ have only finitely many common zeros. Therefore

$$\begin{aligned}N\left(r,\frac{1}{Q_1(z,w_2)}\right)&\le N\left(r,\frac{P_1(z,w_2)}{Q_1(z,w_2)}\right)+O(\log r)\\&\le N\left(r,\sum_{\{I\}}a_I(z)\prod_{i\in I}w_1(z+c_i)\right)+O(\log r)\\&=O(\log r),\end{aligned}\tag{5.18}$$

$$\begin{aligned}N\left(r,\frac{1}{Q_2(z,w_1)}\right)&\le N\left(r,\frac{P_2(z,w_1)}{Q_2(z,w_1)}\right)+O(\log r)\\&\le N\left(r,\sum_{\{J\}}b_J(z)\prod_{j\in J}w_2(z+c_j)\right)+O(\log r)\\&=O(\log r).\end{aligned}\tag{5.19}$$

It follows from (5.16), (5.19), (5.17), and (5.18) that $T(r,w_1)=S(r,w_1)$, $T(r,w_2)=S(r,w_2)$. This is a contradiction. Thus $Q_1(z,w_2)=(w_2(z)-s_2(z))^{q_1}$, $Q_2(z,w_1)=(w_1(z)-s_1(z))^{q_2}$.

Using (5.18) and (5.19), we see that $Q_1(z,w_2)=(w_2(z)-s_2(z))^{q_1}$, $Q_2(z,w_1)=(w_1(z)-s_1(z))^{q_2}$ have finitely many zeros. Since $w_1(z)$, $w_2(z)$ are transcendental meromorphic functions with at most finitely many poles, $Q_1(z,w_2)=(w_2(z)-s_2(z))^{q_1}$, $Q_2(z,w_1)=(w_1(z)-s_1(z))^{q_2}$

have at most finitely many poles, therefore, there are two rational functions $h_1(z)$, $h_2(z)$ and non-constant entire functions $\overline{g_1}(z)$, $\overline{g_2}(z)$ such that

$$Q_1(z,w_2) = \bigl(w_2(z) - s_2(z)\bigr)^{q_1} = h_1(z)e^{\overline{g_1}(z)},$$
$$Q_2(z,w_1) = \bigl(w_1(z) - s_1(z)\bigr)^{q_2} = h_2(z)e^{\overline{g_2}(z)}.$$

That is,

$$w_2(z) - s_2(z) = \alpha_1\bigl(h_1(z)\bigr)^{\frac{1}{q_1}} e^{\frac{\overline{g_1}(z)}{q_1}}, \tag{5.20}$$
$$w_1(z) - s_1(z) = \alpha_2\bigl(h_2(z)\bigr)^{\frac{1}{q_2}} e^{\frac{\overline{g_2}(z)}{q_2}}, \tag{5.21}$$

where α_i, $i = 1,2$, is the q_ith root of unity.

Let

$$g_1(z) = \frac{\overline{g_1}(z)}{q_1}, \qquad g_2(z) = \frac{\overline{g_2}(z)}{q_2}, \tag{5.22}$$
$$r_1(z) = \alpha_1\bigl(h_1(z)\bigr)^{\frac{1}{q_1}}, \qquad r_2(z) = \alpha_2\bigl(h_2(z)\bigr)^{\frac{1}{q_2}}. \tag{5.23}$$

Then $g_1(z)$, $g_2(z)$ are non-constant entire functions, $r_1(z)$, $r_2(z)$ are rational functions.

From (5.20) to (5.23), we have

$$w_2(z) = r_1(z)e^{g_1(z)} + s_2(z), \qquad w_1(z) = r_2(z)e^{g_2(z)} + s_1(z). \tag{5.24}$$

Substituting (5.24) into the system of higher-order complex difference equations (1.4), we get

$$\begin{cases} \sum_{\{I\}} a_I(z)[\prod_{i\in I}(r_2(z+c_i)e^{g_2(z+c_i)} + s_1(z+c_i))](r_1(z)e^{g_1(z)})^{q_1} \\ \quad = \sum_{j_1=0}^{p_1} \overline{p_{1j_1}}(z)e^{j_1 g_1(z)}, \\ \sum_{\{J\}} b_J(z)[\prod_{j\in J}(r_1(z+c_j)e^{g_1(z+c_j)} + s_2(z+c_j))](r_2(z)e^{g_2(z)})^{q_2} \\ \quad = \sum_{j_2=0}^{p_2} \overline{p_{2j_2}}(z)e^{j_2 g_2(z)}. \end{cases} \tag{5.25}$$

Further,

$$\begin{cases} r_1^{q_1}(z)a_{M_1}(z)\prod_{i\in M_1} r_2(z+c_i)e^{\sum_{i\in M_1} g_2(z+c_i)+q_1 g_1(z)} \\ \quad + r_1^{q_1}(z)\sum_{I\in\{\overline{I}\}} H_{1I}(z)e^{\sum_{i\in I} g_2(z+c_i)+q_1 g_1(z)} = \sum_{j_1=0}^{p_1} \overline{p_{1j_1}}(z)e^{j_1 g_1(z)}, \\ r_2^{q_2}(z)b_{M_2}(z)\prod_{j\in M_2} r_1(z+c_j)e^{\sum_{j\in M_2} g_1(z+c_j)+q_2 g_2(z)} \\ \quad + r_2^{q_2}(z)\sum_{J\in\{\overline{J}\}} H_{2J}(z)e^{\sum_{j\in J} g_1(z+c_j)+q_2 g_2(z)} = \sum_{j_2=0}^{p_2} \overline{p_{2j_2}}(z)e^{j_2 g_2(z)}, \end{cases} \tag{5.26}$$

where the cardinality of the set $M_1 \in \{I\}$ is maximal among the sets in the collection $\{I\}$, the cardinality of the set $M_2 \in \{J\}$ is maximal among the sets in the collection $\{J\}$, $\{\overline{I'}\}$ is a collection of non-empty subsets of $\{I'\} = \{1,2,\dots,n\}$, $\{\overline{J'}\}$ is a collection of non-empty subsets of $\{J'\} = \{1,2,\dots,n\}$, and $M_1 \notin \{\overline{I'}\}$, $M_2 \notin \{\overline{J'}\}$, $H_{1I}(z)$, $H_{2J}(z)$ are rational functions for every I, J, $\overline{p_{1j_1}}(z)$ ($j_1 = 0,1,\dots,p_1$), $\overline{p_{2j_2}}(z)$ ($j_2 = 0,1,\dots,p_2$) are rational functions, and $\overline{p_{1p_1}}(z) \not\equiv 0$, $\overline{p_{2p_2}}(z) \not\equiv 0$.

By Lemma 4.5, there must exist at least two exponents in every equation in (5.26) that cancel each other up to a constant, *i.e.* there exist $d_1, d_2 \in C$ such that

$$\sum_{i\in M_1} g_2(z+c_i) - \sum_{i\in I_0} g_2(z+c_i) = d_1, \tag{5.27}$$

$$\sum_{j\in M_2} g_1(z+c_j) - \sum_{j\in J_0} g_1(z+c_j) = d_2, \tag{5.28}$$

or

$$\sum_{i\in I_1} g_2(z+c_i) + (q_1 - j_1)g_1(z) = d_1, \tag{5.29}$$

$$\sum_{j\in J_1} g_1(z+c_j) + (q_2 - j_2)g_2(z) = d_2, \tag{5.30}$$

where $I_0 \in \{\overline{I'}\}$, $J_0 \in \{\overline{J'}\}$, $I_1 \in \{M_1\} \cup \{\overline{I'}\}$, $J_1 \in \{M_2\} \cup \{\overline{J'}\}$.

Next, we prove that $g_1(z), g_2(z)$ must be transcendental.

Suppose that $g_1(z), g_2(z)$ are two non-constant polynomials. Then, for every $i \in \{I\}$, $j \in \{J\}$, we have

$$g_1(z+c_j) = g_1(z) + g_j(z), \qquad g_2(z+c_i) = g_2(z) + g_i(z), \tag{5.31}$$

where $\deg^z_{g_j} < \deg^z_{g_1}$, $\deg^z_{g_i} < \deg^z_{g_2}$.

Case (i): Using (5.27), (5.28), (5.31), and the relationship among M_1, I_0, M_2, J_0, it is easy to obtain a contradiction.

Case (ii): Substituting the expression $w_1(z) = r_2(z)e^{g_2(z)} + s_1(z)$, $w_2(z) = r_1(z)e^{g_1(z)} + s_2(z)$ into the system of higher-order complex difference equations (1.4), we have

$$\begin{cases} \sum_{\{I\}} a_I(z)[\prod_{i\in I}(r_2(z+c_i)e^{g_2(z+c_i)} + s_1(z+c_i))](r_1(z)e^{g_1(z)})^{q_1} \\ \quad = \sum_{j_1=0}^{p_1} p_{1j_1}(z)(r_1(z)e^{g_1(z)} + s_2(z))^{j_1}, \\ \sum_{\{J\}} b_J(z)[\prod_{j\in J}(r_1(z+c_j)e^{g_1(z+c_j)} + s_2(z+c_j))](r_2(z)e^{g_2(z)})^{q_2} \\ \quad = \sum_{j_2=0}^{p_2} p_{2j_2}(z)(r_2(z)e^{g_2(z)} + s_1(z))^{j_2}, \end{cases} \tag{5.32}$$

where the rational functions $p_{1j_1}(z), p_{2j_2}(z)$ are the coefficients of the polynomials $P_1(z, w_2)$, $P_2(z, w_1)$ respectively.

By (5.29), we see that $g_1(z) = \frac{d_1 - \sum_{i\in I_1}(g_2(z)+g_i(z))}{q_1 - j_1}$, substituting this expression into the first equation of the system of higher-order complex difference equations (5.32), we have

$$\begin{aligned} &\sum_{\{I\}} a_I(z)\left[\prod_{i\in I}\left(r_2(z+c_i)e^{g_2(z)+g_i(z)} + s_1(z+c_i)\right)\right] r_1^{q_1}(z)e^{q_1 \frac{d_1 - \sum_{i\in I_1}(g_2(z)+g_i(z))}{q_1 - j_1}} \\ &\quad = \sum_{j_1=0}^{p_1} p_{1j_1}(z)\left(r_1(z)e^{\frac{d_1 - \sum_{i\in I_1}(g_2(z)+g_i(z))}{q_1 - j_1}} + s_2(z)\right)^{j_1}. \end{aligned}$$

Since the polynomials $P_1(z,w_2)$, $Q_1(z,w_2)$ are relatively prime in w_2, there are no common factors of positive degree in w_2 for $P_1(z,w_2)$, $Q_1(z,w_2)$, and we find that

$$\sum_{j_1=0}^{p_1} p_{1j_1}(z)\bigl(s_2(z)\bigr)^{j_1} \neq 0.$$

Therefore we have got above a non-trivial algebraic equation for $e^{g_2(z)}$ with coefficients which are small relative to $e^{g_2(z)}$, this is a contradiction.

Similarly, by (5.30), we obtain $g_2(z) = \frac{d_2-\sum_{j\in J_1}(g_1(z)+g_j(z))}{q_2-j_2}$, substituting this expression into the second equation of the system of higher-order complex difference equations (5.32), we also derive a contradiction.

This completes the proof of Theorem 2.3. □

6 Proof of Theorems 3.1-3.3

Proof of Theorem 3.1 Let $w(z)$ be a zero order transcendental meromorphic solution of equation (1.5). It follows from Lemma 4.3 and Lemma 4.7 that

$$\begin{aligned}\max\{p,q\}T\bigl(r,w(z)\bigr) &= T\bigl(r,R\bigl(z,w(z)\bigr)\bigr)+S(r,w)\\ &= T\bigl(r,\Omega(z,w)\bigr)+S(r,w)\\ &\leq \sum_{i=1}^{k}\lambda_i T(r,w)+S(r,w).\end{aligned}$$

Thus

$$\max\{p,q\} \leq \sum_{i=1}^{k}\lambda_i.$$

Theorem 3.1 is proved. □

Proof of Theorem 3.2 Let $w(z)$ be a zero order admissible entire solution of equation (1.6).

(i): We assume that (1.6) has the following form:

$$\bigl[w(q_1z)\bigr]^m = a_p(z)\bigl(w(z)+b(z)\bigr)^p,$$

where $b(z) = \frac{a_{p-1}(z)}{pa_p(z)}$. By Lemma 4.3 and Lemma 4.6, we have

$$mT(r,w)+S(r,w) = pT(r,w)+S(r,w).$$

Further

$$(m-p)T(r,w) = S(r,w).$$

In other words

$$m-p = 0.$$

This is a contradiction.

(ii): We assume that (1.6) does not have the following form:

$$\bigl[w(q_1z)\bigr]^m = a_p(z)\bigl(w(z)+b(z)\bigr)^p,$$

where $b(z) = \frac{a_{p-1}(z)}{pa_p(z)}$. In other words, (1.6) has the following form:

$$\bigl[w(q_1z)\bigr]^m = a_p(z)\bigl(w(z)+b(z)\bigr)^p + \sum_{i=0}^{p-2} b_i(z)w^i(z), \quad 2 \le p < m, \tag{6.1}$$

and there is at least one i such that $b_i(z) \not\equiv 0$. Then (6.1) becomes

$$\bigl[w(q_1z)\bigr]^m = a_p(z)\bigl(w(z)+b(z)\bigr)^p + \sum_{i=0}^{t} b_i(z)w^i(z), \quad b_t(z) \not\equiv 0, 0 \le t \le p-2.$$

Let $A = -a_p(z)(w(z)+b(z))^p$, $B = [w(q_1z)]^m$, $\Phi = \sum_{i=0}^{t} b_i(z)w^i(z)$, then $A+B=\Phi$.

Case (i): If $A = 0$, then $w(z) = -b(z)$, we obtain

$$T(r,w) = T(r,b) \le T(r,a_p) + T(r,a_{p-1}) + O(1).$$

Case (ii): If $B = 0$, then $w^p(z) = -a_p^{-1}(z)\sum_{i=0}^{p-1} a_i(z)w^i(z)$, we have

$$pT(r,w) \le (p-1)T(r,w) + \sum_{i=0}^{p} T(r,a_i) + O(1).$$

That is,

$$T(r,w) \le \sum_{i=0}^{p} T(r,a_i) + O(1).$$

Case (iii): If $\Phi = 0$, then $w^t(z) = -b_t^{-1}(z)\sum_{i=0}^{t-1} b_i(z)w^i(z)$, we get

$$T(r,w) \le \sum_{i=0}^{t} T(r,b_i) + O(1) \le K_1 \sum_{i=0}^{p} T(r,a_i) + O(1),$$

where K_1 is a positive constant.

Case (iv): If $A \not\equiv 0$, $B \not\equiv 0$, $\Phi \not\equiv 0$, A and B are linearly dependent, then there are constants k_1 and k_2 ($k_1k_2 \neq 0$) such that $k_1A + k_2B = 0$. From $k_1A + k_2(\Phi - A) = 0$, we obtain

$$\left(\frac{k_1}{k_2} - 1\right)a_p(z)\bigl(w(z)+b(z)\bigr)^p = \sum_{i=0}^{t} b_i(z)w^i(z).$$

But $A \not\equiv 0$, $\Phi \not\equiv 0$, thus $\frac{k_1}{k_2} \neq 1$, and we get

$$\begin{aligned} T(r,w) &\le \frac{1}{p-t}\left[\sum_{i=0}^{t} T\bigl(r,b_i(z)\bigr) + T\bigl(r,a_p(z)\bigr) + pT\bigl(r,b(z)\bigr)\right] + O(1) \\ &\le K_2 \sum_{i=0}^{p} T\bigl(r,a_i(z)\bigr) + O(1), \end{aligned}$$

where K_2 is a positive constant.

Case (v): If $A \neq 0$, $B \neq 0$, $\Phi \neq 0$, A and B are linearly independent, then, by Lemma 4.3, we have

$$T(r,A) \leq T(r,\Phi) + \overline{N}(r,\Phi) + \overline{N}(r,A) + 2\overline{N}(r,B) + \overline{N}\left(r,\frac{1}{A}\right) + \overline{N}\left(r,\frac{1}{B}\right) + S(r). \tag{6.2}$$

Next we estimate each term of (6.2).

It follows from Lemma 4.3 that

$$T(r,A) = pT(r,w) - pT\big(r,b(z)\big) - T\big(r,a_p(z)\big), \tag{6.3}$$

$$T(r,\Phi) = tT(r,w) + \sum_{i=0}^{t} T\big(r,b_i(z)\big) + O(1), \tag{6.4}$$

$$mT\big(r,w(q_1z)\big) = T(r,B) = T(r,\Phi - A) = pT(r,w) + \sum_{i=0}^{p} T\big(r,a_i(z)\big),$$

i.e.

$$T\big(r,w(q_1z)\big) = \frac{p}{m}T(r,w) + \frac{1}{m}\sum_{i=0}^{p} T\big(r,a_i(z)\big). \tag{6.5}$$

Obviously,

$$\overline{N}(r,\Phi) \leq \overline{N}(r,w) + \sum_{i=0}^{t} \overline{N}\big(r,b_i(z)\big), \tag{6.6}$$

$$\overline{N}(r,A) \leq \overline{N}(r,w) + \overline{N}(r,a_p) + \overline{N}(r,a_{p-1}) + \overline{N}\left(r,\frac{1}{a_p}\right). \tag{6.7}$$

It follows from Lemma 4.6 that

$$\overline{N}(r,B) = \overline{N}\big(r,w(q_1z)\big) = \overline{N}\big(r,w(z)\big) + S(r,w), \tag{6.8}$$

$$\begin{aligned}\overline{N}\left(r,\frac{1}{A}\right) &\leq \overline{N}\left(r,\frac{1}{a_p(z)}\right) + \overline{N}\left(r,\frac{1}{w(z)+b(z)}\right) \\ &\leq T\big(r,w(z)\big) + T\big(r,a_p(z)\big) + T\big(r,b(z)\big) + O(1),\end{aligned} \tag{6.9}$$

$$\begin{aligned}\overline{N}\left(r,\frac{1}{B}\right) &\leq \overline{N}\left(r,\frac{1}{w(q_1z)}\right) \leq T\big(r,w(q_1z)\big) + O(1) \\ &= \frac{p}{m}T(r,w) + \frac{1}{m}\sum_{i=0}^{p} T\big(r,a_i(z)\big).\end{aligned} \tag{6.10}$$

From (6.2) to (6.10), we obtain

$$\begin{aligned}\left(p-t-1-\frac{p}{m}\right)T(r,w) &\leq 4\overline{N}(r,w) + K_3\sum_{i=0}^{p} T(r,a_i) \\ &\leq 4N(r,w) + K_3\sum_{i=0}^{p} T(r,a_i),\end{aligned} \tag{6.11}$$

where K_3 is a positive constant.

Note that $w(z)$ is an entire solution of (1.6), thus

$$N\big(r, w(z)\big) = 0. \tag{6.12}$$

From (6.11), (6.12), and $p - t - \frac{p}{m} - 1 > 0$, we obtain

$$T\big(r, w(z)\big) \leq K_4 \sum_{i=0}^{p} T(r, a_i) + S(r, w), \tag{6.13}$$

where K_4 is a positive constant.

From case (i) to case (v), and $w(z)$ being admissible, we obtain

$$T\big(r, w(z)\big) \leq K \sum_{i=0}^{p} T(r, a_i) + S(r, w) = S(r, w) = o\big\{T\big(r, w(z)\big)\big\},$$

where K is a positive constant. That is a contradiction.

This completes the proof of Theorem 3.2. □

Proof of Theorem 3.3 Let $w(z)$ be a zero order admissible entire solution of equation (1.7). We assume that (1.7) does not have the following form:

$$\big[\Omega(z, w)\big]^m = a_p(z)\big(w(z) + b(z)\big)^p,$$

where $b(z) = \frac{a_{p-1}(z)}{pa_p(z)}$. In other words, (1.7) has the following form:

$$\big[\Omega(z, w)\big]^m = a_p(z)\big(w(z) + b(z)\big)^p + \sum_{i=0}^{p-2} b_i(z)w^i(z), \quad 2 \leq p < m, \tag{6.14}$$

and there is at least one i such that $b_i(z) \not\equiv 0$. Then (6.14) becomes

$$\big[\Omega(z, w)\big]^m = a_p(z)\big(w(z) + b(z)\big)^p + \sum_{i=0}^{t} b_i(z)w^i(z), \quad b_t(z) \not\equiv 0, 0 \leq t \leq p - 2.$$

Let $A_1 = -a_p(z)(w(z) + b(z))^p$, $B_1 = [\Omega(z, w)]^m$, $\Phi_1 = \sum_{i=0}^{t} b_i(z)w^i(z)$, then $A_1 + B_1 = \Phi_1$.

Case (i): As the proof of Theorem 3.2, if $A_1 = 0$ or $B_1 = 0$ or $\Phi_1 = 0$ or $A_1 \not\equiv 0$, $B_1 \not\equiv 0$, $\Phi_1 \not\equiv 0$, and A_1 and B_1 are linearly dependent, we obtain

$$T\big(r, w(z)\big) \leq K_5 \sum_{i=0}^{p} T(r, a_i) + S(r, w),$$

where K_5 is a positive constant.

Case (ii): If $A_1 \not\equiv 0$, $B_1 \not\equiv 0$, $\Phi_1 \not\equiv 0$, A_1 and B_1 are linearly independent, then by Lemma 4.3, we have

$$\begin{aligned} T(r, A_1) \leq{}& T(r, \Phi_1) + \overline{N}(r, \Phi_1) + \overline{N}(r, A_1) + 2\overline{N}(r, B_1) \\ &+ \overline{N}\left(r, \frac{1}{A_1}\right) + \overline{N}\left(r, \frac{1}{B_1}\right) + S(r). \end{aligned} \tag{6.15}$$

Next we estimate each term of (6.15).
It follows from Lemma 4.3 that

$$T(r,A_1)=pT(r,w)-pT\bigl(r,b(z)\bigr)-T\bigl(r,a_p(z)\bigr), \tag{6.16}$$

$$T(r,\Phi_1)=tT(r,w)+\sum_{i=0}^{t}T\bigl(r,b_i(z)\bigr)+O(1), \tag{6.17}$$

$$mT\bigl(r,\Omega(z,w)\bigr)=T(r,B_1)=T(r,\Phi_1-A_1)=pT(r,w)+\sum_{i=0}^{p}T\bigl(r,a_i(z)\bigr),$$

i.e.

$$T\bigl(r,\Omega(z,w)\bigr)=\frac{p}{m}T(r,w)+\frac{1}{m}\sum_{i=0}^{p}T\bigl(r,a_i(z)\bigr). \tag{6.18}$$

Obviously,

$$\overline{N}(r,\Phi_1)\le\overline{N}(r,w)+\sum_{i=0}^{t}\overline{N}\bigl(r,b_i(z)\bigr), \tag{6.19}$$

$$\overline{N}(r,A_1)\le\overline{N}(r,w)+\overline{N}(r,a_p)+\overline{N}(r,a_{p-1})+\overline{N}\left(r,\frac{1}{a_p}\right). \tag{6.20}$$

It follows from Lemma 4.6 that

$$\begin{aligned}\overline{N}(r,B_1)&=\overline{N}\bigl(r,\Omega(z,w)\bigr)\\&=\overline{N}\left(r,\sum_{i=1}^{k}a_{(i)}(z)w^{i_0}\bigl(w(q_1z)\bigr)^{i_1}\cdots\bigl(w(q_nz)\bigr)^{i_n}\right)\\&\le\sum_{i=1}^{k}\overline{N}\bigl(r,a_{(i)}(z)w^{i_0}\bigl(w(q_1z)\bigr)^{i_1}\cdots\bigl(w(q_nz)\bigr)^{i_n}\bigr)\\&\le\sum_{i=1}^{k}\bigl[\overline{N}\bigl(r,a_{(i)}(z)\bigr)+\overline{N}\bigl(r,w(z)\bigr)+\cdots+\overline{N}\bigl(r,w(q_nz)\bigr)\bigr]\\&=\sum_{i=1}^{k}\bigl[(n+1)\overline{N}(r,w)\bigr]+S(r,w)\\&=k(n+1)\overline{N}(r,w)+S(r,w),\end{aligned} \tag{6.21}$$

$$\begin{aligned}\overline{N}\left(r,\frac{1}{A_1}\right)&\le\overline{N}\left(r,\frac{1}{a_p(z)}\right)+\overline{N}\left(r,\frac{1}{w(z)+b(z)}\right)\\&\le T\bigl(r,w(z)\bigr)+T\bigl(r,a_p(z)\bigr)+T\bigl(r,b(z)\bigr)+O(1),\end{aligned} \tag{6.22}$$

$$\begin{aligned}\overline{N}\left(r,\frac{1}{B_1}\right)&\le\overline{N}\left(r,\frac{1}{\Omega(z,w)}\right)\\&\le T\bigl(r,\Omega(z,w)\bigr)+O(1)\\&=\frac{p}{m}T(r,w)+\frac{1}{m}\sum_{i=0}^{p}T\bigl(r,a_i(z)\bigr).\end{aligned} \tag{6.23}$$

From (6.15) to (6.23), we obtain

$$\begin{aligned}\left(p-t-1-\frac{p}{m}\right)T(r,w) &\le \left[2+2k(n+1)\right]\overline{N}(r,w)+K_6\sum_{i=0}^{p}T(r,a_i)\\ &\le \left[2+2k(n+1)\right]N(r,w)+K_6\sum_{i=0}^{p}T(r,a_i),\end{aligned} \tag{6.24}$$

where K_6 is a positive constant.

Note that $w(z)$ is an entire solution of (1.7), thus

$$N(r,w)=0. \tag{6.25}$$

From (6.24), (6.25), and $p-t-\frac{p}{m}-1>0$, we obtain

$$T\big(r,w(z)\big)\le K_7\sum_{i=0}^{p}T(r,a_i)+S(r,w), \tag{6.26}$$

where K_7 is a positive constant.

By case (i) and case (ii), and $w(z)$ being admissible, we have

$$T\big(r,w(z)\big)\le K_8\sum_{i=0}^{p}T(r,a_i)+S(r,w)=S(r,w)=o\big\{T\big(r,w(z)\big)\big\},$$

where K_8 is a positive constant. This is a contradiction.

This completes the proof of Theorem 3.3. □

Competing interests
The author declares to have no competing interests.

Author's contributions
The author drafted the manuscript and read and approved the final manuscript.

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