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Existence of periodic solutions for a class of second order discrete Hamiltonian systems

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Abstract
By using the variational minimizing method and the saddle point theorem, the periodic solutions for non-autonomous second-order discrete Hamiltonian systems are considered. The results obtained in this paper complete and extend previous results.

Keywords: periodic solutions; second-order discrete Hamiltonian systems; saddle point theorem; least action principle

1 Introduction and main results

Consider the second-order discrete Hamiltonian system

$$\Delta^2 u(n-1) = \nabla F\bigl(n, u(n)\bigr), \tag{1.1}$$

where $\Delta^2 u(n) = \Delta(\Delta u(n))$ and $\nabla F(n,x)$ denotes the gradient of F with respect to the second variable. F satisfies the following assumption:

(A) $F(n,x) \in C^1(\mathbb{R}^N, \mathbb{R})$ for any $n \in \mathbb{Z}$, $F(n+T,x) = F(n,x)$ for $(n,x) \in \mathbb{Z} \times \mathbb{R}^N$, T is a positive integer.

Since Guo and Yu developed a new method to study the existence and multiplicity of periodic solutions of difference equations by using critical point theory (see [1–4]), the existence and multiplicity of periodic solutions for system (1.1) have been extensively studied and lots of interesting results have been worked out; see [5–16] and the references therein. System (1.1) is a discrete form of classical second-order Hamiltonian systems, which has been paid much attention to by many mathematicians in the past 30 years; see [17–24] for example.

In particular, when the nonlinearity $\nabla F(n,x)$ is bounded, Guo and Yu [3] obtained one periodic solution to system (1.1). When the gradient of the potential energy does not exceed sublinear growth, *i.e.* there exist $M_1 > 0$, $M_2 > 0$, and $\alpha \in [0,1)$, such that

$$\bigl|\nabla F(n,x)\bigr| \le M_1|x|^\alpha + M_2, \quad \forall (n,x) \in \mathbb{Z}[1,T] \times \mathbb{R}^N, \tag{1.2}$$

where $\mathbb{Z}[a,b] := \mathbb{Z} \cap [a,b]$ for every $a, b \in \mathbb{Z}$ with $a \le b$, Xue and Tang [12, 13] considered the periodic solutions of system (1.1), which completed and extended the results in [3] under the condition where

$$\lim_{|x|\to\infty} |x|^{-2\alpha} \sum_{n=1}^{T} F(n,x) = +\infty, \tag{1.3}$$

or

$$\lim_{|x|\to\infty} |x|^{-2\alpha} \sum_{n=1}^{T} F(n,x) = -\infty. \tag{1.4}$$

Under weaker conditions on $\nabla F(n,x)$, *i.e.*,

$$\lim_{|x|\to\infty} |x|^{-2\alpha} \sum_{n=1}^{T} F(n,x) < +\infty, \tag{1.5}$$

or

$$\lim_{|x|\to\infty} |x|^{-2\alpha} \sum_{n=1}^{T} F(n,x) > -\infty, \tag{1.6}$$

Tang and Zhang [11] considered the periodic solutions of system (1.1), which completed and extended the results in [12, 13].

In this paper, we will further investigate periodic solutions to the system (1.1) under the conditions of (1.5) or (1.6). Our main results are the following theorems.

Theorem 1.1 *Suppose that* $F(n,x) = F_1(n,x) + F_2(x)$, *where* F_1 *and* F_2 *satisfy* (A) *and the following conditions*:

(1) *there exist* $f, g : \mathbb{Z}[1,T] \to \mathbb{R}^+$ *and* $\alpha \in [0,1)$ *such that*

$$\left|\nabla F_1(n,x)\right| \le f(n)|x|^{\alpha} + g(n), \quad \textit{for all } (n,x) \in \mathbb{Z}[1,T] \times \mathbb{R}^N;$$

(2) *there exist constants* $r > 0$ *and* $\gamma \in [0,2)$ *such that*

$$\left(\nabla F_2(x) - \nabla F_2(y), x - y\right) \ge -r|x-y|^{\gamma}, \quad \textit{for all } x, y \in \mathbb{R}^N;$$

(3)

$$\liminf_{|x|\to+\infty} |x|^{-2\alpha} \sum_{n=1}^{T} F(n,x) > \frac{1}{8\sin^2\frac{\pi}{T}} \sum_{n=1}^{T} f^2(n).$$

Then system (1.1) *has at least one* T*-periodic solution*.

Theorem 1.2 *Suppose that* $F(n,x) = F_1(n,x) + F_2(x)$, *where* F_1 *and* F_2 *satisfy* (A), (1), (2), *and the following conditions*:

(4) *there exist* $\delta \in [0,2)$ *and* $C > 0$ *such that*

$$\left(\nabla F_2(x) - \nabla F_2(y), x - y\right) \le C|x-y|^{\delta}, \quad \textit{for all } x, y \in \mathbb{R}^N;$$

(5)

$$\limsup_{|x|\to+\infty} |x|^{-2\alpha} \sum_{n=1}^{T} F(n,x) < -\frac{3}{8\sin^2\frac{\pi}{T}} \sum_{n=1}^{T} f^2(n).$$

Then system (1.1) *has at least one T-periodic solution.*

Theorem 1.3 *Suppose that* $F(n,x) = F_1(n,x) + F_2(x)$, *where* F_1 *and* F_2 *satisfy* (A), (1), *and the following conditions*:

(6) *there exists a constant* $0 < r < \frac{6}{T^2-1}$, *such that*

$$\big(\nabla F_2(x) - \nabla F_2(y), x-y\big) \geq -r|x-y|^2, \quad \text{for all } x,y \in \mathbb{R}^N;$$

(7)

$$\liminf_{|x|\to+\infty} |x|^{-2\alpha} \sum_{n=1}^{T} F(n,x) > \frac{3}{(24-4(T^2-1)r)\sin^2\frac{\pi}{T}} \sum_{n=1}^{T} f^2(n).$$

Then system (1.1) *has at least one T-periodic solution.*

Theorem 1.4 *Suppose that* $F = F_1 + F_2$, *where* F_1 *and* F_2 *satisfy* (A), (1), *and the following conditions*:

(8) *there exist* $h: \mathbb{Z}[1,T] \to \mathbb{R}^+$ *and* (λ, u)*-subconvex potential* $G: \mathbb{R}^N \to \mathbb{R}$ *with* $\lambda > 1/2$ *and* $1/2 < \mu < 2\lambda^2$, *such that*

$$\big(\nabla F_2(n,x), y\big) \geq -h(n)G(x-y), \quad \text{for all } x,y \in \mathbb{R}^N \text{ and } n \in \mathbb{Z}[1,T];$$

(9)

$$\limsup_{|x|\to+\infty} |x|^{-2\alpha} \sum_{n=1}^{T} F_1(n,x) < -\frac{3}{8\sin^2\frac{\pi}{T}} \sum_{n=1}^{T} f^2(n),$$

$$\limsup_{|x|\to+\infty} |x|^{-\beta} \sum_{n=1}^{T} F_2(n,x) < -8\mu \max_{|s|\leq 1} G(s) \sum_{n=1}^{T} h(n),$$

where $\beta = \log_{2\lambda}(2\mu)$.

Then system (1.1) *has at least one T-periodic solution.*

Remark 1.5 Theorems 1.1-1.3 extend some existing results. On the one hand, we decomposed the potential F into F_1 and F_2. On the other hand, if $F_2 = 0$, the theorems in [11], Theorems 1 and 2, are special cases of Theorem 1.1 and Theorem 1.2, respectively. Some examples of F are given in Section 4, which are not covered in the references. Moreover, our Theorem 1.4 is a new result.

2 Some important lemmas

H_T can be equipped with the inner product

$$\langle u,v\rangle = \sum_{n=1}^{T} \big[\big(\Delta u(n), \Delta v(n)\big) + \big(u(n), v(n)\big)\big], \quad \forall u,v \in H_T,$$

by which the norm $\|\cdot\|$ can be induced by

$$\|u\| = \left(\sum_{n=1}^{T}\left[\left|\Delta u(n)\right|^2 + \left|u(n)\right|^2\right]\right)^{\frac{1}{2}}, \quad \forall u \in H_T.$$

Define

$$\Phi(u) = \frac{1}{2}\sum_{t=1}^{T}\left|\Delta u(t)\right|^2 - \sum_{t=1}^{T}F\big(t, u(t)\big)$$

and

$$\left\langle\Phi'(u), v\right\rangle = \sum_{t=1}^{T}\big(\Delta u(t), \Delta v(t)\big) - \sum_{t=1}^{T}\big(\nabla F\big(t, u(t)\big), v(t)\big),$$

for $u, v \in H_T$.

By (A), it is easy to see that Φ is continuously differentiable, and the critical points of Φ are the T-periodic solutions of system (1.1).

The following lemma is a discrete form of Wirtinger's inequality and Sobolev's inequality (see [19]).

Lemma 2.1 [11] *If $u \in H_T$ and $\sum_{t=1}^{T} u(t) = 0$, then*

$$\sum_{t=1}^{T}\left|u(t)\right|^2 \leq \frac{1}{4\sin^2\frac{\pi}{T}}\sum_{t=1}^{T}\left|\Delta u(t)\right|^2,$$

$$\|u\|_\infty^2 := \Big(\max_{t\in\mathbb{Z}[1,T]}\left|u(t)\right|\Big)^2 \leq \frac{T^2-1}{6T}\sum_{t=1}^{T}\left|\Delta u(t)\right|^2.$$

Lemma 2.2 [25] *Let $E = V \oplus X$, where E is a real Banach space and $V \neq \{0\}$ and is finite dimensional. Suppose $I \in C^1(E, \mathbb{R})$, it satisfies (PS), and*

(i) *there is a constant α and a bounded neighborhood D of 0 in V such that $I|_{\partial D} \leq \gamma$, and*

(ii) *there is a constant $\beta > \gamma$ such that $I|_X \geq \beta$.*

Then I possesses a critical value $c \geq \beta$. Moreover, c can be characterized as

$$c = \inf_{h\in\Gamma}\max_{s\in\overline{D}} I\big(h(s)\big),$$

where

$$\Gamma = \big\{h \in C(\overline{D}, E) \mid h(s) = s, s \in \partial D\big\}.$$

3 Proof of theorems

For convenience, we denote

$$R_1 = \left(\sum_{n=1}^{T}f^2(n)\right)^{1/2}, \qquad R_2 = \sum_{n=1}^{T}f(n), \quad \text{and} \quad R_3 = \sum_{n=1}^{T}g(n).$$

Proof of Theorem 1.1 According to (3), there exists $a_1 > \frac{1}{4\sin^2\frac{\pi}{T}}$ satisfying

$$\liminf_{x\longrightarrow+\infty} |x|^{-2\alpha}\sum_{n=1}^{T} F(n,x) > \frac{a_1}{2}R_1^2.$$

From (1) and Lemma 2.1, for any $u \in H_T$, one has

$$\begin{aligned}
&\left|\sum_{n=1}^{T}\left[F_1\left(n,u(n)\right) - F_1(n,\bar{u})\right]\right| \\
&\quad = \left|\sum_{n=1}^{T}\int_0^1 \left(\nabla F_1\left(n,\bar{u}+s\tilde{u}(n)\right),\tilde{u}(n)\right)ds\right| \\
&\quad \le \sum_{n=1}^{T}\int_0^1 f(n)\left|\bar{u}+s\tilde{u}(n)\right|^{\alpha}\left|\tilde{u}(n)\right|ds + \sum_{n=1}^{T}\int_0^1 g(n)\left|\tilde{u}(n)\right|ds \\
&\quad \le \sum_{n=1}^{T} f(n)\left(|\bar{u}| + \left|\tilde{u}(n)\right|\right)^{\alpha}\left|\tilde{u}(n)\right| + \sum_{n=1}^{T} g(n)\left|\tilde{u}(n)\right| \\
&\quad \le \sum_{n=1}^{T} f(n)|\bar{u}|^{\alpha}\left|\tilde{u}(n)\right| + \sum_{n=1}^{T} f(n)\left|\tilde{u}(n)\right|^{\alpha+1} + \sum_{n=1}^{T} g(n)\left|\tilde{u}(n)\right| \\
&\quad \le |\bar{u}|^{\alpha}\left(\sum_{n=1}^{T} f^2(n)\right)^{1/2}\left(\sum_{n=1}^{T}\left|\tilde{u}(n)\right|^2\right)^{1/2} + \|\tilde{u}\|_\infty^{\alpha+1}\sum_{n=1}^{T} f(n) + \|\tilde{u}\|_\infty\sum_{n=1}^{T} g(n) \\
&\quad \le \frac{1}{2a_1}\sum_{n=1}^{T}\left|\tilde{u}(n)\right|^2 + \frac{a_1}{2}R_1^2|\bar{u}|^{2\alpha} + R_2\|\tilde{u}\|_\infty^{\alpha+1} + R_3\|\tilde{u}\|_\infty \\
&\quad \le \frac{1}{8a_1\sin^2\frac{\pi}{T}}\sum_{n=1}^{T}\left|\Delta u(n)\right|^2 + \frac{a_1}{2}R_1^2|\bar{u}|^{2\alpha} + \left(\frac{T^2-1}{6T}\right)^{(\alpha+1)/2} R_2\left(\sum_{n=1}^{T}\left|\Delta u(n)\right|^2\right)^{(\alpha+1)/2} \\
&\qquad + \left(\frac{T^2-1}{6T}\right)^{1/2} R_3\left(\sum_{n=1}^{T}\left|\Delta u(n)\right|^2\right)^{1/2}.
\end{aligned} \tag{3.1}$$

From (2) and Lemma 2.1, for any $u \in H_T$, we have

$$\begin{aligned}
&\sum_{n=1}^{T}\left[F_2\left(u(n)\right) - F_2(\bar{u})\right] \\
&\quad = \sum_{n=1}^{T}\int_0^1 \frac{1}{s}\left(\nabla F_2\left(\bar{u}+s\tilde{u}(n)\right) - \nabla F_2(\bar{u}), s\tilde{u}(n)\right)ds \\
&\quad \ge -\sum_{n=1}^{T}\int_0^1 rs^{\gamma-1}\left|\tilde{u}(n)\right|^{\gamma}ds \\
&\quad \ge -\frac{rT}{\gamma}\|\tilde{u}\|_\infty^{\gamma} \\
&\quad \ge -\frac{rT}{\gamma}\left(\frac{T^2-1}{6T}\right)^{\gamma/2}\left(\sum_{n=1}^{T}\left|\Delta u(n)\right|^2\right)^{\gamma/2}.
\end{aligned} \tag{3.2}$$

Combining (3.1) with (3.2), for all $u \in H_T^1$ one has

$$\begin{aligned}
\varphi(u) &= \frac{1}{2}\sum_{n=1}^{T}\left|\Delta u(n)\right|^2 + \sum_{n=1}^{T}\left[F_1(n,u(n)) - F_1(n,\bar{u})\right] \\
&\quad + \sum_{n=1}^{T}\left[F_2(u(n)) - F_2(\bar{u})\right] + \sum_{n=1}^{T}F(n,\bar{u}) \\
&\geq \left(\frac{1}{2} - \frac{1}{8a_1\sin^2\frac{\pi}{T}}\right)\sum_{n=1}^{T}\left|\Delta u(n)\right|^2 - \left(\frac{T^2-1}{6T}\right)^{(\alpha+1)/2}R_2\left(\sum_{n=1}^{T}\left|\Delta u(n)\right|^2\right)^{(\alpha+1)/2} \\
&\quad - \left(\frac{T^2-1}{6T}\right)^{1/2}R_3\left(\sum_{n=1}^{T}\left|\Delta u(n)\right|^2\right)^{1/2} - \frac{rT}{\gamma}\left(\frac{T^2-1}{6T}\right)^{\gamma/2}\left(\sum_{n=1}^{T}\left|\Delta u(n)\right|^2\right)^{\gamma/2} \\
&\quad + |\bar{u}|^{2\alpha}\left(|\bar{u}|^{-2\alpha}\sum_{n=1}^{T}F(n,\bar{u}) - \frac{a_1}{2}R_1^2\right).
\end{aligned}$$

Hence, $\varphi(u) \to \infty$ as $\|u\| \to \infty$. From this result, if $\{u_k\} \subset H_T$ is a minimizing sequence for φ, *i.e.*, $\varphi(u_k) \to \inf\varphi$, $k \to \infty$, then $\{u_k\}$ is bounded. Since H_T is finite dimensional, going if necessary to a subsequence, we can assume that $\{u_k\}$ converges to some $u_0 \in H_T$. Because of φ is continuously differentiable on H_T, one has

$$\varphi(u_0) = \inf\varphi \quad \text{and} \quad \varphi'(u_0).$$

Obviously, $u_0 \in H_T$ is a T-periodic solution of system (1.1). □

Proof of Theorem 1.2 *Step* 1. To prove φ satisfies the (PS) condition. Suppose that u_k is a (PS) sequence, that is, $\varphi'(u_k) \to 0$ as $k \to \infty$ and $\varphi(u_k)$ is bounded. According to (5), there exists $a_2 > \frac{1}{4\sin^2\frac{\pi}{T}}$ satisfying

$$\limsup_{x\to+\infty}|x|^{-2\alpha}\sum_{n=1}^{T}F(n,x) < -\left(a_2 + \frac{1}{8\sin^2\frac{\pi}{T}}\right)R_1^2.$$

In the same way as (3.1), for any $u \in H_T$, one has

$$\begin{aligned}
\left|\sum_{n=1}^{T}\left(\nabla F_1(n,u_k(n)),\tilde{u}_k(n)\right)\right| &\leq \frac{1}{8a_2\sin^2\frac{\pi}{T}}\sum_{n=1}^{T}\left|\Delta u_k(n)\right|^2 + \frac{a_2}{2}R_1^2|\bar{u}_k|^{2\alpha} \\
&\quad + \left(\frac{T^2-1}{6T}\right)^{(\alpha+1)/2}R_2\left(\sum_{n=1}^{T}\left|\Delta u_k(n)\right|^2\right)^{(\alpha+1)/2} \\
&\quad + \left(\frac{T^2-1}{6T}\right)^{1/2}R_3\left(\sum_{n=1}^{T}\left|\Delta u_k(n)\right|^2\right)^{1/2}
\end{aligned} \tag{3.3}$$

and

$$\sum_{n=1}^{T}\left(\nabla F_2(u_k(n)),\tilde{u}_k(n)\right) \geq -\frac{rT}{\gamma}\left(\frac{T^2-1}{6T}\right)^{\gamma/2}\left(\sum_{n=1}^{T}\left|\Delta u_k(n)\right|^2\right)^{\gamma/2}.$$

Hence, we have

$$
\begin{aligned}
\|\tilde{u}_k\| &\geq \langle \varphi'(u_k), \tilde{u}_k \rangle \\
&= \sum_{n=1}^{T} \left|\Delta u_k(n)\right|^2 + \sum_{n=1}^{T} \left(\nabla F\left(n, u_k(n)\right), \tilde{u}_k(n)\right) \\
&\geq \left(1 - \frac{1}{8a_2 \sin^2 \frac{\pi}{T}}\right) \sum_{n=1}^{T} \left|\Delta u_k(n)\right|^2 - \left(\frac{T^2-1}{6T}\right)^{(\alpha+1)/2} R_2 \left(\sum_{n=1}^{T} \left|\Delta u_k(n)\right|^2\right)^{(\alpha+1)/2} \\
&\quad + \left(\frac{T^2-1}{6T}\right)^{1/2} R_3 \left(\sum_{n=1}^{T} \left|\Delta u_k(n)\right|^2\right)^{1/2} - \frac{a_2}{2} R_1^2 |\bar{u}_k|^{2\alpha} \\
&\quad - \frac{rT}{\gamma} \left(\frac{T^2-1}{6T}\right)^{\gamma/2} \left(\sum_{n=1}^{T} \left|\Delta u_k(n)\right|^2\right)^{\gamma/2}
\end{aligned}
\tag{3.4}
$$

for all large k.

By Lemma 2.1, one has

$$
\|\tilde{u}_k\| \leq \frac{(4\sin^2 \frac{\pi}{T} + 1)^{1/2}}{2\sin \frac{\pi}{T}} \left(\sum_{n=1}^{T} \left|\Delta u_k(n)\right|^2\right)^{1/2}.
\tag{3.5}
$$

By (3.4) and (3.5), for all $u \in H_T^1$ one has

$$
\begin{aligned}
\frac{a_2}{2} R_1^2 |\bar{u}_k|^{2\alpha} &\geq \left(1 - \frac{1}{8a_2 \sin^2 \frac{\pi}{T}}\right) \sum_{n=1}^{T} \left|\Delta u_k(n)\right|^2 \\
&\quad - \left(\frac{T^2-1}{6T}\right)^{(\alpha+1)/2} R_2 \left(\sum_{n=1}^{T} \left|\Delta u_k(n)\right|^2\right)^{(\alpha+1)/2} \\
&\quad - \left[\frac{(4\sin^2 \frac{\pi}{T} + 1)^{1/2}}{2\sin \frac{\pi}{T}} + \left(\frac{T^2-1}{6T}\right)^{1/2} R_3\right] \left(\sum_{n=1}^{T} \left|\Delta u_k(n)\right|^2\right)^{1/2} \\
&\quad - \frac{rT}{\gamma} \left(\frac{T^2-1}{6T}\right)^{\gamma/2} \left(\sum_{n=1}^{T} \left|\Delta u_k(n)\right|^2\right)^{\gamma/2} \\
&\geq \frac{1}{2} \sum_{n=1}^{T} \left|\Delta u_k(n)\right|^2 + C_1,
\end{aligned}
\tag{3.6}
$$

where

$$
\begin{aligned}
C_1 = \min_{s \in [0,+\infty)} \Bigg\{ &\frac{4a_2 \sin^2 \frac{\pi}{T} - 1}{8a_2 \sin^2 \frac{\pi}{T}} s^2 - \left(\frac{T^2-1}{6T}\right)^{(\alpha+1)/2} R_2 s^{\alpha+1} - \frac{rT}{\gamma} \left(\frac{T^2-1}{6T}\right)^{\gamma/2} s^{\gamma} \\
&- \left[\frac{(4\sin^2 \frac{\pi}{T} + 1)^{1/2}}{2\sin \frac{\pi}{T}} + \left(\frac{T^2-1}{6T}\right)^{1/2} R_3\right] s \Bigg\}.
\end{aligned}
$$

By the choice of $a_2 > \frac{1}{4\sin^2\frac{\pi}{T}}$, $-\infty < C_1 < 0$. Hence

$$\sum_{n=1}^{T}\left|\Delta u_k(n)\right|^2 \leq a_2 R_1^2 |\bar{u}_k|^{2\alpha} - 2C_1, \tag{3.7}$$

and then

$$\left(\sum_{n=1}^{T}\left|\Delta u_k(n)\right|^2\right)^{1/2} \leq \sqrt{a_2} R_1 |\bar{u}_k|^{\alpha} + C_2, \tag{3.8}$$

where $0 < C_2 < +\infty$.

From Theorem 1.1, one has

$$\begin{aligned}\left|\sum_{n=1}^{T}\left[F_1\bigl(n,u_k(n)\bigr) - F_1(n,\bar{u}_k)\right]\right| &\leq \frac{1}{8a_2\sin^2\frac{\pi}{T}}\sum_{n=1}^{T}\left|\Delta u_k(n)\right|^2 + \frac{a_2}{2}R_1^2|\bar{u}_k|^{2\alpha} \\ &\quad + \left(\frac{T^2-1}{6T}\right)^{(\alpha+1)/2} R_2\left(\sum_{n=1}^{T}\left|\Delta u_k(n)\right|^2\right)^{(\alpha+1)/2} \\ &\quad + \left(\frac{T^2-1}{6T}\right)^{1/2} R_3\left(\sum_{n=1}^{T}\left|\Delta u_k(n)\right|^2\right)^{1/2}.\end{aligned} \tag{3.9}$$

By (4), we obtain

$$\begin{aligned}&\sum_{n=1}^{T}\left[F_2\bigl(u_k(n)\bigr) - F_2(\bar{u}_k)\right] \\ &\quad = \sum_{n=1}^{T}\int_0^1 \frac{1}{s}\bigl(\nabla F_2\bigl(\bar{u}_k + s\tilde{u}_k(n)\bigr) - \nabla F_2(\bar{u}_k), s\tilde{u}_k(n)\bigr)\,ds \\ &\quad \leq \sum_{n=1}^{T}\int_0^1 Cs^{\delta-1}\left|\tilde{u}_k(t)\right|^{\delta}\,ds \leq \frac{CT}{\delta}\|\tilde{u}_k\|_\infty^{\delta} \\ &\quad \leq \frac{CT}{\delta}\left(\frac{T^2-1}{6T}\right)^{\delta/2}\left(\sum_{=1}^{T}\left|\Delta u_k(n)\right|^2\right)^{\delta/2}.\end{aligned}$$

Combining the boundedness of $\{\varphi(u_k)\}$ and (3.7)-(3.9), one has

$$\begin{aligned}&C_3 \leq \varphi(u_k) \\ &\quad = \frac{1}{2}\sum_{n=1}^{T}\left|\Delta u_k(n)\right|^2 + \sum_{n=1}^{T}\left[F_1\bigl(n,u_k(n)\bigr) - F_1(n,\bar{u}_k)\right] \\ &\qquad + \sum_{n=1}^{T}\left[F_2\bigl(u_k(n)\bigr) - F_2(\bar{u}_k)\right] + \sum_{n=1}^{T}F(n,\bar{u}_k) \\ &\quad \leq \left(\frac{1}{2} + \frac{1}{8a_2\sin^2\frac{\pi}{T}}\right)\sum_{n=1}^{T}\left|\Delta u_k(n)\right|^2 + \left(\frac{T^2-1}{6T}\right)^{(\alpha+1)/2} R_2\left(\sum_{n=1}^{T}\left|\Delta u_k(n)\right|^2\right)^{(\alpha+1)/2}\end{aligned}$$

$$
\begin{aligned}
&+\left(\frac{T^2-1}{6T}\right)^{1/2} R_3 \left(\sum_{n=1}^{T}\left|\Delta u_k(n)\right|^2\right)^{1/2} + \frac{a_2}{2}R_1^2|\bar{u}_k|^{2\alpha} \\
&+\frac{CT}{\delta}\left(\frac{T^2-1}{6T}\right)^{\delta/2}\left(\sum_{n=1}^{T}\left|\Delta u_k(n)\right|^2\right)^{\delta/2} + \sum_{n=1}^{T} F(n,\bar{u}_k) \\
\leq{}& \left(\frac{1}{2}+\frac{1}{8a_2\sin^2\frac{\pi}{T}}\right)\left(a_2R_1^2|\bar{u}_k|^{2\alpha}-2C_1\right) + \frac{a_2}{2}R_1^2|\bar{u}_k|^{2\alpha} + \sum_{n=1}^{T}F(n,\bar{u}_k) \\
&+\left(\frac{T^2-1}{6T}\right)^{(\alpha+1)/2} R_2\left(\sqrt{a_2}R_1|\bar{u}_k|^{\alpha}+C_2\right)^{\alpha+1} \\
&+\frac{CT}{\delta}\left(\frac{T^2-1}{6T}\right)^{\delta/2}\left(\sqrt{a_2}R_1|\bar{u}_k|^{\alpha}+C_2\right)^{\delta} \\
&+\left(\frac{T^2-1}{6T}\right)^{1/2} R_3\left(\sqrt{a_2}R_1|\bar{u}_k|^{\alpha}+C_2\right) \\
\leq{}& |\bar{u}_k|^{2\alpha}\left[|\bar{u}_k|^{-2\alpha}\sum_{n=1}^{T}F(n,\bar{u}_k) + \left(a_2+\frac{1}{8\sin^2\frac{\pi}{T}}\right)R_1^2\right] \\
&+C_4|\bar{u}_k|^{\alpha(\alpha+1)} + C_5|\bar{u}_k|^{\alpha} + C_6|\bar{u}_k|^{\alpha\delta} + C_7
\end{aligned}
$$

for large k. By the choice of a_2, $\{\bar{u}_k\}$ is bounded. From (3.7), $\{u_k\}$ is bounded. In view of H_T is finite dimensional Hilbert space, φ satisfies the (PS) condition.

Step 2. Let $\tilde{H}_T = \{u \in H_T : \bar{u} = 0\}$. We show that, for $u \in \tilde{H}_T$,

$$
\varphi(u) \to +\infty, \quad \|u\| \to \infty. \tag{3.10}
$$

From (1) and Lemma 2.1, one has

$$
\begin{aligned}
\left|\sum_{n=1}^{T}\left[F_1\left(n,u(n)\right)-F(n,0)\right]\right| &= \left|\sum_{n=1}^{T}\int_0^1\left(\nabla F_1\left(n,su(n)\right),u(n)\right)ds\right| \\
&\leq \sum_{n=1}^{T} f(n)\left|u(n)\right|^{\alpha+1} + \sum_{n=1}^{T} g(n)\left|u(n)\right| \\
&\leq R_2\|u\|_\infty^{\alpha+1} + R_3\|u\|_\infty \\
&\leq \left(\frac{T^2-1}{6T}\right)^{(\alpha+1)/2} R_2\left(\sum_{n=1}^{T}\left|\Delta u(n)\right|^2\right)^{(\alpha+1)/2} \\
&\quad + \left(\frac{T^2-1}{6T}\right)^{1/2} R_3\left(\sum_{n=1}^{T}\left|\Delta u(n)\right|^2\right)^{1/2}
\end{aligned}
$$

for all $u \in \tilde{H}_T$. It follows from (2) that

$$
\begin{aligned}
&\sum_{n=1}^{T}\left[F_2\left(u(n)\right)-F_2(0)\right] \\
&\quad = \sum_{n=1}^{T}\int_0^1\left(\nabla F_2\left(su(n)\right)-\nabla F_2(0),u(n)\right)ds
\end{aligned}
$$

$$\begin{aligned}
&\geq -\sum_{n=1}^{T}\int_0^1 rs^{\gamma-1}\left|u(n)\right|^{\gamma}\,ds \geq -\frac{rT}{\gamma}\|u\|_\infty^{\gamma} \\
&\geq -\frac{rT}{\gamma}\left(\frac{T^2-1}{6T}\right)^{\gamma/2}\left(\sum_{n=1}^{T}\left|\Delta u(n)\right|^2\right)^{\gamma/2}.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\varphi(u) &= \frac{1}{2}\sum_{n=1}^{T}\left|\Delta u(n)\right|^2 + \sum_{n=1}^{T}\left[F\left(n,u(n)\right) - F(n,0)\right] + \sum_{n=1}^{T}F(n,0) \\
&\geq \frac{1}{2}\sum_{n=1}^{T}\left|\Delta u(n)\right|^2 - \left(\frac{T^2-1}{6T}\right)^{(\alpha+1)/2}R_2\left(\sum_{n=1}^{T}\left|\Delta u(n)\right|^2\right)^{(\alpha+1)/2} \\
&\quad - \left(\frac{T^2-1}{6T}\right)^{1/2}R_3\left(\sum_{n=1}^{T}\left|\Delta u(n)\right|^2\right)^{1/2} + \sum_{n=1}^{T}F(n,0) \\
&\quad - \frac{rT}{\gamma}\left(\frac{T^2-1}{6T}\right)^{\gamma/2}\left(\sum_{n=1}^{T}\left|\Delta u(n)\right|^2\right)^{\gamma/2}.
\end{aligned}$$

In view of Lemma 2.1, $\|u\| \to +\infty$ in $\tilde{H}_T$ if and only if $(\sum_{n=1}^{T}|\Delta u(n)|^2)^{1/2} \to \infty$. Hence (3.10) is satisfied.

Step 3. By (5), for all $u \in (\tilde{H}_T)^{\perp} = \mathbb{R}^N$, one has

$$\varphi(u) = -\sum_{n=1}^{T}F\left(n,u(n)\right) \to -\infty, \quad \|u\| \to \infty.$$

Above all, all conditions of Lemma 2.2 are satisfied. So, by Lemma 2.2, system (1.1) has at least one T-periodic solution. □

Proof of Theorem 1.3 By (7), there exists $a_3 > \frac{3}{(12-2(T^2-1)r)\sin^2\frac{\pi}{T}}$ satisfying

$$\liminf_{|x|\longrightarrow+\infty}|x|^{-2\alpha}\sum_{n=1}^{T}F(n,x) > \frac{a_3}{2}R_1^2.$$

Similar to (3.1), we have

$$\begin{aligned}
&\sum_{n=1}^{T}\left[F_1\left(n,u(n)\right) - F_1(n,\bar{u})\right] \\
&\quad\geq -\frac{1}{8a_3\sin^2\frac{\pi}{T}}\sum_{n=1}^{T}\left|\Delta u(n)\right|^2 - \frac{a_3}{2}R_1^2|\bar{u}|^{2\alpha} - \left(\frac{T^2-1}{6T}\right)^{(\alpha+1)/2}R_2\left(\sum_{n=1}^{T}\left|\Delta u(n)\right|^2\right)^{(\alpha+1)/2} \\
&\qquad - \left(\frac{T^2-1}{6T}\right)^{1/2}R_3\left(\sum_{n=1}^{T}\left|\Delta u(n)\right|^2\right)^{1/2}.
\end{aligned}$$

By (6) and Lemma 2.1, one has

$$\begin{aligned}\sum_{n=1}^{T}\bigl[F_2\bigl(u(n)\bigr)-F_2(\bar{u})\bigr] &= \sum_{n=1}^{T}\int_0^1 \frac{1}{s}\bigl(\nabla F_2\bigl(\bar{u}+s\tilde{u}(n)\bigr)-\nabla F_2(\bar{u}), s\tilde{u}(n)\bigr)\,ds \\ &\geq -\sum_{n=1}^{T}\int_0^1 rs\bigl|\tilde{u}(n)\bigr|^2\,ds \geq -\frac{(T^2-1)r}{12}\sum_{n=1}^{T}\bigl|\Delta u(n)\bigr|^2.\end{aligned}$$

So, for any $u \in H_T$, we have

$$\begin{aligned}\varphi(u) &= \frac{1}{2}\sum_{n=1}^{T}\bigl|\Delta u(n)\bigr|^2 + \sum_{n=1}^{T}\bigl[F\bigl(n,u(n)\bigr)-F(n,\bar{u})\bigr] + \sum_{n=1}^{T}F(n,\bar{u}) \\ &\geq \left(\frac{1}{2}-\frac{1}{8a_3\sin^2\frac{\pi}{T}}-\frac{(T^2-1)r}{12}\right)\sum_{n=1}^{T}\bigl|\Delta u(n)\bigr|^2 \\ &\quad - \left(\frac{T^2-1}{6T}\right)^{(\alpha+1)/2} R_2\left(\sum_{n=1}^{T}\bigl|\Delta u(n)\bigr|^2\right)^{(\alpha+1)/2} - \left(\frac{T^2-1}{6T}\right)^{1/2} R_3\left(\sum_{n=1}^{T}\bigl|\Delta u(n)\bigr|^2\right)^{1/2} \\ &\quad + |\bar{u}|^{2\alpha}\left(|\bar{u}|^{-2\alpha}\sum_{n=1}^{T}F(n,\bar{u})-\frac{a_3}{2}R_1^2\right).\end{aligned}$$

Therefore, $\varphi(u)\to+\infty$ as $\|u\|\to+\infty$ due to the choice of a_3 and $r<\frac{6}{T^2-1}$. The rest is similar to the proof of Theorem 1.1. □

Proof of Theorem 1.4 First, we prove that φ satisfies the (PS) condition. Suppose that $\{u_k\}\subset H_T$ is a (PS) sequence of φ, that is, $\varphi'(u_k)\to 0$ as $k\to\infty$ and $\{\varphi(u_k)\}$ is bounded. By (9), there exists $a_4>\frac{1}{4\sin^2\frac{\pi}{T}}$ satisfying

$$\limsup_{|x|\to+\infty}|x|^{-2\alpha}\sum_{n=1}^{T}F(n,x) < -\left(a_4+\frac{1}{8\sin^2\frac{\pi}{T}}\right)R_1^2. \tag{3.11}$$

By the (λ,μ)-subconvexity of $G(x)$, we have

$$G(x)\leq\bigl(2\mu|x|^{\beta}+1\bigr)G_0 \tag{3.12}$$

for all $x\in\mathbb{R}^N$, where $G_0=\max_{|s|\leq 1}G(s)$, $\beta=\log_{2\lambda}(2\mu)<2$.

Then

$$\begin{aligned}\sum_{n=1}^{T}\bigl(\nabla F_2\bigl(n,u_k(n)\bigr),\tilde{u}_k(n)\bigr) &\geq -\sum_{n=1}^{T}h(n)G(\bar{u}_k) \\ &\geq -\sum_{n=1}^{T}h(n)\bigl(2\mu|\bar{u}_k|^{\beta}+1\bigr)G_0 \\ &= -2\mu R_4|\bar{u}_k|^{\beta}-R_4,\end{aligned} \tag{3.13}$$

where $R_4 = G_0 \sum_{n=1}^{T} h(n)$. For large k, according to (3.3) and (3.13) we have

$$\begin{aligned}
\|\tilde{u}_k\| &\geq \langle \varphi'(u_k), \tilde{u}_k \rangle \\
&= \sum_{n=1}^{T} \left|\Delta u_k(n)\right|^2 + \sum_{n=1}^{T} \left(\nabla F\left(n, u_k(n)\right), \tilde{u}_k(n)\right) \\
&\geq \left(1 - \frac{1}{8a_4 \sin^2 \frac{\pi}{T}}\right) \sum_{n=1}^{T} \left|\Delta u_k(n)\right|^2 - \frac{a_4}{2} R_1^2 |\bar{u}_k|^{2\alpha} \\
&\quad - \left(\frac{T^2-1}{6T}\right)^{(\alpha+1)/2} R_2 \left(\sum_{n=1}^{T} \left|\Delta u_k(n)\right|^2\right)^{(\alpha+1)/2} \\
&\quad - \left(\frac{T^2-1}{6T}\right)^{1/2} R_3 \left(\sum_{n=1}^{T} \left|\Delta u_k(n)\right|^2\right)^{1/2} - 2\mu R_4 |\bar{u}_k|^{\beta} - R_4.
\end{aligned} \tag{3.14}$$

By (3.5) and (3.14), one has

$$\begin{aligned}
&\frac{a_4}{2} R_1^2 |\bar{u}_k|^{2\alpha} + 2\mu R_4 |\bar{u}_k|^{\beta} \\
&\quad\geq \left(1 - \frac{1}{8a_4 \sin^2 \frac{\pi}{T}}\right) \sum_{n=1}^{T} \left|\Delta u_k(n)\right|^2 \\
&\qquad - \left(\frac{T^2-1}{6T}\right)^{(\alpha+1)/2} R_2 \left(\sum_{n=1}^{T} \left|\Delta u_k(n)\right|^2\right)^{(\alpha+1)/2} - R_4 \\
&\qquad - \left[\frac{(4\sin^2 \frac{\pi}{T} + 1)^{1/2}}{2\sin \frac{\pi}{T}} + \left(\frac{T^2-1}{6T}\right)^{1/2} R_3\right] \left(\sum_{n=1}^{T} \left|\Delta u_k(n)\right|^2\right)^{1/2} \\
&\quad\geq \frac{1}{2} \sum_{n=1}^{T} \left|\Delta u_k(n)\right|^2 + C_8,
\end{aligned} \tag{3.15}$$

where

$$\begin{aligned}
C_8 = \min_{s \in [0,+\infty)} \biggl\{ &\left(\frac{1}{2} - \frac{1}{8a_4 \sin^2 \frac{\pi}{T}}\right) s^2 - \left(\frac{T^2-1}{6T}\right)^{(\alpha+1)/2} R_2 s^{\alpha+1} \\
&- R_4 - \left[\frac{(4\sin^2 \frac{\pi}{T} + 1)^{1/2}}{2\sin \frac{\pi}{T}} + \left(\frac{T^2-1}{6T}\right)^{1/2} R_3\right] s \biggr\}.
\end{aligned}$$

By the choice of a_4, $-\infty < C_8 < 0$. By (3.15), we have

$$\sum_{n=1}^{T} \left|\Delta u_k(n)\right|^2 \leq a_4 R_1^2 |\bar{u}_k|^{2\alpha} + 4\mu R_4 |\bar{u}_k|^{\beta} - 2C_8, \tag{3.16}$$

and then

$$\left(\sum_{n=1}^{T} \left|\Delta u_k(n)\right|^2\right)^{1/2} \leq \sqrt{a_4} R_1 |\bar{u}_k|^{\alpha} + 2\sqrt{\mu R_4} |\bar{u}_k|^{\beta/2} + C_9, \tag{3.17}$$

where $C_9 > 0$. By (8) and (3.12), for any $u \in H_T$, we get

$$\begin{aligned}
&\sum_{n=1}^{T}\bigl[F_2\bigl(n,u(n)\bigr)-F_2(n,\bar{u})\bigr] \\
&\quad= -\sum_{n=1}^{T}\int_0^1\bigl(\nabla F_2\bigl(n,\bar{u}_k+s\tilde{u}_k(n)\bigr),\tilde{u}_k(n)\bigr)\,ds \\
&\quad\le \sum_{n=1}^{T}\int_0^1 h(n)G\bigl(\bar{u}_k+(s+1)\tilde{u}_k(n)\bigr)\,ds \\
&\quad\le \sum_{n=1}^{T}\int_0^1 h(n)\bigl(2\mu\bigl|\bar{u}_k+(s+1)\tilde{u}_k(n)\bigr|^{\beta}+1\bigr)G_0\,ds \\
&\quad\le 4\mu\sum_{n=1}^{T}h(n)\bigl(|\bar{u}_k|^{\beta}+2^{\beta}\bigl|\tilde{u}_k(n)\bigr|^{\beta}\bigr)G_0+R_4 \\
&\quad\le 2^{\beta+2}\mu R_4\|\tilde{u}_k\|_{\infty}^{\beta}+4\mu R_4|\bar{u}_k|^{\beta}+R_4 \\
&\quad\le \left(\frac{T^2-1}{6T}\right)^{\beta/2}2^{\beta+2}\mu R_4\left(\sum_{n=1}^{T}\bigl|\Delta u_k(n)\bigr|^2\right)^{\beta/2}+4\mu R_4|\bar{u}_k|^{\beta}+R_4.
\end{aligned} \tag{3.18}$$

Combining the boundedness of $\{\varphi(u_k)\}$ and (3.16)-(3.18), one has

$$\begin{aligned}
C_{10} &\le \varphi(u_k) \\
&= \frac{1}{2}\sum_{n=1}^{T}\bigl|\Delta u_k(n)\bigr|^2+\sum_{n=1}^{T}\bigl[F\bigl(n,u_k(n)\bigr)-F(n,\bar{u}_k)\bigr]+\sum_{n=1}^{T}F(n,\bar{u}_k) \\
&\le \left(\frac{1}{2}+\frac{1}{8a_4\sin^2\frac{\pi}{T}}\right)\sum_{n=1}^{T}\bigl|\Delta u_k(n)\bigr|^2+\frac{a_4}{2}R_1^2|\bar{u}_k|^2 \\
&\quad+\left(\frac{T^2-1}{6T}\right)^{(\alpha+1)/2}R_2\left(\sum_{n=1}^{T}\bigl|\Delta u_k(n)\bigr|^2\right)^{(\alpha+1)/2} \\
&\quad+\left(\frac{T^2-1}{6T}\right)^{1/2}R_3\left(\sum_{n=1}^{T}\bigl|\Delta u_k(n)\bigr|^2\right)^{1/2} \\
&\quad+\left(\frac{T^2-1}{6T}\right)^{\beta/2}2^{\beta+2}\mu R_4\left(\sum_{n=1}^{T}\bigl|\Delta u_k(n)\bigr|^2\right)^{\beta/2}+4\mu R_4|\bar{u}_k|^{\beta}+R_4+\sum_{n=1}^{T}F(n,\bar{u}_k) \\
&\le \left(\frac{1}{2}+\frac{1}{8a_4\sin^2\frac{\pi}{T}}\right)\bigl(a_4R_1^2|\bar{u}_k|^{2\alpha}+4\mu R_4|\bar{u}_k|^{\beta}-2C_8\bigr)+\frac{a_4}{2}R_1^2|\bar{u}_k|^{2\alpha} \\
&\quad+\left(\frac{T^2-1}{6T}\right)^{(\alpha+1)/2}R_2\bigl(\sqrt{a_4}R_1|\bar{u}_k|^{\alpha}+2\sqrt{\mu R_4}|\bar{u}_k|^{\beta/2}+C_9\bigr)^{\alpha+1} \\
&\quad+\left(\frac{T^2-1}{6T}\right)^{1/2}R_3\bigl(\sqrt{a_4}R_1|\bar{u}_k|^{\alpha}+2\sqrt{\mu R_4}|\bar{u}_k|^{\beta/2}+C_9\bigr) \\
&\quad+\left(\frac{T^2-1}{6T}\right)^{\beta/2}2^{\beta+2}\mu R_4\bigl(\sqrt{a_4}R_1|\bar{u}_k|^{\alpha}+2\sqrt{\mu R_4}|\bar{u}_k|^{\beta/2}+C_9\bigr)^{\beta}+4\mu R_4|\bar{u}_k|^{\beta}
\end{aligned}$$

$$
\begin{aligned}
&+ R_4 + \sum_{n=1}^{T} F(n,\bar{u}_k) \\
\le{}& \left(1 + \frac{1}{8a_4 \sin^2 \frac{\pi}{T}}\right) a_4 R_1^2 |\bar{u}_k|^{2\alpha} + \left(6 + \frac{1}{2a_4 \sin^2 \frac{\pi}{T}}\right) \mu R_4 |\bar{u}_k|^{\beta} \\
&- \left(1 + \frac{1}{4a_4 \sin^2 \frac{\pi}{T}}\right) C_8 \\
&+ \left(\frac{T^2-1}{6T}\right)^{(\alpha+1)/2} R_2 \bigl(2^{\alpha} a_4^{\frac{\alpha+1}{2}} R_1^{\alpha+1} |\bar{u}_k|^{\alpha(\alpha+1)} + 2^{3\alpha+1} \mu^{\frac{\alpha+1}{2}} R_4^{\frac{\alpha+1}{2}} |\bar{u}_k|^{\frac{\beta(\alpha+1)}{2}} + 2^{2\alpha} C_9^{\alpha+1}\bigr) \\
&+ \left(\frac{T^2-1}{6T}\right)^{1/2} R_3 \bigl(\sqrt{a_4} R_1 |\bar{u}_k|^{\alpha} + 2\sqrt{\mu R_4} |\bar{u}_k|^{\beta/2} + C_9\bigr) \\
&+ \left(\frac{T^2-1}{6T}\right)^{\beta/2} 2^{\beta+2} \mu R_4 \bigl(2^{\beta-1} a_4^{\frac{\beta}{2}} R_1^{\beta} |\bar{u}_k|^{\alpha\beta} + 2^{3\beta-2} \mu^{\frac{\beta}{2}} R_4^{\frac{\beta}{2}} |\bar{u}_k|^{\frac{\beta^2}{2}} + 2^{2(\beta-1)} C_9^{\beta}\bigr) \\
&+ R_4 + \sum_{n=1}^{T} F(n,\bar{u}_k) \\
={}& |\bar{u}_k|^{2\alpha} \Biggl[|\bar{u}_k|^{-2\alpha} \sum_{n=1}^{T} F_1(n,\bar{u}_k) + \left(a_4 + \frac{1}{8\sin^2 \frac{\pi}{T}}\right) R_1^2 + \left(\frac{T^2-1}{6T}\right)^{1/2} \sqrt{a_4} R_1 R_3 |\bar{u}_k|^{-\alpha} \\
&+ \left(\frac{T^2-1}{6T}\right)^{(\alpha+1)/2} 2^{\alpha} a_4^{\frac{\alpha+1}{2}} R_1^{\alpha+1} |\bar{u}_k|^{\alpha(\alpha-1)} + \left(\frac{T^2-1}{6T}\right)^{\beta/2} 2^{2\beta+1} \mu a_4^{\frac{\beta}{2}} R_1^{\beta} R_4 |\bar{u}_k|^{\alpha(\beta-2)} \Biggr] \\
&+ |\bar{u}_k|^{\beta} \Biggl[|\bar{u}_k|^{-\beta} \sum_{n=1}^{T} F_2(n,\bar{u}_k) + \left(6 + \frac{1}{2a_4 \sin^2 \frac{\pi}{T}}\right) \mu R_4 \\
&+ \left(\frac{T^2-1}{6T}\right)^{\beta/2} 2^{4\beta} \mu^{\frac{\beta+2}{2}} R_4^{\frac{\beta+2}{2}} |\bar{u}_k|^{\frac{1}{2}\beta^2-\beta} \\
&+ \left(\frac{T^2-1}{6T}\right)^{(\alpha+1)/2} 2^{3\alpha+1} \mu^{\frac{\alpha+2}{2}} R_2 R_4^{\frac{\alpha+1}{2}} |\bar{u}_k|^{\frac{\beta(\alpha-1)}{2}} + \left(\frac{T^2-1}{6T}\right)^{1/2} 2R_3 \sqrt{\mu R_4} |\bar{u}_k|^{-\beta/2} \Biggr] \\
&- \left(1 + \frac{1}{4a_4 \sin^2 \frac{\pi}{T}}\right) C_8 + \left(\frac{T^2-1}{6T}\right)^{(\alpha+1)/2} 2^{2\alpha} R_2 C_9^{\alpha+1} + \left(\frac{T^2-1}{6T}\right)^{1/2} R_3 C_9 \\
&+ \left(\frac{T^2-1}{6T}\right)^{\beta/2} 2^{3\beta} \mu R_4 C_9^{\beta} + R_4.
\end{aligned}
$$

Combining (3.11) and the above inequality, we see that $\{|\bar{u}|\}$ is bounded. By (3.16), $\{u_k\}$ is bounded. Since H_T is a finite dimensional Hilbert space, φ satisfies the (PS) condition.

Similar to the proof of Theorem 1.2, all conditions of Lemma 2.2 are satisfied. So, the proof of Theorem 1.4 is completed. □

4 Examples

In this section, we give some examples to illustrate our results.

Example 4.1 Let $F = F_1 + F_2$, with

$$
F_1(n,x) = \left(\frac{T+1}{2} - n\right) |x|^{7/4} + (2T-n)|x|^{3/2} + \bigl(k(n),x\bigr),
$$

$$F_2(x) = C(x) - \frac{3r}{4}|x|^{4/3},$$

where $k : \mathbb{Z}[1, T] \longrightarrow \mathbb{R}$ and $k(n + T) = k(n)$, for all $n \in \mathbb{Z}$, $r > 0$, $C(x) = \frac{3r}{4}(|x_1|^{4/3} + |x_2|^{4/3} + \cdots + |x_N|^{4/3})$. It is easy to see that

$$\begin{aligned}\left|\nabla F_1(n,x)\right| &\le \frac{7}{8}|T+1-2n||x|^{3/4} + \frac{3}{2}|2T-n||x|^{1/2} + \left|k(n)\right| \\ &\le \frac{7}{8}\bigl(|T+1-2n| + \varepsilon\bigr)|x|^{3/4} + \frac{9T^2}{\varepsilon^2} + \left|k(n)\right|.\end{aligned}$$

For all $(n,x) \in \mathbb{Z}[1, T] \times \mathbb{R}^N$, where $\varepsilon > 0$,

$$\bigl(\nabla F_2(x) - \nabla F_2(y), x - y\bigr) \ge -r|x-y|^{4/3}.$$

Thus, (1), (2) hold with $\alpha = 3/4$, $\gamma = 4/3$, and

$$f(n) = \frac{7}{8}\bigl(|T+1-2n| + \varepsilon\bigr), \qquad g(n) = \frac{9T^2}{\varepsilon^2} + \left|k(n)\right|.$$

So, we have

$$\begin{aligned}&|x|^{-2\alpha}\sum_{n=1}^{T} F(n,x) \\ &\quad= |x|^{-3/2}\sum_{n=1}^{T}\left[\left(\frac{T+1}{2} - n\right)|x|^{7/4} + (2T-n)|x|^{3/2} + C(x) - \frac{3r}{4}|x|^{4/3} + \bigl(k(n),x\bigr)\right] \\ &\quad= \frac{T(3T-1)}{2} + \frac{T(C(x) - \frac{3r}{4}|x|^{4/3})}{|x|^{3/2}} + \left(\sum_{n=1}^{T} k(n), |x|^{-3/2}x\right).\end{aligned}$$

On the other hand, one has

$$\frac{1}{8\sin^2\frac{\pi}{T}}\sum_{n=1}^{T} f^2(n) = \frac{1}{8\sin^2\frac{\pi}{T}}\sum_{n=1}^{T}\left[\frac{7}{8}\bigl(|T+1-2n| + \varepsilon\bigr)\right]^2 \le \frac{49[T(T^2-1+6\varepsilon T+2\varepsilon^2)]}{1{,}536\sin^2\frac{\pi}{T}}.$$

If $T \in \{2,3,4,5,6,7\}$, we can choose $\varepsilon > 0$ such that

$$\liminf_{|x|\longrightarrow+\infty} |x|^{-2\alpha}\sum_{n=1}^{T} F(n,x) = \frac{T(3T-1)}{2} > \frac{1}{8\sin^2\frac{\pi}{T}}\sum_{n=1}^{T} f^2(n).$$

So, (3) holds. By Theorem 1.1, system (1.1) has at least one T-periodic solution.

Example 4.2 Let $F = F_1 + F_2$, with

$$\begin{aligned}F_1(n,x) &= \left(\frac{T+1}{2} - n\right)|x|^{7/4} - (2T-n)|x|^{3/2} + \bigl(k(n),x\bigr), \\ F_2(x) &= -\frac{4r}{5}|x|^{5/4},\end{aligned}$$

where $k : \mathbb{Z}[1, T] \longrightarrow \mathbb{R}^N$ and $k(n + T) = k(n)$ for all $n \in \mathbb{Z}$, $r > 0$.

In a way similar to Example 4.1, it is easy to see that condition (1) and (4) are satisfied with $\alpha = 3/4$. So,

$$\begin{aligned}
&|x|^{-2\alpha}\sum_{n=1}^{T}F(n,x)\\
&= |x|^{-3/2}\sum_{n=1}^{T}\left[\left(\frac{T+1}{2}-n\right)|x|^{7/4}-(2T-n)|x|^{3/2}-\frac{4r}{5}|x|^{5/4}+\bigl(k(n),x\bigr)\right]\\
&= -\frac{T(3T-1)}{2}-\frac{4r}{5}|x|^{-1/4}+\left(\sum_{n=1}^{T}k(n),|x|^{-3/2}x\right).
\end{aligned}$$

If $T\in\{2,3,4,5\}$, we can choose $\varepsilon>0$ small enough such that

$$\limsup_{|x|\longrightarrow+\infty}|x|^{-2\alpha}\sum_{n=1}^{T}F(n,x) = -\frac{T(3T-1)}{2} < -\frac{3}{8\sin^2\frac{\pi}{T}}\sum_{n=1}^{T}f^2(n),$$

which implies that (5) holds. By Theorem 1.2, system (1.1) has at least one T-periodic solution.

Example 4.3 Let $F = F_1 + F_2$, with

$$\begin{aligned}
&F_1(n,x) = \left(\frac{T+1}{2}-n\right)|x|^{7/4}-\left(\frac{T-3n}{2}\right)|x|^{3/2}+\bigl(k(n),x\bigr),\\
&F_2(x) = C(x)-\frac{r}{2}|x|^2,
\end{aligned}$$

where $k:\mathbb{Z}[1,T]\longrightarrow\mathbb{R}$ and $k(n+T)=k(n)$ for all $n\in\mathbb{Z}$, $r>0$, $C(x)=\frac{r}{2}(|x_1|^4+|x_2|^2+\cdots+|x_N|^2)$, $0<r<\frac{6}{T^2-1}$.

In a way similar to Example 4.1, it is easy to see that conditions (1) and (6) are satisfied with $\alpha = 3/4$. So

$$\begin{aligned}
&|x|^{-2\alpha}\sum_{n=1}^{T}F(n,x)\\
&= |x|^{-3/2}\sum_{n=1}^{T}\left[\left(\frac{T+1}{2}-n\right)|x|^{7/4}-\left(\frac{T-3n}{2}\right)|x|^{3/2}+C(x)-\frac{r}{2}|x|^2+\bigl(k(n),x\bigr)\right]\\
&= \frac{T(T+3)}{4}+\frac{T(C(x)-\frac{r}{2}|x|^2)}{|x|^{3/2}}+\left(\sum_{n=1}^{T}k(n),|x|^{-3/2}x\right)\\
&= \frac{T(T+3)}{4}+\frac{rT(|x|_1^4-|x|_1^2)}{2|x|^{3/2}}+\left(\sum_{n=1}^{T}k(n),|x|^{-3/2}x\right).
\end{aligned}$$

If $T\in\{2,3\}$, we choose $\varepsilon>0$, such that

$$\liminf_{|x|\longrightarrow+\infty}|x|^{-2\alpha}\sum_{n=1}^{T}F(n,x) = \frac{T(T+3)}{4} > \frac{3}{(24-4(T^2-1)r)\sin^2\frac{\pi}{T}}\sum_{n=1}^{T}f^2(n),$$

which implies that (7) holds. By Theorem 1.3, system (1.1) has at least one T-periodic solution.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The main idea of this paper was proposed by KY and WG, WG prepared the manuscript initially and KY performed a part of the steps of the proofs in this research. All authors read and approved the final manuscript.

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