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Darboux transformation of the general Hirota equation: multisoliton solutions, breather solutions, and rogue wave solutions

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Abstract

In this paper, we investigate the exact solutions and conservation laws of a general Hirota equation. Firstly, the N -fold Darboux transformation of this equation is proposed. Then by choosing three kinds of seed solutions, the multisoliton solutions, breather solutions, and rogue wave solutions of the general Hirota equation are obtained based on the Darboux transformation. Finally, the conservation laws of this equation are derived by using its linear spectral problem. The results in this paper may be useful in the study of ultrashort optical solitons in optical fibers.

Keywords: Darboux transformation; multisoliton solutions; breather solutions; rogue wave solutions; optical fibers

1 Introduction

During the past decades, many researches have been focused on optical solitons due to their potential applications in optical fiber long-distance transmission systems. According to the theoretical report and experimental results in [1, 2], optical solitons are based on the balance between the group velocity dispersion and self-phase modulation in the picosecond regime, and the propagation of such a soliton is governed by the standard nonlinear Schrödinger equation (NLS) [3, 4],

$$iu_t + \frac{1}{2}u_{xx} + |u|^2u = 0, \quad (1)$$

where $u = u(x, t)$ denotes the slowly varying complex envelope of the wave, and subscripts x and t are the longitudinal distance and retarded time, respectively.

In recent years, the ultrashort pulses have attracted much interest because of their applications in optical fibers. When they are considered, the NLS equation (1) cannot describe the corresponding physical characteristics, and it accounts for the following three points: first, the fourth-order dispersion should be considered when the pulse width is below 10 femtoseconds [5, 6]; second, the higher-order nonlinearities should not be neglected when the optical field frequency approaches a resonant frequency of the optical fibers material [6, 7]; third, the self-steepening and self-frequency shift should be included when extremely narrow pulse has very high optical intensity as the fourth-order dispersion

and cubic-quintic nonlinearities are considered [4, 8]. Comprehensively considering these three factors, in this paper, we study the higher-order nonlinear Schrödinger equation

$$u_t = i\alpha_1 u_{xx} + i\alpha_2 |u|^2 u + \alpha_3 \epsilon u_{xxx} + 3\alpha_4 \epsilon |u|^2 u_x + 2\alpha_5 \epsilon |u|^2 u_x, \quad (2)$$

where u is the complex amplitude of the pulse envelope, the parameter ϵ denotes the relative width of the spectrum that arises due to the quasi-monochromaticity, and α_1 , α_2 , α_3 , α_4 , and α_5 are real constants. It is shown in [9] that when $\alpha_2 = 2\delta^2\alpha_3$, $\alpha_4 = 6\delta^2\alpha_3$, and $\alpha_5 = -6\delta^2\alpha_3$, Eq. (2) becomes the bright soliton version of the general Hirota equation

$$u_t = \alpha_1 (iu_{xx} + 2i\delta^2 |u|^2 u) + \epsilon \alpha_3 (u_{xxx} + 6\delta^2 |u|^2 u_x), \quad (3)$$

which further becomes the following general Hirota equation by putting $\alpha_1 = \alpha$ and $\alpha_3 = 1$:

$$u_t = \alpha (iu_{xx} + 2i\delta^2 |u|^2 u) + \epsilon (u_{xxx} + 6\delta^2 |u|^2 u_x). \quad (4)$$

Note that when $\epsilon = 0$, $\alpha = \frac{1}{2}$, and $\delta^2 = 1$, the general Hirota equation (4) becomes the standard focusing NLS equation (1). When $\alpha = 0$, it becomes the complex modified Korteweg-de Vries equation [10], namely

$$u_t = \epsilon (u_{xxx} + 6\delta^2 |u|^2 u_x). \quad (5)$$

The aim of this paper is to find some new-type explicitly exact solutions and conservation laws of the general Hirota equation (4). In recent years, some work [11–18] has been done to investigate the exact solutions and numerical solutions of various differential equation. Especially, some results on the Hirota equation are proposed by analytical [11–14] and numerical [18] methods. However, here we consider the general Hirota equation (4) in a different way and derive the multisoliton solutions, breather solutions, rogue wave solutions, and conservation laws of this equation. This paper is organized as follows. The Lax pair and N -fold Darboux transformation of Eq. (4) are presented in Section 2. In Section 3, the multisoliton solutions, breather solutions, and rogue wave solutions are obtained via the proposed Darboux transformation, and the dynamics of these exact solutions is analyzed by their density distributions. In Section 4, some conservation laws for the Eq. (4) are listed explicitly. Conclusions are addressed in the last section.

2 Lax pair and Darboux transformation

The Lax pair (*i.e.*, linear spectral problem) [9] of the general Hirota equation (4) is

$$\varphi_x = U\varphi, \quad \varphi_t = V\varphi, \quad (6)$$

with matrices U and V of the forms

$$U = -i\lambda U_0 + U_1, \quad V = i\lambda^3 V_3 - \lambda^2 V_2 + \lambda V_1 + V_0, \quad (7)$$

where $\varphi = (f, g)^T$ (here T denotes the transpose), λ is a spectral parameter, and

$$U_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad U_1 = \begin{pmatrix} 0 & \delta u \\ -\delta u^* & 0 \end{pmatrix}, \quad (8)$$

$$V_3 = 4\epsilon U_0, \quad V_2 = 2i\alpha U_0 + 4\epsilon U_1,$$

$$V_1 = \begin{pmatrix} -2i\epsilon\delta^2|u|^2 & 2(\alpha\delta u - i\epsilon\delta u_x) \\ -2(\alpha\delta u^* - i\epsilon\delta u_x^*) & 2i\epsilon\delta^2|u|^2 \end{pmatrix}, \quad V_0 = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & -a_{11} \end{pmatrix}, \quad (9)$$

with

$$a_{11} = \epsilon\delta^2 u_x u^* - \epsilon\delta^2 u_x^* u + i\alpha\delta^2 |u|^2, \quad a_{12} = \epsilon\delta u_{xx} + i\alpha\delta u_x + 2\epsilon\delta^3 |u|^2 u, \quad (10)$$

$$a_{21} = -\epsilon\delta u_{xx}^* + i\alpha\delta u_x^* - 2\epsilon\delta^3 |u|^2 u^*. \quad (11)$$

Through direct calculations, we can verify that the zero-curvature equation $U_t - V_x + [U, V] = 0$ exactly gives rise to the general Hirota equation (4).

Darboux transformation (DT) technique is a method that can derive multisoliton solutions from trivial seed solutions in a purely algebraic procedure for the integrable nonlinear wave equations [19]. Main feature of DT is that the Lax pair associated with the nonlinear wave equations remains covariant under the gauge transformation. By using the Lax pair (6) we construct the gauge transformation of Eq. (4) as

$$\varphi^{(1)} = T^{(1)}\varphi = (\lambda I - S)\varphi, \quad (12)$$

where I denotes the identity matrix, and $S = (s_{ij}^{(0)})_{2 \times 2}$ is a 2×2 matrix. Furthermore, if the matrix S takes the form

$$S = H\Lambda H^{-1} \quad (13)$$

with

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1^* \end{pmatrix}, \quad H = \begin{pmatrix} f_1 & -g_1^* \\ g_1 & f_1^* \end{pmatrix}, \quad (14)$$

$$S = \frac{1}{\Delta} \begin{pmatrix} \lambda_1 f_1 f_1^* + \lambda_1^* g_1 g_1^* & f_1 g_1^* (\lambda_1 - \lambda_1^*) \\ f_1^* g_1 (\lambda_1 - \lambda_1^*) & \lambda_1^* f_1 f_1^* + \lambda_1 g_1 g_1^* \end{pmatrix}, \quad (15)$$

$$\det H = \Delta = |f_1|^2 + |g_1|^2, \quad (16)$$

then we can verify that if $(f_1, g_1)^T$ is a solution of the Lax pair (6) with $\lambda = \lambda_1$, then $(-g_1^*, f_1^*)^T$ is also a solution of the Lax pair (6) corresponding to $\lambda = \lambda_1^*$. Thus, under the gauge transformation $\varphi^{(1)} = T^{(1)}\varphi$, the Lax pair (6) becomes

$$\varphi_x^{(1)} = U^{(1)}\varphi^{(1)}, \quad \varphi_t^{(1)} = V^{(1)}\varphi^{(1)}, \quad (17)$$

with

$$U^{(1)} = -i\lambda U_0^{(1)} + U_1^{(1)}, \quad V^{(1)} = i\lambda^3 V_3^{(1)} - \lambda^2 V_2^{(1)} + \lambda V_1^{(1)} + V_0^{(1)}, \quad (18)$$

where $U_0^{(1)}$, $U_1^{(1)}$, and $V_i^{(1)}$ ($j = 0, 1, 2, 3$) have the same forms as U_0 , U_1 , and V_i ($j = 0, 1, 2, 3$) in Eqs. (8)-(11) with u replaced by $u^{(1)}$.

Through the gauge transformation (12), the relationships among $T^{(1)}$, U , and $U^{(1)}$ can be found:

$$T_x^{(1)} + T^{(1)}U = U^{(1)}T^{(1)}, \quad (19)$$

$$T_t^{(1)} + T^{(1)}V = V^{(1)}T^{(1)}. \quad (20)$$

By direct computation based on Eqs. (19)-(20), we can obtain a relation between potential functions $u^{(1)}$ and u :

$$u^{(1)} = u - \frac{2is_{21}^{(0)}}{\delta}. \quad (21)$$

So the DT of the general Hirota equation (4) is defined as

$$\varphi^{(1)} = T^{(1)}\varphi, \quad u^{(1)} = u - \frac{2is_{21}^{(0)}}{\delta}, \quad (22)$$

where

$$T^{(1)} = \lambda_1 I - S = \begin{pmatrix} \lambda_1 - s_{11}^{(0)} & -s_{12}^{(0)} \\ -s_{21}^{(0)} & \lambda_1 - s_{22}^{(0)} \end{pmatrix}, \quad s_{21}^{(0)} = \frac{f_1^* g_1 (\lambda_1 - \lambda_1^*)}{|f_1|^2 + |g_1|^2}. \quad (23)$$

If N distinct seed solutions φ_k ($k = 1, 2, \dots, N$) of Eq. (6) are given, then the basic DT may be iterated. To do the second step of transformation, we employ φ_2 , which is mapped to $\varphi_2^{(1)} = T^{(1)}|_{\lambda=\lambda_2} \varphi_2$; therefore,

$$\varphi^{(2)} = T^{(2)}\varphi^{(1)} = T^{(2)}T^{(1)}\varphi, \quad u^{(2)} = u^{(1)} - \frac{2is_{21}^{(1)}}{\delta} = u - \frac{2i}{\delta}(s_{21}^{(0)} + s_{21}^{(1)}), \quad (24)$$

where

$$T^{(2)} = \begin{pmatrix} \lambda_2 - s_{11}^{(1)} & -s_{12}^{(1)} \\ -s_{21}^{(1)} & \lambda_2 - s_{22}^{(1)} \end{pmatrix}, \quad s_{21}^{(1)} = \frac{f_2^{(1)*} g_2^{(1)} (\lambda_2 - \lambda_2^*)}{|f_2^{(1)}|^2 + |g_2^{(1)}|^2}. \quad (25)$$

In the general case, we have the following theorem.

Theorem 1 Let $\varphi_1, \varphi_2, \dots, \varphi_N$ be N distinct solutions of the spectral problem (6) at $\lambda_1, \dots, \lambda_N$, respectively. Then the N -fold DT for the general Hirota equation (4) is

$$\varphi^{(n)} = T^{(n)}T^{(n-1)} \dots T^{(1)}\varphi, \quad u^{(n)} = u - \frac{2i}{\delta} \sum_{j=0}^{n-1} s_{21}^{(j)}, \quad (26)$$

with

$$T^{(j+1)} = \begin{pmatrix} \lambda_{j+1} - s_{11}^{(j)} & -s_{12}^{(j)} \\ -s_{21}^{(j)} & \lambda_{j+1} - s_{22}^{(j)} \end{pmatrix}, \quad s_{21}^{(j)} = \frac{f_{j+1}^{(j)*} g_{j+1}^{(j)} (\lambda_{j+1} - \lambda_{j+1}^*)}{|f_{j+1}^{(j)}|^2 + |g_{j+1}^{(j)}|^2}, \quad (27)$$

$$\varphi^{(j-1)} = (T^{(j-1)} T^{(j-2)} \dots T^{(1)})|_{\lambda=\lambda_j} \varphi, \quad (28)$$

where $\varphi^{(j+1)} = (f_{j+1}^j, g_{j+1}^j)^T$ is the solution of the spectral problem (6) at $\lambda = \lambda_{j+1}$.

3 Multisoliton solutions, breather solutions, and rogue wave solutions

In this section, we give some explicit solutions of Eq. (4) via the DT in Section 2. Now we take the nonzero continuous wave (cw) solution $u = ce^{i(ax+bt)}$ as the initial seed for Eq. (4), where a, b, c are all real parameters. Equation (4) requires the frequency b to satisfy the following nonlinear dispersion relation:

$$b = \epsilon(6a\delta^2 c^2 - a^3) + \alpha(2\delta^2 c^2 - a^2). \quad (29)$$

Thus, the cw seed solution of the general Hirota equation (4) is

$$u = ce^{i(ax + (\epsilon(6a\delta^2 c^2 - a^3) + \alpha(2\delta^2 c^2 - a^2))t)}. \quad (30)$$

Setting $(f_1, g_1)^T$ to be the solution of the spectral problem (6), we have

$$f_{1x} = -i\lambda f_1 + \delta u g_1, \quad g_{1x} = i\lambda g_1 - \delta u^* f_1, \quad (31)$$

$$f_{1t} = \nu_{11} f_1 + \nu_{12} g_1, \quad g_{1t} = \nu_{21} f_1 + \nu_{22} g_1, \quad (32)$$

where

$$\nu_{11} = 4i\epsilon\lambda^3 - 2i\alpha\lambda^2 - 2i\epsilon\delta^2 u^2 \lambda + i\alpha\delta^2 u^2, \quad (33)$$

$$\nu_{22} = -4i\epsilon\lambda^3 + 2i\alpha\lambda^2 + 2i\epsilon\delta^2 u^2 \lambda - i\alpha\delta^2 u^2, \quad (34)$$

$$\nu_{12} = -4\epsilon\delta u \lambda^2 + (2\alpha\delta u - 2i\epsilon\delta u_x)\lambda + \epsilon\delta u_{xx} + i\alpha\delta u_x + 2\epsilon\delta^3 u^3, \quad (35)$$

$$\nu_{21} = 4\epsilon\delta u \lambda^2 + (-2\alpha\delta u - 2i\epsilon\delta u_x)\lambda - \epsilon\delta u_{xx} + i\alpha\delta u_x - 2\epsilon\delta^3 u^3. \quad (36)$$

After tedious computations, we obtain that

$$f_1 = (C_1 e^A + C_2 e^{-A}) e^B, \quad (37)$$

$$g_1 = i \left[C_2 \frac{\sqrt{(a+2\lambda)^2 + 4\delta^2 c^2} + (a+2\lambda)}{2\delta c} - C_1 \frac{\sqrt{(a+2\lambda)^2 + 4\delta^2 c^2} - (a+2\lambda)}{2\delta c} \right] e^{-B}, \quad (38)$$

with

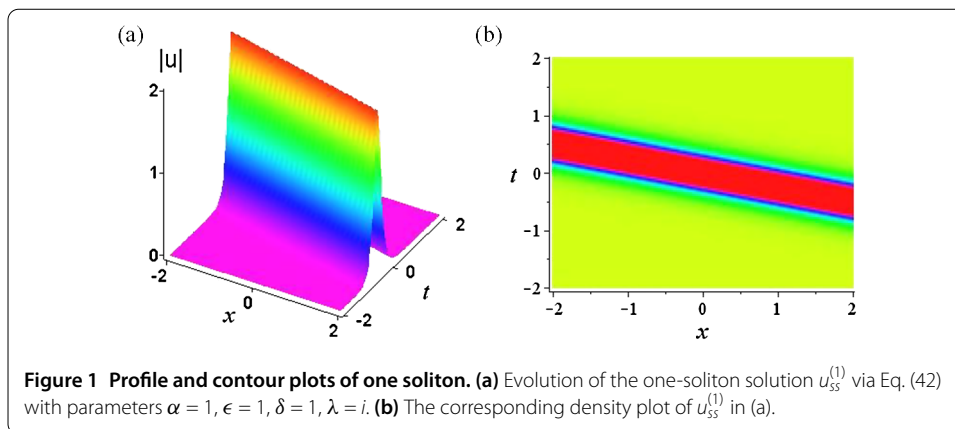
$$A = \frac{i\sqrt{(a+2\lambda)^2 + 4\delta^2 c^2}}{2} [x - (a^2\epsilon + 4\lambda^2\epsilon - 2\delta^2 c^2\epsilon - 2\lambda a\epsilon + \alpha a - 2\lambda\alpha)t], \quad (39)$$

$$B = \frac{i}{2} [ax - (a^3\epsilon - 6\epsilon\delta^2 c^2 a + a^2\alpha - 2\alpha\delta^2 c^2)t]. \quad (40)$$

3.1 Multisoliton solutions

In this subsection, we present the multisoliton solutions of Eq. (4) explicitly. To do so, take $c = 0$, that is, zero initial seed solution for Eq. (4). The solutions of the spectral problem (6) in Eqs. (37) and (38) are reduced to the following forms:

$$f_1 = e^{-i\lambda x + (4i\epsilon\lambda^3 - 2i\alpha\lambda^2)t}, \quad g_1 = e^{i\lambda x - (4i\epsilon\lambda^3 - 2i\alpha\lambda^2)t}. \quad (41)$$



(1) When $N = 1$ in the Darboux transformation of Theorem 1, after some calculations, we have

$$u_{ss}^{(1)} = -\frac{2\eta_1}{\delta} e^{-2i(\epsilon_1 x + t(4\epsilon\epsilon_1(3\eta_1^2 - \epsilon_1^2) + 2\alpha(\epsilon_1^2 - \eta_1^2)))} \operatorname{sech}(2\eta_1(x + 4(\epsilon\eta_1^2 - 3\epsilon\epsilon_1^2 + \epsilon_1\alpha)t)), \quad (42)$$

where $\lambda = \epsilon_1 + i\eta_1$, and $u_{ss}^{(N)}$ is the N th-order soliton solution.

The solution $u_{ss}^{(1)}$ represents a bright single soliton whose dynamic features are delineated in Figure 1. We can conclude some physical quantities from this bright single soliton: the maximum amplitude is $|\frac{2\eta_1}{\delta}|$; the width is $\frac{1}{2\eta_1}$; the envelop velocity is $-4\eta_1(\epsilon\eta_1^2 - 3\epsilon\epsilon_1^2 + \epsilon_1\alpha)$; the frequency is $-4\epsilon\epsilon_1(3\eta_1^2 - \epsilon_1^2) - 2\alpha(\epsilon_1^2 - \eta_1^2)$, and the energy E is equal to $|\frac{4\eta_1}{\delta^2}|$.

(2) When $N = 2$ in the Darboux transformation of Theorem 1, we obtain the following two-soliton solution of Eq. (4):

$$u_{ss}^{(2)} = u_{ss}^{(1)} - \frac{2if_2^{(1)*}g_2^{(1)}(\lambda_2 - \lambda_2^*)}{\delta(|f_2^{(1)}|^2 + |g_2^{(1)}|^2)}, \quad (43)$$

where $u_{ss}^{(1)}$ is given in Eq. (42), and $(f_2^{(1)}, g_2^{(1)})^T$ is the solution of the spectral problem (6) at $\lambda = \lambda_2$.

The dynamics of the two-soliton solution $u_{ss}^{(2)}$ is delineated in Figure 2 with parameters $\alpha = 1$, $\epsilon = 1$, $\delta = 1$, $\lambda_1 = i$, $\lambda_2 = 2i$. The solution $u_{ss}^{(2)}$ represents the elastic interaction between two bright solitons whose dynamic features are delineated in Figure 2. Figure 2 shows the overtaking interaction between two bell-shape solitons for $u_{ss}^{(2)}$ in Eq. (43), from which we can find that the solitonic shapes and amplitudes have not changed after the interaction.

(3) When $N = 3$, we obtain the following three-soliton solution of Eq. (4):

$$u_{ss}^{(3)} = u_{ss}^{(2)} - \frac{2if_3^{(2)*}g_3^{(2)}(\lambda_3 - \lambda_3^*)}{\delta(|f_3^{(2)}|^2 + |g_3^{(2)}|^2)}, \quad (44)$$

where $u_{ss}^{(2)}$ is given in Eq. (43), and $(f_3^{(2)}, g_3^{(2)})^T$ is the solution of the spectral problem (6) at $\lambda = \lambda_3$.

The dynamic features of the three-soliton solution $u_{ss}^{(3)}$ is delineated in Figure 3 with parameters $\alpha = 1$, $\epsilon = 1$, $\delta = 1$, $\lambda_1 = i$, $\lambda_2 = 2i$, $\lambda_3 = 4i$. The solution $u_{ss}^{(3)}$ represents the elastic interaction among three bright solitons whose dynamic features are delineated in

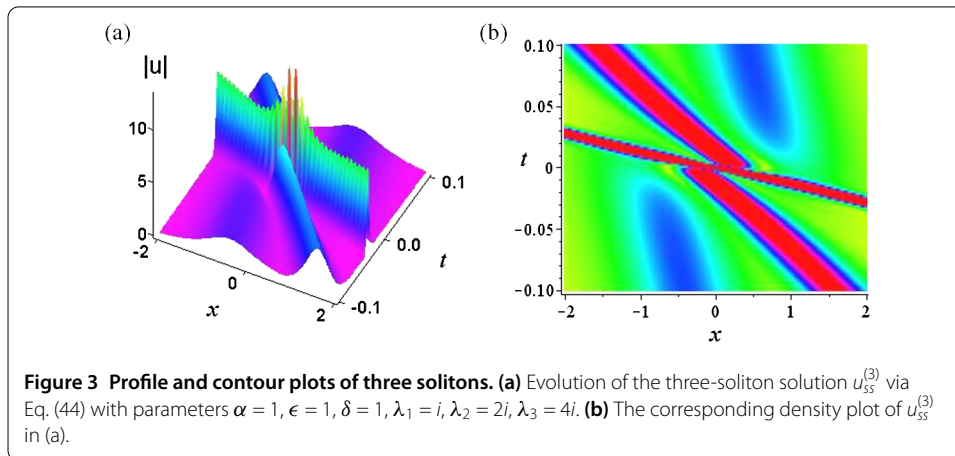
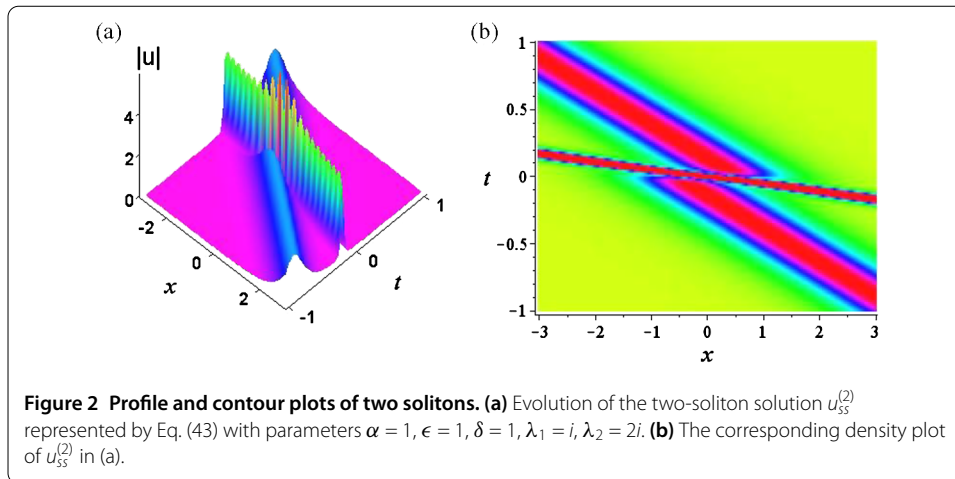


Figure 3. Figure 3 shows the overtaking interaction among three bell-shape bright solitons for $u_{ss}^{(3)}$ in Eq. (44), from which we can find that the solitonic shapes and amplitudes have not changed after the interaction.

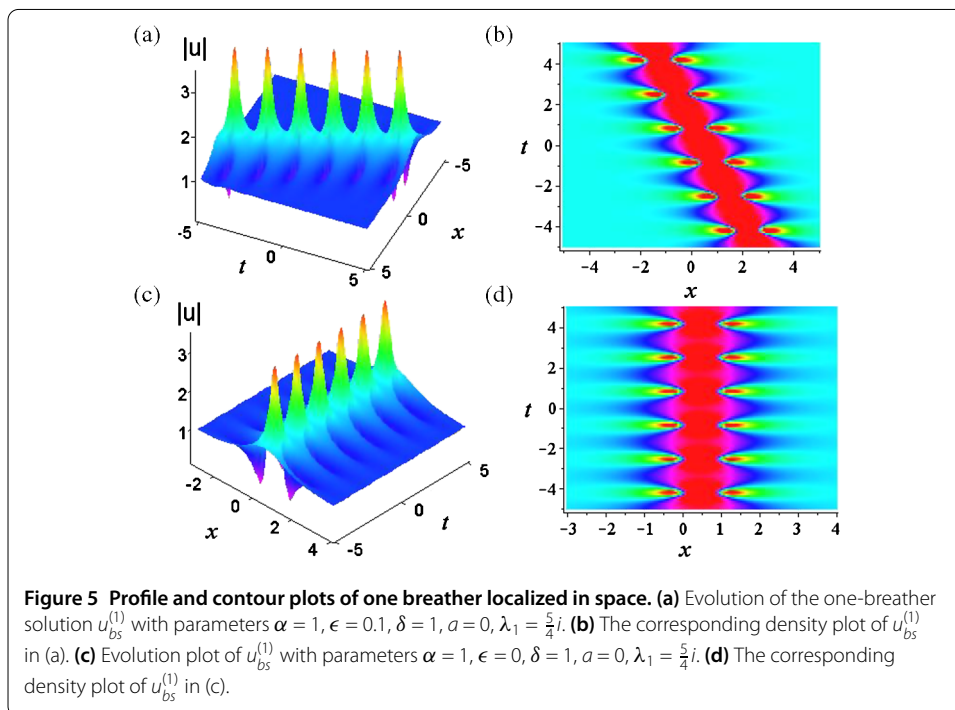
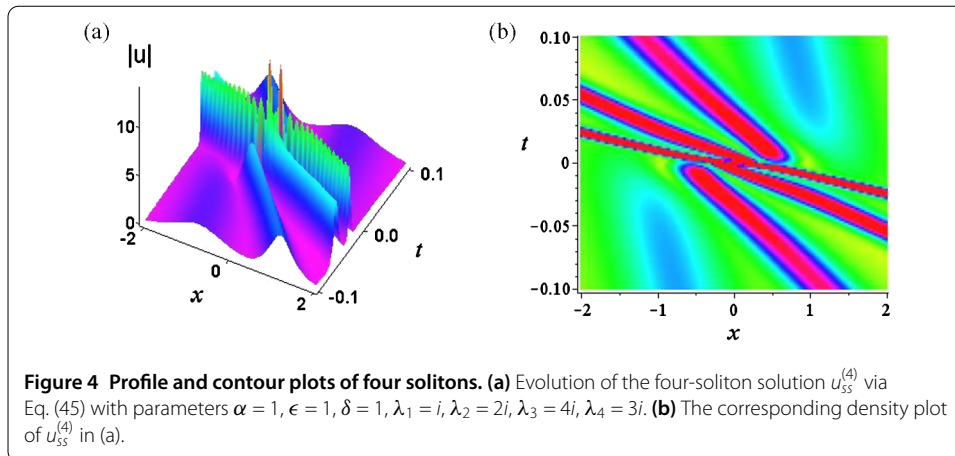
(4) When $N = 4$, we also obtain the four-soliton solution of Eq. (4) by the same procedure:

$$u_{ss}^{(4)} = u_{ss}^{(3)} - \frac{2i f_4^{(3)*} g_4^{(3)} (\lambda_4 - \lambda_4^*)}{\delta |f_4^{(3)}|^2 + |g_4^{(3)}|^2}, \quad (45)$$

where $u_{ss}^{(3)}$ is given in Eq. (44), and $(f_4^{(3)}, g_4^{(3)})^T$ is the solution of the spectral problem (6) at $\lambda = \lambda_4$.

The dynamic features of the four-soliton solution $u_{ss}^{(4)}$ is delineated in Figure 4 with parameters $\alpha = 1, \epsilon = 1, \delta = 1, \lambda_1 = i, \lambda_2 = 2i, \lambda_3 = 4i, \lambda_4 = 3i$. The solution $u_{ss}^{(4)}$ represents the elastic interaction among four bright solitons whose dynamic features are shown in Figure 4. Figure 4 shows the overtaking interaction among four bell-shape bright solitons for solution $u_{ss}^{(4)}$ in Eq. (45), from which we can find that the solitonic shapes and amplitudes have not changed after the interaction.

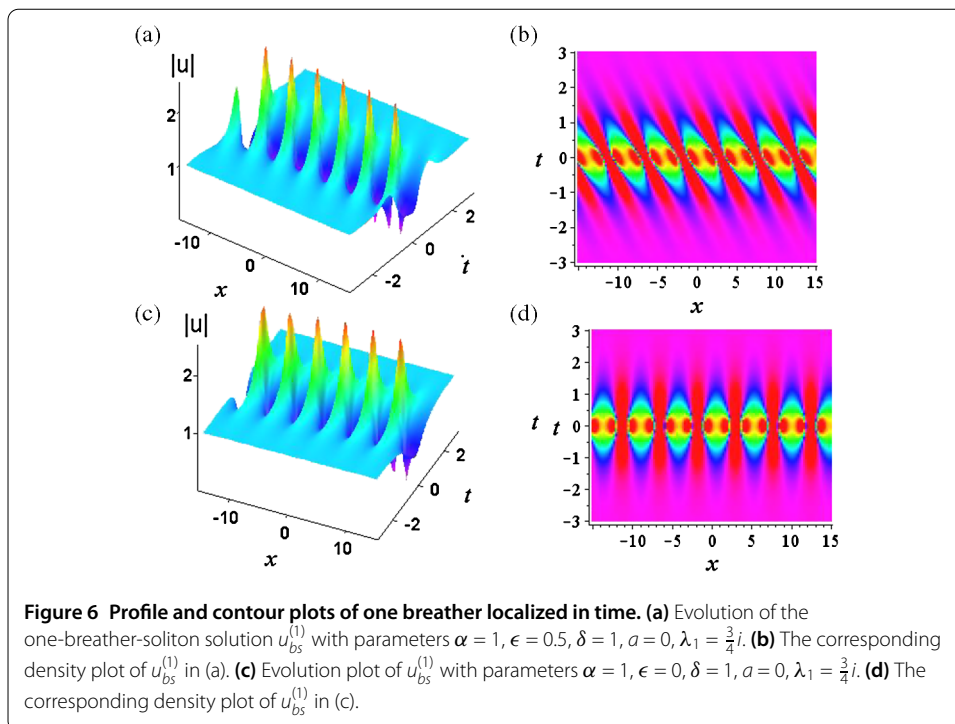
We remark that we can also get the high-order soliton solutions of the general Hirota equation (4) by continuing iteration of the DT in Theorem 1.



3.2 Multibreather solutions

In this subsection, we consider the case of parameter $c \neq 0$ in the seed solution of the general Hirota equation (4), which we will start with the plane wave seed solution. Without loss of generality, assuming that $c = 1$ and substituting the solution of the spectral problem (6) given by Eqs. (37) and (38) into the 1-fold DT in Eq. (22), the one-breather solution of the general Hirota equation (4) can be obtained immediately. Hereby, iterating the DT by using Theorem 1 we can derive the multibreather solutions of Eq. (4), where $u_{bs}^{(N)}$ represents the N th-order breather solution for simplicity. In the following, we only discuss two cases, $N = 1$ and $N = 2$. Moreover, we omit the expressions of the breather solutions because they are too long to write down.

(1) When $N = 1$, the one-breather solution $u_{bs}^{(1)}$ is derived from the plane wave initial solution and 1-fold DT in Eq. (22), whose dynamic features are delineated in Figures 5 and 6.



Figures 5(a), (b) and 6(a), (b) display the structures of the one-breather solution $u_{bs}^{(1)}$ for the general Hirota equation (4) with $\epsilon \neq 0$, and Figures 5(c), (d) and 6(c), (d) display the same case except the parameter $\epsilon = 0$. We see that the one-breather solution is periodic both in space and time for the parameter $\epsilon \neq 0$. Moreover, comparing Figure 5 with Figure 6, we observe that when $|\text{Im}(\lambda)| > 1$, the evolution and density plots for the one-breather solution $u_{bs}^{(1)}$ are periodic in time (*i.e.*, Ma soliton), however, they are periodic in space (*i.e.*, Akhmediev soliton) for $0 < |\text{Im}(\lambda)| < 1$ under the same parameters. We can see that the parameter ϵ can change the shape and the arrangement of bright solitons.

(2) When $N = 2$, the two-breather solution $u_{bs}^{(2)}$ is also derived from the plane wave initial solution and 2-fold DT in Theorem 1, whose dynamic features are delineated in Figures 7 and 8.

From Figures 7 and 8 we know that: when $\epsilon = 0$, $\text{Re}(\lambda_1) = 0$, $\text{Re}(\lambda_2) = 0$, $0 < |\text{Im}(\lambda_1)| < 1$, $|\text{Im}(\lambda_2)| > 1$, the solution $u_{bs}^{(2)}$ represents the elastic interaction between a space periodic breather solution and a time-periodic breather solution (see Figure 7(a), (b)); when $\epsilon = 0$, $\text{Re}(\lambda_1) = 0$, $\text{Re}(\lambda_2) \neq 0$, $0 < |\text{Im}(\lambda_1)| < 1$, the solution $u_{bs}^{(2)}$ represents the elastic interaction between two breather solutions, in which one is a space-periodic breather solution, the other is a periodic breather solution in time and space (see Figure 7(c), (d)); when $\epsilon = 0$, $\text{Re}(\lambda_1) = 0$, $\text{Re}(\lambda_2) = 0$, $0 < |\text{Im}(\lambda_1)| < 1$, $0 < |\text{Im}(\lambda_2)| < 1$, the solution $u_{bs}^{(2)}$ represents the elastic interaction between two space periodic breather soliton solutions (see Figure 8(a), (b)); when $\epsilon = 0$, $\text{Re}(\lambda_1) = 0$, $\text{Re}(\lambda_2) = 0$, $|\text{Im}(\lambda_1)| > 1$, $|\text{Im}(\lambda_2)| > 1$, the solution $u_{bs}^{(2)}$ represents the interaction between two time-periodic breather soliton solutions (see Figure 8(c), (d)).

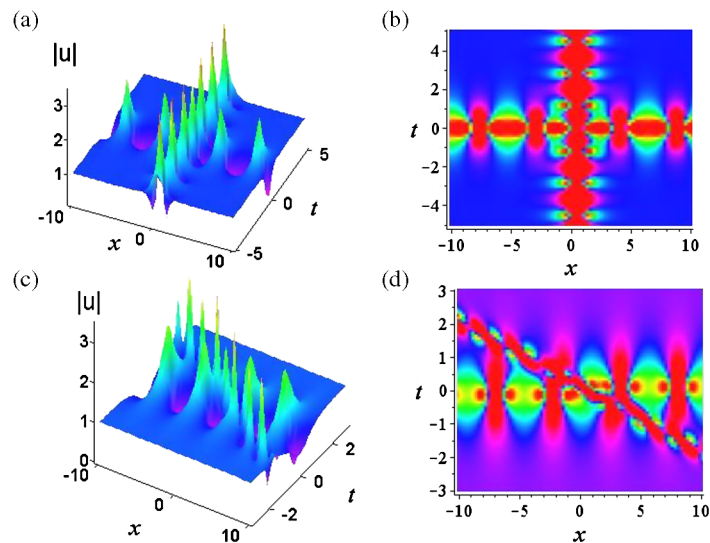


Figure 7 Profile and contour plots of two breathers of first kind. **(a)** Evolution of the two-breather solution $u_{bs}^{(2)}$ with parameters $\alpha = 1, \epsilon = 0, \delta = 1, a = 0, \lambda_1 = \frac{3}{4}i, \lambda_2 = \frac{5}{4}i$. **(b)** The corresponding density plot of $u_{bs}^{(2)}$ in (a). **(c)** Evolution plot of $u_{bs}^{(2)}$ with parameters $\alpha = 1, \epsilon = 0, \delta = 1, a = 0, \lambda_1 = \frac{3}{4}i, \lambda_2 = 1 + i$. **(d)** The corresponding density plot of $u_{bs}^{(2)}$ in (c).

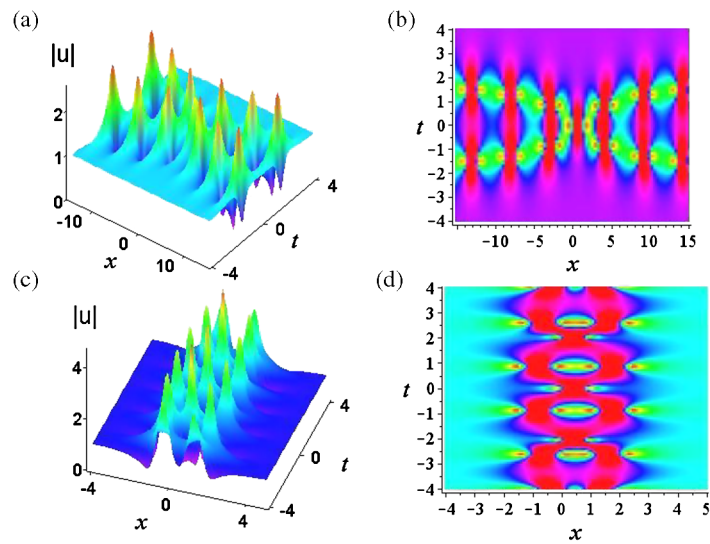


Figure 8 Profile and contour plots of two breathers of second kind. **(a)** Evolution of the two-breather solution $u_{bs}^{(2)}$ with parameters $\alpha = 1, \epsilon = 0, \delta = 1, a = 0, \lambda_1 = \frac{3}{4}i, \lambda_2 = \frac{4}{5}i$. **(b)** The corresponding density plot of $u_{bs}^{(2)}$ in (a). **(c)** Evolution of the two-breather solution $u_{bs}^{(2)}$ with parameters $\alpha = 1, \epsilon = 0, \delta = 1, a = 0, \lambda_1 = \frac{5}{4}i, \lambda_2 = \frac{3}{2}i$. **(d)** The corresponding density plot of $u_{bs}^{(2)}$ in (c).

3.3 Rogue wave solutions

Next, we manage to search a generalized DT. Suppose that $\varphi_2 = \varphi_1(\lambda_1 + \sigma)$ is a special solution for Eq. (6). Then after transformation we have $\varphi_2^{(1)} = T_1^{(1)} \varphi_2$. Expanding φ_2 at λ_1 , we have from [20]

$$\varphi_1(\lambda_1 + \sigma) = \varphi_1 + \varphi_1^{[1]} \sigma + \varphi_1^{[2]} \sigma^2 + \cdots + \varphi_1^{[N]} \sigma^N + \cdots, \quad (46)$$

where $\varphi_1^{[k]} = \frac{1}{k!} \frac{\partial^k}{\partial \lambda^k} \varphi_1(\lambda)|_{\lambda=\lambda_1}$, and σ is a small parameter. Through the limit process

$$\lim_{\sigma \rightarrow 0} \frac{[T_1^{(1)}|_{\lambda=\lambda_1+\sigma}] \varphi_2}{\sigma} = \lim_{\sigma \rightarrow 0} \frac{[\sigma + T_1^{(1)}|_{\lambda=\lambda_1}] \varphi_2}{\sigma} = \varphi_1 + T_1^{(1)}|_{\lambda=\lambda_1} \varphi_1^{[1]} \equiv \varphi_1^{(1)}, \quad (47)$$

we can find another solution to the linear system Eq. (6) with $u^{(1)}$ and spectral parameter $\lambda = \lambda_1$. This allows us to go to the next step of the DT, namely, Eq. (24).

Similarly, the limit

$$\begin{aligned} \lim_{\sigma \rightarrow 0} \frac{[\sigma + T_1^{(2)}|_{\lambda=\lambda_1}][\sigma + T_1^{(1)}|_{\lambda=\lambda_1}] \varphi_2}{\sigma^2} \\ = \varphi_1 + [T_1^{(1)} + T_1^{(2)}]|_{\lambda=\lambda_1} \varphi_1^{[1]} + T_1^{(2)}|_{\lambda=\lambda_1} T_1^{(1)}|_{\lambda=\lambda_1} \varphi_1^{[2]} \equiv \varphi_1^{(2)} \end{aligned} \quad (48)$$

provides a nontrivial solution for the linear spectral problem with $u^{(2)}$ and $\lambda = \lambda_1$. Thus, we may do the third-step iteration of the DT as follows:

$$\varphi^{(3)} = T_1^{(3)} \varphi^{(2)}, \quad u^{(3)} = u^{(2)} - \frac{2is_{21}^{(2)}}{\delta}. \quad (49)$$

Continuing this process and combining all the DT, a generalized DT is constructed. Thus, we have the following theorem.

Theorem 2 Let $\varphi_1, \varphi_2, \dots, \varphi_N$ be N distinct solutions of the spectral problem Eq. (6) at $\lambda_1, \dots, \lambda_N$, respectively, and

$$\varphi_i(\lambda_i + \sigma) = \varphi_i + \varphi_i^{[1]} \sigma + \varphi_i^{[2]} \sigma^2 + \dots + \varphi_i^{[N]} \sigma^N + \dots \quad (i = 1, 2, \dots, n) \quad (50)$$

be their expansions, where

$$\varphi_i^{[j]} = \frac{1}{j!} \frac{\partial^j}{\partial \lambda^j} \varphi_i(\lambda)|_{\lambda=\lambda_i} \quad (j = 1, 2, \dots). \quad (51)$$

Define

$$T = \Gamma_n \Gamma_{n-1} \cdots \Gamma_1 \Gamma_0, \quad \Gamma_k = T_k^{(m_k)} \cdots T_k^{(1)} \quad (i \geq 1), \quad \Gamma_0 = I, \quad (52)$$

where

$$T_k^{(j+1)} = \begin{pmatrix} \lambda_k - s_{11}^{(j)} & -s_{12}^{(j)} \\ -s_{21}^{(j)} & \lambda_k - s_{22}^{(j)} \end{pmatrix}, \quad (53)$$

$$\varphi_k^{(j)} = \lim_{\sigma \rightarrow 0} \frac{[\sigma + T_k^{(j)}|_{\lambda=\lambda_k}] \cdots [\sigma + T_k^{(2)}|_{\lambda=\lambda_k}] [\sigma + T_k^{(1)}|_{\lambda=\lambda_k}] \Gamma_{k-1}(\lambda_k + \sigma) \cdots \Gamma_1(\lambda_k + \sigma) \Gamma_0 \varphi_k(\lambda_k + \sigma)}{\sigma^j} \quad (54)$$

($1 \leq j < m_i$). Then the transformations

$$\varphi^{(N)} = T\varphi, \quad u^{(N)} = u + \frac{2i}{\delta} \sum_{j=0}^{n-1} \sum_{k=1}^{m_i} s_{21}^{(j)}[m_k] \quad \left(N = n - 1 + \sum_{k=1}^{n-1} m_k \right) \quad (55)$$

are the generalized DT for the general Hirota equation (4).

We remark that the solution expressed in Eq. (55) is written in terms of summations, which is very easy to understand. In fact, for nonzero $\varphi^{(k)}$ ($k = 1, 2, \dots, N$), all the denominators of $s_{21}^{(j)}$ are easily seen to be nonzero in these forms; therefore, Eq. (55) provides some nonsingular solutions.

Let us consider an example to illustrate the application of Theorem 2 to the construction of rogue wave solutions. To do so, starting with the seed solution $u = e^{8it}$, the solution of the linear spectral problem (6) at $\lambda = ih$ is

$$\varphi_1(f) = \begin{pmatrix} e^B(C_1 e^A + C_2 e^{-A}) \\ ie^{-B}(C_1 e^{-A} - C_2 e^A) \end{pmatrix}, \quad (56)$$

where

$$C_1 = \frac{\sqrt{(2\sqrt{(4+\lambda^2)}+2\lambda)}}{2}, \quad C_2 = \frac{\sqrt{(2\sqrt{(4+\lambda^2)}-2\lambda)}}{2}, \quad B = 4it. \quad (57)$$

Let $h = 2 + f^2$. Expanding the vector function $\varphi_1(f)$ at $f = 0$, we have

$$\varphi_1(f) = \varphi_1^{[0]} + \varphi_1^{[1]}f^2 + \varphi_1^{[2]}f^4 + \varphi_1^{[3]}f^6 + \dots, \quad (58)$$

where

$$\varphi_1^{[0]} = \begin{pmatrix} \sqrt{2}e^{4it} \\ -\sqrt{2}e^{-4it} \end{pmatrix}, \quad (59)$$

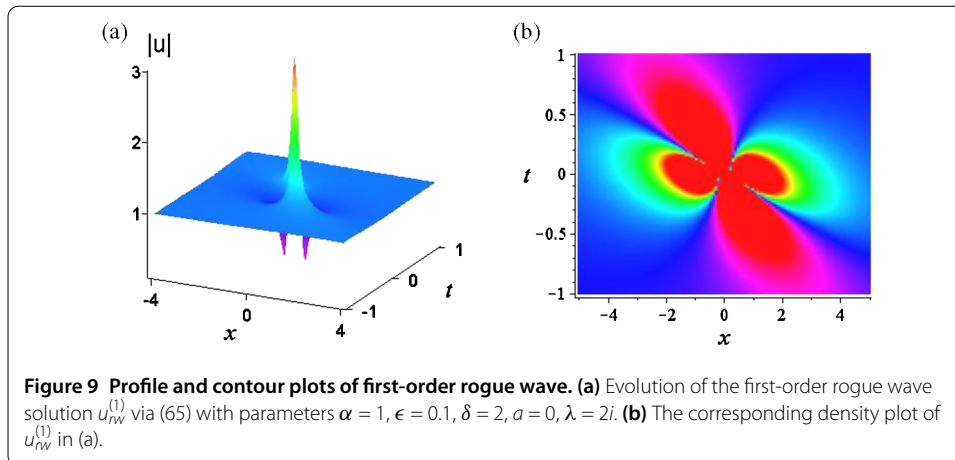
$$\varphi_1^{[1]} = \begin{pmatrix} \frac{\sqrt{2}}{200}e^{4it}((7,680 + 4,096i)t^2 + (800 - 480i)t - 400ix^2 - 200ix + (3,200 - 1,920i)xt - 25i) \\ -\frac{\sqrt{2}}{200}e^{-4it}((7,680 + 4,096i)t^2 + (480i - 800)t - 400ix^2 + 200ix + (3,200 - 1,920i)xt - 25i) \end{pmatrix}, \quad (60)$$

$$\varphi_1^{[2]} = \begin{pmatrix} P \\ Q \end{pmatrix}, \quad (61)$$

with

$$\begin{aligned} P = & \frac{\sqrt{2}e^{it}}{240,000} (1,875 - 456,000t - 30,000x - 1,152,000x^2t + 64,880,640t^3x \\ & - 1,536,000tx^3 + 9,830,400t^2x^2 + 4,915,200xt^2 - 2,400,000tx + 5,836,800t^2 \\ & - 180,000x^2 - 160,000x^4 - 160,000x^3 + 16,220,160t^3 + 42,205,184t^4 \\ & - 1,920,000itx^2 - 4,608,000itb_1 + 480,000d_1 - 2,560,000ix^3t + 1,920,000xd_1 \\ & - 3,360,000ixt - 1,920,000ixb_1 + 7,680,000tb_1 + 4,608,000td_1 - 819,200it^3 \\ & - 9,216,000it^2x - 600,000it - 18,432,000it^2x^2 - 3,276,800it^3x \\ & + 7,680,000itd_1 - 14,208,000it^2 + 62,914,560it^4 - 480,000ib_1), \end{aligned} \quad (62)$$

$$\begin{aligned} Q = & \frac{\sqrt{2}e^{it}}{240,000} (1,875 + 456,000t + 30,000x - 1,920,000ixb_1 - 3,276,800it^3x \\ & + 480,000ib_1 + 1,152,000x^2t + 64,880,640t^3x - 1,536,000tx^3 + 9,830,400t^2x^2 \end{aligned}$$



$$\begin{aligned}
 & -4,915,200xt^2 - 2,400,000tx + 7,680,000itd_1 - 3,360,000ixt + 5,836,800t^2 \\
 & -180,000x^2 - 160,000x^4 + 160,000x^3 - 16,220,160t^3 + 42,205,184t^4 \\
 & -2,560,000ix^3t - 4,608,000itb_1 + 1,920,000itx^2 + 819,200it^3 + 1,920,000xd_1 \\
 & + 600,000it - 14,208,000it^2 + 62,914,560it^4 + 9,216,000it^2x \\
 & + 7,680,000tb_1 + 4,608,000td_1 - 18,432,000it^2x^2 - 480,000d_1). \quad (63)
 \end{aligned}$$

It is clear that $\varphi_1^{[0]}$ is a solution for Eq. (6) at $\lambda = 2i$. In the following, we only discuss two cases, $N = 1$ and $N = 2$.

(1) When $N = 1$, by means of formula (47) we can obtain

$$\varphi_1^{(1)} = \lim_{f \rightarrow 0} \frac{[f^2 + T_1^{(1)}]\varphi_1(f)}{f^2} = T_1^{(1)}\varphi_1^{[1]} + i\varphi_1^{[0]}. \quad (64)$$

From Eqs. (59), (60), and (64) we derive a simplified form of the first-order rogue wave solution:

$$u_{rw}^{(1)} = -\frac{e^{8it}(75 - 8,704t^2 - 1,920xt - 400x^2 + 1,600it)}{25 + 8,704t^2 + 1,920xt + 400x^2}, \quad (65)$$

where $u_{rw}^{(N)}$ is the N th-order rogue wave solution.

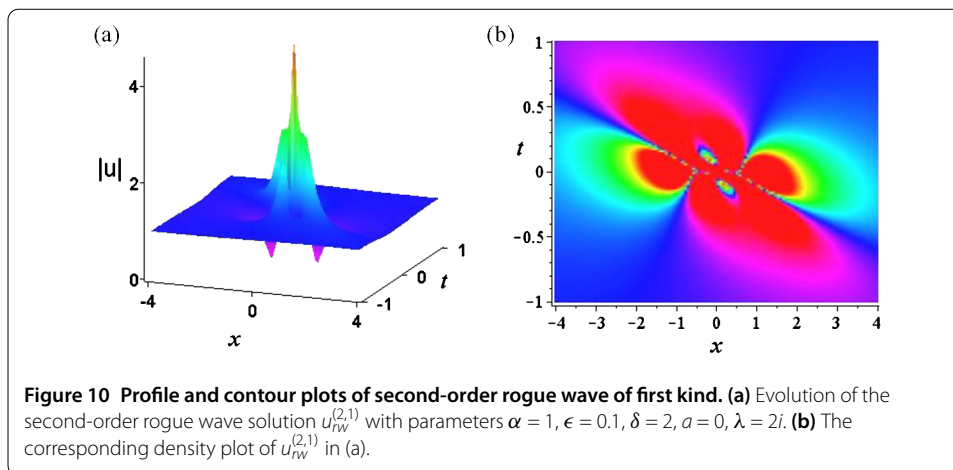
Figure 9 displays the evolution and density plots of $u_{rw}^{(1)}$ in Eq. (65), which shows that the first-order rogue wave solution is localized in both space and time.

(2) When $N = 2$, by means of formula (54) we have

$$\varphi_1^{(2)} = \lim_{f \rightarrow 0} \frac{[f^2 + T_1^{(2)}][f^2 + T_1^{(1)}]\varphi_1(f)}{f^4} = T_1^{(2)}T_1^{(1)}\varphi_1^{[2]} + (T_1^{(1)} + T_1^{(2)})\varphi_1^{[1]} + \varphi_1^{[0]}, \quad (66)$$

which helps to derive the second-order rogue wave solution with parameters $b_1 = 0$, $d_1 = 0$ in Eqs. (62) and (63) as

$$u_{rw}^{(2,1)} = \frac{M_1}{N_1} \quad (67)$$



with

$$\begin{aligned}
 M_1 = & e^{8it} \left(-659,411,697,664t^6 + (363,646,156,800i - 436,375,388,160x)t^5 \right. \\
 & + (160,432,128,000ix + 52,592,640,000 - 187,170,816,000x^2)t^4 \\
 & + (4,300,800,000i - 47,185,920,000x^3)t^3 + (8,110,080,000x^2 \\
 & - 8,601,600,000x^4 + 460,800,000ix + 7,372,800,000ix^3 + 648,000,000)t^2 \\
 & + (140,400,000x + 499,200,000x^3 - 45,000,000i - 288,000,000ix^2 \\
 & \left. - 921,600,000x^5)t - 64,000,000x^6 - 703,125 \right), \quad (68)
 \end{aligned}$$

$$\begin{aligned}
 N_1 = & 659,411,697,664t^6 + 436,375,388,160xt^5 + (187,170,816,000x^2 \\
 & + 36,981,964,800)t^4 - (147,456,000x - 47,185,920,000x^3)t^3 \\
 & + (596,160,000 - 1,843,200,000x^2 + 8,601,600,000x^4)t^2 \\
 & + (921,600,000x^5 - 38,400,000x^3 + 61,200,000x)t \\
 & + 140,625 + 64,000,000x^6 + 12,000,000x^4 + 6,750,000x^2. \quad (69)
 \end{aligned}$$

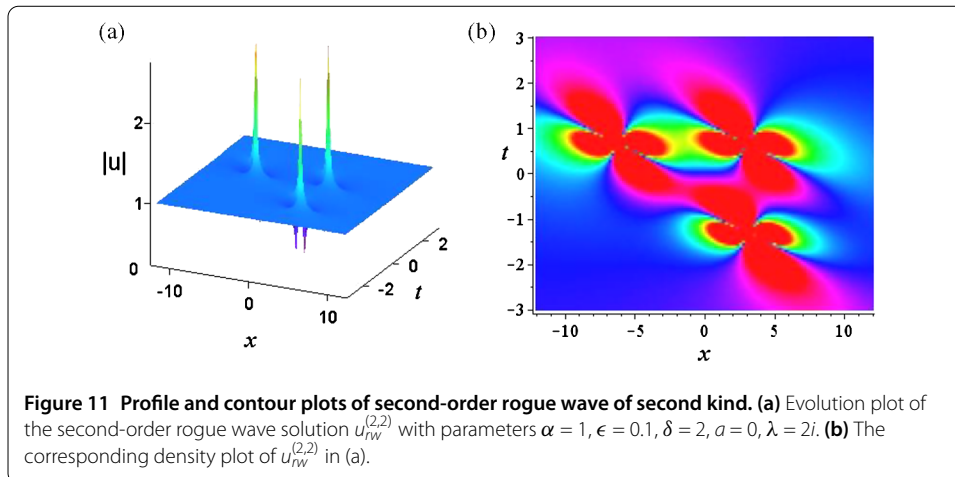
We show the structure of the second-order rogue wave solution Eq. (67) in Figure 10 under the parameters $\alpha = 1, \epsilon = 0.1, \delta = 2, a = 0, \lambda = 2i$.

When choosing parameters $b_1 = 100, d_1 = 0$ in Eqs. (62) and (63), we can obtain another second-order rogue wave solution of the general Hirota equation (4) as

$$u_{rw}^{(2,2)} = \frac{M_2}{N_2} \quad (70)$$

with

$$\begin{aligned}
 M_2 = & (-659,411,697,664t^6 + (363,646,156,800i - 436,375,388,160x)t^5 \\
 & + (52,592,640,000 - 187,170,816,000x^2 + 160,432,128,000ix)t^4 \\
 & + (98,304,000,000 + 24,920,064,000x - 47,185,920,000x^3 \\
 & + 4,300,800,000i + 51,118,080,000ix^2)t^3 + (648,000,000
 \end{aligned}$$



$$\begin{aligned}
& + 8,110,080,000x^2 - 8,601,600,000x^4 + 1,105,920,000,000x \\
& + 294,912,000,000i + 7,372,800,000ix^3 + 460,800,000ix)t^2 \\
& + (14,400,000,000 + 140,400,000x + 499,200,000x^3 - 921,600,000x^5 \\
& + 230,400,000,000x^2 - 45,000,000i + 768,000,000ix^4 \\
& - 138,240,000,000ix - 288,000,000ix^2)t + (1,125,000 - 28,800,000,000i)x^2 \\
& + 36,000,000x^4 - 64,000,000x^6 - 1,440,000,703,125 \\
& - 1,800,000,000i)e^{8it}, \tag{71}
\end{aligned}$$

$$\begin{aligned}
N_2 = & 659,411,697,664t^6 + 436,375,388,160xt^5 + (36,981,964,800 \\
& + 187,170,816,000x^2)t^4 - (98,304,000,000 + 147,456,000x \\
& - 47,185,920,000x^3)t^3 + (596,160,000 - 1,105,920,000,000x \\
& - 1,843,200,000x^2 + 8,601,600,000x^4)t^2 \\
& + (921,600,000x^5 - 230,400,000,000x^2 - 38,400,000x^3 \\
& + 61,200,000x + 43,200,000,000)t + 12,000,000x^4 \\
& + 64,000,000x^6 + 1,440,000,140,625 + 6,750,000x^2. \tag{72}
\end{aligned}$$

We show the structure of the second-order rogue wave solution in Eq. (70) by Figure 11 under the parameters $\alpha = 1, \epsilon = 0.1, \delta = 2, a = 0, \lambda = 2i$.

4 Conservation laws

Conservation laws play a key role in discussing the integrability for the nonlinear wave equations. In what follows, we would like to derive the conservation laws of the general Hirota equation (4) by using the linear spectral problem (6), that is,

$$\varphi_{1x} + i\lambda\varphi_1 = \delta u\varphi_2, \quad \varphi_{2x} - i\lambda\varphi_2 = -\delta u^*\varphi_1, \tag{73}$$

$$\varphi_{1t} = A\varphi_1 + B\varphi_2, \quad \varphi_{2t} = C\varphi_1 - A\varphi_2. \tag{74}$$

Setting $\Gamma_1 = \frac{\varphi_2}{\varphi_1}$, we have the following equations from Eqs. (73) and (74):

$$\Gamma_{1x} = 2i\lambda\Gamma_1 - \delta u^* - \delta u\Gamma_1^2, \quad (75)$$

$$\Gamma_{1t} = C - 2A\Gamma_1 - B\Gamma_1^2. \quad (76)$$

From Eqs. (73)-(76) it is easy to get the following equation:

$$\delta u\Gamma_1 = \frac{-\delta^2 uu^* - (\delta u\Gamma_1)^2 - \delta u(\frac{\delta u\Gamma_1}{\delta u})_x}{-2i\lambda}. \quad (77)$$

Expanding $\delta u\Gamma_1$ into the power series with respect to $(-i\lambda)$,

$$\delta u\Gamma_1 = \sum_{n=1}^{\infty} \frac{f_n}{(-i\lambda)^n}, \quad (78)$$

and substituting Eq. (78) into Eq. (77), we can get the following recursion relations:

$$\sum_{n=1}^{\infty} \frac{f_n}{(-i\lambda)^n} = \frac{-\delta^2 uu^* - [\sum_{n=1}^{\infty} \frac{f_n}{(-i\lambda)^n}]^2 - \delta u[\sum_{n=1}^{\infty} \frac{f_n}{(-i\lambda)^n \delta u}]_x}{-2i\lambda}. \quad (79)$$

Furthermore, the following conservation law equation can be obtained from Eqs. (73)-(76):

$$(\delta u\Gamma_1)_t = (A + B\Gamma_1)_x. \quad (80)$$

Substituting Eq. (79) into Eq. (80), we derive an infinite number of conservation laws for Eq. (4) immediately. The first three conservation laws describe the energy conservation, momentum conservation, and Hamiltonian conservation. We list them as follows:

$$(uu^*)_t = [3\delta^2 \epsilon u^2 u^{*2} + \epsilon(uu_{xx}^* - u_x u_x^* + u_{xx} u^*) - i\alpha(uu_x^* - u_x u^*)]_x, \quad (81)$$

$$(uu_x^*)_t = [4\delta^2 \epsilon u^2 u_x^* u_x^* + \epsilon(uu_{xxx}^* - u_x u_{xx}^* + 2u_{xx} u_x^* + 4u^2 u_x^*) - \delta^2 i\alpha u^2 u^{*2} + i\alpha(2u_x u_x^* - uu_{xx}^*)]_x, \quad (82)$$

$$\begin{aligned} & (u^2 u^* + uu_{xx}^*)_t \\ &= [\delta^2 \alpha_3 (8u^2 u^* u_{xx}^* + 5u^2 u_x^{*2} + 2uu_x u_x^* u_x^* + uu_{xx} u_x^{*2} + uu_{xxx}^* - u_x^2 u^{*2} + uu^{*2} u_{xx}) \\ & \quad + 4\delta^4 \epsilon u^3 u^{*3} - 4\delta^2 i\alpha u^2 u^* u_x^* + i\alpha(u_x u_{xx}^* - uu_{xxx}^*) + \epsilon(u_{xx} u_{xx}^* - u_x u_{xxx}^*)]_x. \end{aligned} \quad (83)$$

5 Conclusions

In summary, we have studied the exact solutions and conservation laws of the general Hirota equation (4) based on its linear spectral problem (6). We have derived an N -fold Darboux transformation of this equation and then obtained some exact multisoliton solutions and breather solutions. Introducing a generalized Darboux transformation, some rogue wave solutions of this equation have been also proposed explicitly. Moreover, the dynamical features of these exact solutions are analyzed by their density distributions. At last, the conservation laws of this general Hirota equation are proposed by means of its linear spectral problem.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have made equal contributions. All authors have read and approved the final manuscript.

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