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A Philos-type theorem for second-order neutral delay dynamic equations with damping

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Abstract

We establish a Philos-type oscillation theorem for a class of nonlinear second-order neutral delay dynamic equations with damping on a time scale by using the Riccati transformation and integral averaging technique. An illustrative example is provided to show that our theorem has practicability and maneuverability.

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1 Introduction

Oscillation, as a kind of physical phenomena, widely exists in the natural sciences and engineering. For instance, self-excited vibration in control system, beam vibration in synchrotron accelerator, the complicated oscillation in chemical reaction, and so forth. The assorted phenomena can be unified into oscillation theory of equations; see [1]. On the basis of these background details, we are concerned with the oscillation of a nonlinear second-order damped delay dynamic equation of neutral type

$$[A(t)\varphi(y^\Delta(t))]^\Delta + b(t)\varphi(y^\Delta(t)) + P(t)F(\varphi(x(\delta(t)))) = 0, \quad (1.1)$$

where $t \in [t_0, +\infty)_{\mathbf{T}} := [t_0, +\infty) \cap \mathbf{T}$, \mathbf{T} is a time scale satisfying $\sup \mathbf{T} = +\infty$, $t_0 \in \mathbf{T}$, $\varphi(u) := |u|^{\lambda-1}u$, $0 < \lambda \leq 1$, $y(t) := x(t) + B(t)g(x(\tau(t)))$, and

(A₁) $A, B, b, P \in C_{rd}([t_0, +\infty)_{\mathbf{T}}, \mathbf{R})$, $A(t) > 0$, $0 \leq B(t) \leq b_0 < +\infty$, $b(t) \geq 0$, $-b/A \in \mathfrak{R}^+$ (i.e., $A(t) - \mu(t)b(t) > 0$), $\int_{t_0}^{+\infty} [e_{-b/A}(t, t_0)/A(t)]^{1/\lambda} \Delta t = +\infty$, $P(t) > 0$, $F \in C(\mathbf{R}, \mathbf{R})$, $uF(u) > 0$, and $F(u)/u \geq L > 0$ for all $u \neq 0$, where b_0 and L are constants;

(A₂) $\tau \in C_{rd}^1([t_0, +\infty)_{\mathbf{T}}, \mathbf{T})$, $\delta \in C_{rd}([t_0, +\infty)_{\mathbf{T}}, \mathbf{T})$, $\tau(t) \leq t$, $\delta(t) \leq t$, $\tau([t_0, +\infty)_{\mathbf{T}}) = [\tau(t_0), +\infty)_{\mathbf{T}}$, $\lim_{t \rightarrow +\infty} \delta(t) = +\infty$, $\tau^\Delta(t) \geq \tau_0 > 0$, $\tau \circ \delta = \delta \circ \tau$, and $\delta(t) \geq \tau(t)$, where τ_0 is a constant;

(A₃) $g \in C(\mathbf{R}, \mathbf{R})$, $ug(u) > 0$, and $g(u)/u \leq \eta \leq 1$ for all $u \neq 0$, where $\eta > 0$ is a constant.

By a solution to equation (1.1) we mean a nontrivial real-valued function x satisfying (1.1) for $t \in [t_0, +\infty)_{\mathbf{T}}$. A solution of (1.1) is said to be oscillatory if it has arbitrarily large generalized zeros on $[t_0, +\infty)_{\mathbf{T}}$; otherwise, it is termed nonoscillatory. Equation (1.1) is called

oscillatory if all its solutions are oscillatory. It should be noted that the solutions vanishing in some neighborhood of infinity will be excluded from our consideration.

The theory of oscillation is an important branch of the qualitative theory of dynamic equations. In recent years, there has been a great deal of interest in studying oscillatory behavior of solutions to dynamic equations on various classes of time scales; see, for example, [1–26] and the references therein. To establish sufficient conditions for oscillation of dynamic equations, one usually uses either an integral averaging technique involving integrals and weighted integrals of coefficients of a given dynamic equation (see, e.g., [1–4, 6, 8–22, 24–26]) or comparison methods and linearization techniques; see, for instance, the papers by Agarwal *et al.* [5], Zhang *et al.* [23], and the references therein. For some concepts related to the notion of time scales, see [27–29].

Let us briefly comment on a number of closely related background details, which strongly motivated our research in this paper. Assuming that $B(t) = 0$, Agarwal *et al.* [2], Qiu and Wang [14], Şenel [16], Zhang [25], and Zhang and Gao [26] investigated oscillatory properties of (1.1), whereas Bohner and Li [8] and Erbe *et al.* [9] considered oscillation of solutions to a nonlinear second-order dynamic equation

$$[A(t)(x^\Delta(t))^\gamma]^\Delta + b(t)(x^{\Delta^\sigma}(t))^\gamma + P(t)F(x(\tau(t))) = 0$$

under the assumption that γ is a ratio of two odd positive integers. In a special case where $g(u) = u$, $b(t) = 0$, and λ is the quotient of two odd positive integers, oscillatory behavior of (1.1) was studied by Li and Saker [12] and Saker and O’Regan [15]. Agarwal *et al.* [5] and Zhang *et al.* [22] established several oscillation results for a second-order linear neutral dynamic equation

$$[r(t)(x(t) + p(t)x(\tau(t)))^\Delta]^\Delta + q(t)x(\delta(t)) = 0$$

assuming that

$$0 \leq p(t) < 1 \quad \text{or} \quad p(t) > 1$$

and

$$0 \leq p(t) \leq p_0 < +\infty, \tag{1.2}$$

respectively. Very recently, assuming that (1.2) is satisfied and using the double Riccati substitutions, Li and Rogovchenko [11] obtained two oscillation theorems for the second-order neutral delay differential equation

$$[r(t)|z'(t)|^{\lambda-1}z'(t)]' + q(t)f(x(\delta(t))) = 0, \quad z := x + p \cdot x \circ \tau.$$

Our principal goal is to analyze oscillation of (1.1) assuming that conditions (A_1) – (A_3) are satisfied. It should be noted that the topic of this paper is new for dynamic equations on time scales due to the fact that the results reported in [2, 5, 8, 9, 12, 14–16, 22, 25, 26] cannot be applied to a general equation (1.1) in the case where $B(t) \neq 0$ and $b(t) \neq 0$.

2 Philos-type oscillation result

In the section, we employ the integral averaging technique to establish a Philos-type (see [13]) oscillation theorem for (1.1). To this end, let

$$D := \{(t, s) : t \geq s \geq t_0, t, s \in [t_0, +\infty)_{\mathbf{T}}\} \quad \text{and} \quad D_0 := \{(t, s) : t > s \geq t_0, t, s \in [t_0, +\infty)_{\mathbf{T}}\}.$$

We say that a function $H \in C_{rd}(D, \mathbf{R})$ belongs to the class Ω (denoted by $H \in \Omega$) if

$$H(t, t) = 0 \quad \text{for } t \geq t_0, \quad H(t, s) > 0 \quad \text{for } (t, s) \in D_0,$$

and it has a nonpositive rd-continuous Δ -partial derivative $H^{\Delta_s}(t, s)$ with respect to the second variable satisfying the condition

$$\sqrt{H(t, s)}h(t, s) = H^{\Delta_s}(t, s) + H(t, s) \frac{\phi^{\Delta}(s)}{\phi(\sigma(s))}$$

for some function $h \in C_{rd}(D_0, \mathbf{R})$, where the meaning of ϕ will be explained later.

We need the following auxiliary results.

Lemma 2.1 (see [27], Theorem 1.93) *Assume that $v : \mathbf{T} \rightarrow \mathbf{R}$ is strictly increasing, $\tilde{\mathbf{T}} := v(\mathbf{T})$ is a time scale, and let $w : \tilde{\mathbf{T}} \rightarrow \mathbf{R}$. If $v^{\Delta}(t)$ and $w^{\tilde{\Delta}}(v(t))$ exist for $t \in \mathbf{T}^k$, then*

$$(w \circ v)^{\Delta} = (w^{\tilde{\Delta}} \circ v)v^{\Delta}.$$

Lemma 2.2 (see [18], Lemma 2.3) *If $0 < \lambda \leq 1$ and $x_1, x_2 \in [0, +\infty)$, then $x_1^{\lambda} + x_2^{\lambda} \geq (x_1 + x_2)^{\lambda}$.*

In what follows, all functional inequalities are tacitly supposed to hold for all t large enough.

Theorem 2.1 *Assume that conditions (A₁)-(A₃) are satisfied. If there exist two functions $\phi \in C^1_{rd}([t_0, +\infty)_{\mathbf{T}}, (0, +\infty))$ and $H \in \Omega$ such that*

$$\limsup_{t \rightarrow +\infty} \frac{1}{H(t, t_0)} \int_{t_1}^t \left\{ L\xi(s)\phi(s)H(t, s) - \psi(t, s) \left[(h_1(t, s))^{\lambda+1} + \frac{(b_0\eta)^{\lambda}}{\tau_0} (h_2(t, s))^{\lambda+1} \right] \right\} \Delta s = +\infty \tag{2.1}$$

for all sufficiently large $t_1 \in [t_0, +\infty)_{\mathbf{T}}$, where

$$\xi(s) := \min\{P(s), P(\tau(s))\}, \quad \psi(t, s) := \frac{A(\tau(s))(\phi(\sigma(s)))^{\lambda+1}}{(\lambda + 1)^{\lambda+1} \tau_0^{\lambda} \phi^{\lambda}(s)(H(t, s))^{\lambda}},$$

$$h_1(t, s) := \max \left\{ 0, \sqrt{H(t, s)}h(t, s) - \frac{H(t, s)b(s)\phi(s)}{\phi(\sigma(s))A(s)} \right\},$$

and

$$h_2(t, s) := \max \left\{ 0, \sqrt{H(t, s)}h(t, s) - \frac{\tau_0 H(t, s)b(\tau(s))\phi(s)}{\phi(\sigma(s))A(\tau(s))} \right\},$$

then (1.1) is oscillatory.

Proof Suppose to the contrary that x is a nonoscillatory solution of (1.1) on $[t_0, +\infty)_T$. Without loss of generality, we may assume that there is a $t_1 \in [t_0, +\infty)_T$ such that $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\delta(t)) > 0$ for $t \in [t_1, +\infty)_T$ (since the proof of the case where x is eventually negative is similar). Then $y(t) > 0$. An application of (1.1) implies that, for $t \in [t_1, +\infty)_T$,

$$[A(t)\varphi(y^\Delta(t))]^\Delta + b(t)\varphi(y^\Delta(t)) \leq -LP(t)\varphi(x(\delta(t))) = -LP(t)(x(\delta(t)))^\lambda < 0. \tag{2.2}$$

Thus, using ([27], Theorem 2.33), we conclude that

$$\begin{aligned} \left[\frac{A(t)\varphi(y^\Delta(t))}{e_{-b/A}(t, t_0)} \right]^\Delta &= \frac{[A(t)\varphi(y^\Delta(t))]^\Delta e_{-b/A}(t, t_0) - A(t)\varphi(y^\Delta(t))[e_{-b/A}(t, t_0)]^\Delta}{e_{-b/A}(t, t_0)e_{-b/A}(\sigma(t), t_0)} \\ &= \frac{[A(t)\varphi(y^\Delta(t))]^\Delta + b(t)\varphi(y^\Delta(t))}{e_{-b/A}(\sigma(t), t_0)} \\ &\leq -\frac{LP(t)(x(\delta(t)))^\lambda}{e_{-b/A}(\sigma(t), t_0)} < 0, \end{aligned} \tag{2.3}$$

and so $A\varphi(y^\Delta)/e_{-b/A}(\cdot, t_0)$ is decreasing and eventually of one sign. That is, y^Δ is either eventually positive or eventually negative. We assert that

$$y^\Delta > 0 \text{ eventually.} \tag{2.4}$$

If (2.4) does not hold, then there exists a $t_2 \in [t_1, +\infty)_T$ such that $y^\Delta(t) < 0$ for $t \in [t_2, +\infty)_T$. From (2.3), for $t \in [t_2, +\infty)_T$, we obtain

$$\frac{A(t)\varphi(y^\Delta(t))}{e_{-b/A}(t, t_0)} \leq \frac{A(t_2)\varphi(y^\Delta(t_2))}{e_{-b/A}(t_2, t_0)} = -M < 0, \tag{2.5}$$

where

$$M := -\frac{A(t_2)\varphi(y^\Delta(t_2))}{e_{-b/A}(t_2, t_0)} = \frac{A(t_2)|y^\Delta(t_2)|^{\lambda-1}(-y^\Delta(t_2))}{e_{-b/A}(t_2, t_0)} > 0.$$

By virtue of (2.5),

$$y^\Delta(t) \leq -M^{1/\lambda} \left[\frac{e_{-b/A}(t, t_0)}{A(t)} \right]^{1/\lambda}.$$

An integration of the latter inequality yields

$$y(t) \leq y(t_2) - M^{1/\lambda} \int_{t_2}^t \left[\frac{e_{-b/A}(s, t_0)}{A(s)} \right]^{1/\lambda} \Delta s \rightarrow -\infty \text{ as } t \rightarrow +\infty,$$

which contradicts the fact that $y(t) > 0$. Using Lemma 2.1 and the condition $\tau([t_0, +\infty)_T) = [\tau(t_0), +\infty)_T$, we get $[A(\tau(t))\varphi(y^\Delta(\tau(t)))]^\Delta = [A\varphi(y^\Delta)]^\Delta(\tau(t))\tau^\Delta(t)$. Hence, by (2.2), we have

$$\frac{[A(\tau(t))\varphi(y^\Delta(\tau(t)))]^\Delta}{\tau^\Delta(t)} + b(\tau(t))\varphi(y^\Delta(\tau(t))) + LP(\tau(t))(x(\delta(\tau(t))))^\lambda \leq 0. \tag{2.6}$$

Combining (2.2) and (2.6), we deduce that

$$\begin{aligned}
 & [A(t)\varphi(y^\Delta(t))]^\Delta + b(t)\varphi(y^\Delta(t)) + LP(t)(x(\delta(t)))^\lambda \\
 & + (b_0\eta)^\lambda \left\{ \frac{[A(\tau(t))\varphi(y^\Delta(\tau(t)))]^\Delta}{\tau^\Delta(t)} + b(\tau(t))\varphi(y^\Delta(\tau(t))) + LP(\tau(t))(x(\delta(\tau(t))))^\lambda \right\} \\
 & \leq 0.
 \end{aligned}$$

Note that $\tau^\Delta \geq \tau_0 > 0$, $\tau \circ \delta = \delta \circ \tau$, and $y(t) \leq x(t) + b_0\eta x(\tau(t))$. It follows from the definition of ξ , (2.4), $\delta(t) \geq \tau(t)$, and Lemma 2.2 that

$$\begin{aligned}
 & [A(t)\varphi(y^\Delta(t))]^\Delta + \frac{(b_0\eta)^\lambda}{\tau_0} [A(\tau(t))\varphi(y^\Delta(\tau(t)))]^\Delta \\
 & + b(t)\varphi(y^\Delta(t)) + (b_0\eta)^\lambda b(\tau(t))\varphi(y^\Delta(\tau(t))) \\
 & \leq -L\xi(t)[(x(\delta(t)))^\lambda + (b_0\eta x(\delta(\tau(t))))^\lambda] \leq -L\xi(t)[x(\delta(t)) + b_0\eta x(\delta(\tau(t)))]^\lambda \\
 & \leq -L\xi(t)(y(\delta(t)))^\lambda \leq -L\xi(t)(y(\tau(t)))^\lambda.
 \end{aligned} \tag{2.7}$$

An application of $[A\varphi(y^\Delta)]^\Delta < 0$ implies that

$$\begin{aligned}
 & A(t)\varphi(y^\Delta(t)) \geq A(\sigma(t))\varphi(y^\Delta(\sigma(t))) \quad \text{and} \\
 & A(\tau(t))\varphi(y^\Delta(\tau(t))) \geq A(\tau(\sigma(t)))\varphi(y^\Delta(\tau(\sigma(t)))).
 \end{aligned}$$

Using these inequalities in (2.7), we conclude that

$$\begin{aligned}
 & [A(t)\varphi(y^\Delta(t))]^\Delta + \frac{(b_0\eta)^\lambda}{\tau_0} [A(\tau(t))\varphi(y^\Delta(\tau(t)))]^\Delta \\
 & \leq -L\xi(t)(y(\tau(t)))^\lambda - b(t)\varphi(y^\Delta(t)) - (b_0\eta)^\lambda b(\tau(t))\varphi(y^\Delta(\tau(t))) \\
 & \leq -L\xi(t)(y(\tau(t)))^\lambda - \frac{b(t)}{A(t)} A(\sigma(t))\varphi(y^\Delta(\sigma(t))) \\
 & \quad - \frac{(b_0\eta)^\lambda b(\tau(t))}{A(\tau(t))} A(\tau(\sigma(t)))\varphi(y^\Delta(\tau(\sigma(t)))).
 \end{aligned} \tag{2.8}$$

Define the function W by

$$W(t) := \phi(t) \frac{A(t)\varphi(y^\Delta(t))}{\varphi(y(\tau(t)))} = \phi(t) \frac{A(t)(y^\Delta(t))^\lambda}{(y(\tau(t)))^\lambda}. \tag{2.9}$$

Then $W > 0$. From (2.4) and ([27], Theorems 1.90 and 1.93), we see that

$$[(y(\tau(t)))^\lambda]^\Delta \geq \lambda (y(\tau(\sigma(t))))^{\lambda-1} y^\Delta(\tau(t)) \tau^\Delta(t).$$

By virtue of (2.9),

$$\begin{aligned}
 W^\Delta(t) & = \phi(t) \frac{[A(t)\varphi(y^\Delta(t))]^\Delta}{\varphi(y(\tau(t)))} + A(\sigma(t))(y^\Delta(\sigma(t)))^\lambda \frac{\phi^\Delta(t)(y(\tau(t)))^\lambda - \phi(t)[(y(\tau(t)))^\lambda]^\Delta}{(y(\tau(t)))^\lambda (y(\tau(\sigma(t))))^\lambda} \\
 & \leq \phi(t) \frac{[A(t)\varphi(y^\Delta(t))]^\Delta}{\varphi(y(\tau(t)))} + \frac{\phi^\Delta(t)W(\sigma(t))}{\phi(\sigma(t))}
 \end{aligned}$$

$$\begin{aligned}
 & - \frac{\lambda \phi(t) A(\sigma(t)) (y^\Delta(\sigma(t)))^\lambda (y(\tau(\sigma(t))))^{\lambda-1} y^\Delta(\tau(t)) \tau^\Delta(t)}{(y(\tau(t)))^\lambda (y(\tau(\sigma(t))))^\lambda} \\
 \leq & \phi(t) \frac{[A(t)\varphi(y^\Delta(t))]^\Delta}{\varphi(y(\tau(t)))} + \frac{\phi^\Delta(t) W(\sigma(t))}{\phi(\sigma(t))} - \frac{\lambda \tau_0 \phi(t) A(\sigma(t)) (y^\Delta(\sigma(t)))^\lambda y^\Delta(\tau(t))}{(y(\tau(t)))^\lambda y(\tau(\sigma(t)))}.
 \end{aligned}$$

The inequality $[A\varphi(y^\Delta)]^\Delta < 0$ yields $A(\tau(t))(y^\Delta(\tau(t)))^\lambda \geq A(\sigma(t))(y^\Delta(\sigma(t)))^\lambda$, and hence

$$y^\Delta(\tau(t)) \geq \left(\frac{A(\sigma(t))}{A(\tau(t))} \right)^{1/\lambda} y^\Delta(\sigma(t)).$$

Then, using the condition $y(\tau(t)) \leq y(\tau(\sigma(t)))$, we have

$$\begin{aligned}
 W^\Delta(t) & \leq \phi(t) \frac{[A(t)\varphi(y^\Delta(t))]^\Delta}{\varphi(y(\tau(t)))} + \frac{\phi^\Delta(t) W(\sigma(t))}{\phi(\sigma(t))} - \frac{\lambda \tau_0 \phi(t) (A(\sigma(t)))^{(\lambda+1)/\lambda} (y^\Delta(\sigma(t)))^{\lambda+1}}{(A(\tau(t)))^{1/\lambda} (y(\tau(\sigma(t))))^{\lambda+1}} \\
 & = \frac{\phi(t)}{(y(\tau(t)))^\lambda} [A(t)\varphi(y^\Delta(t))]^\Delta + \frac{\phi^\Delta(t) W(\sigma(t))}{\phi(\sigma(t))} \\
 & \quad - \frac{\lambda \tau_0 \phi(t) (W(\sigma(t)))^{(\lambda+1)/\lambda}}{(A(\tau(t)))^{1/\lambda} (\phi(\sigma(t)))^{(\lambda+1)/\lambda}}. \tag{2.10}
 \end{aligned}$$

Define another function V by

$$V(t) := \phi(t) \frac{A(\tau(t))\varphi(y^\Delta(\tau(t)))}{\varphi(y(\tau(t)))} = \phi(t) \frac{A(\tau(t))(y^\Delta(\tau(t)))^\lambda}{(y(\tau(t)))^\lambda}. \tag{2.11}$$

Then $V > 0$. With a similar proof as before, we conclude that

$$\begin{aligned}
 V^\Delta(t) & = \phi(t) \frac{[A(\tau(t))\varphi(y^\Delta(\tau(t)))]^\Delta}{(y(\tau(t)))^\lambda} \\
 & \quad + A(\tau(\sigma(t)))\varphi(y^\Delta(\tau(\sigma(t)))) \frac{\phi^\Delta(t)(y(\tau(t)))^\lambda - \phi(t)[(y(\tau(t)))^\lambda]^\Delta}{(y(\tau(t)))^\lambda (y(\tau(\sigma(t))))^\lambda} \\
 \leq & \phi(t) \frac{[A(\tau(t))\varphi(y^\Delta(\tau(t)))]^\Delta}{(y(\tau(t)))^\lambda} + \frac{\phi^\Delta(t) V(\sigma(t))}{\phi(\sigma(t))} \\
 & \quad - A(\tau(\sigma(t)))\varphi(y^\Delta(\tau(\sigma(t)))) \frac{\lambda \phi(t) (y(\tau(\sigma(t))))^{\lambda-1} y^\Delta(\tau(t)) \tau^\Delta(t)}{(y(\tau(t)))^\lambda (y(\tau(\sigma(t))))^\lambda} \\
 \leq & \phi(t) \frac{[A(\tau(t))\varphi(y^\Delta(\tau(t)))]^\Delta}{(y(\tau(t)))^\lambda} + \frac{\phi^\Delta(t) V(\sigma(t))}{\phi(\sigma(t))} \\
 & \quad - \frac{\lambda \tau_0 \phi(t) A(\tau(\sigma(t)))\varphi(y^\Delta(\tau(\sigma(t))))}{(y(\tau(\sigma(t))))^{\lambda+1}} \left(\frac{A(\tau(\sigma(t)))}{A(\tau(t))} \right)^{1/\lambda} y^\Delta(\tau(\sigma(t))) \\
 = & \phi(t) \frac{[A(\tau(t))\varphi(y^\Delta(\tau(t)))]^\Delta}{(y(\tau(t)))^\lambda} + \frac{\phi^\Delta(t) V(\sigma(t))}{\phi(\sigma(t))} \\
 & \quad - \frac{\lambda \tau_0 \phi(t) (A(\tau(\sigma(t))))^{(\lambda+1)/\lambda} (y^\Delta(\tau(\sigma(t))))^{\lambda+1}}{(A(\tau(t)))^{1/\lambda} (y(\tau(\sigma(t))))^{\lambda+1}} \\
 = & \frac{\phi(t)}{(y(\tau(t)))^\lambda} [A(\tau(t))\varphi(y^\Delta(\tau(t)))]^\Delta + \frac{\phi^\Delta(t) V(\sigma(t))}{\phi(\sigma(t))} \\
 & \quad - \frac{\lambda \tau_0 \phi(t) (V(\sigma(t)))^{(\lambda+1)/\lambda}}{(A(\tau(t)))^{1/\lambda} (\phi(\sigma(t)))^{(\lambda+1)/\lambda}}. \tag{2.12}
 \end{aligned}$$

Applications of (2.8), (2.10), (2.12), and the condition $y(\tau(t)) \leq y(\tau(\sigma(t)))$ imply that

$$\begin{aligned}
 &W^\Delta(t) + \frac{(b_0\eta)^\lambda}{\tau_0} V^\Delta(t) \\
 &\leq \frac{\phi(t)}{(y(\tau(t)))^\lambda} \left\{ [A(t)\varphi(y^\Delta(t))]^\Delta + \frac{(b_0\eta)^\lambda}{\tau_0} [A(\tau(t))\varphi(y^\Delta(\tau(t)))]^\Delta \right\} \\
 &\quad + \frac{\phi^\Delta(t)W(\sigma(t))}{\phi(\sigma(t))} - \frac{\lambda\tau_0\phi(t)(W(\sigma(t)))^{(\lambda+1)/\lambda}}{(A(\tau(t)))^{1/\lambda}(\phi(\sigma(t)))^{(\lambda+1)/\lambda}} + \frac{(b_0\eta)^\lambda}{\tau_0} \frac{\phi^\Delta(t)V(\sigma(t))}{\phi(\sigma(t))} \\
 &\quad - \frac{(b_0\eta)^\lambda}{\tau_0} \frac{\lambda\tau_0\phi(t)(V(\sigma(t)))^{(\lambda+1)/\lambda}}{(A(\tau(t)))^{1/\lambda}(\phi(\sigma(t)))^{(\lambda+1)/\lambda}} \\
 &\leq -L\xi(t)\phi(t) - \frac{b(t)}{A(t)} \frac{\phi(t)A(\sigma(t))\varphi(y^\Delta(\sigma(t)))}{(y(\tau(t)))^\lambda} \\
 &\quad - \frac{(b_0\eta)^\lambda b(\tau(t))}{A(\tau(t))} \frac{\phi(t)A(\tau(\sigma(t)))\varphi(y^\Delta(\tau(\sigma(t))))}{(y(\tau(t)))^\lambda} \\
 &\quad + \frac{\phi^\Delta(t)W(\sigma(t))}{\phi(\sigma(t))} - \frac{\lambda\tau_0\phi(t)(W(\sigma(t)))^{(\lambda+1)/\lambda}}{(A(\tau(t)))^{1/\lambda}(\phi(\sigma(t)))^{(\lambda+1)/\lambda}} + \frac{(b_0\eta)^\lambda}{\tau_0} \frac{\phi^\Delta(t)V(\sigma(t))}{\phi(\sigma(t))} \\
 &\quad - \frac{(b_0\eta)^\lambda}{\tau_0} \frac{\lambda\tau_0\phi(t)(V(\sigma(t)))^{(\lambda+1)/\lambda}}{(A(\tau(t)))^{1/\lambda}(\phi(\sigma(t)))^{(\lambda+1)/\lambda}} \\
 &\leq -L\xi(t)\phi(t) - \frac{b(t)\phi(t)W(\sigma(t))}{A(t)\phi(\sigma(t))} - \frac{(b_0\eta)^\lambda b(\tau(t))\phi(t)V(\sigma(t))}{A(\tau(t))\phi(\sigma(t))} + \frac{\phi^\Delta(t)W(\sigma(t))}{\phi(\sigma(t))} \\
 &\quad - \frac{\lambda\tau_0\phi(t)(W(\sigma(t)))^{(\lambda+1)/\lambda}}{(A(\tau(t)))^{1/\lambda}(\phi(\sigma(t)))^{(\lambda+1)/\lambda}} + \frac{(b_0\eta)^\lambda}{\tau_0} \frac{\phi^\Delta(t)V(\sigma(t))}{\phi(\sigma(t))} \\
 &\quad - \frac{(b_0\eta)^\lambda}{\tau_0} \frac{\lambda\tau_0\phi(t)(V(\sigma(t)))^{(\lambda+1)/\lambda}}{(A(\tau(t)))^{1/\lambda}(\phi(\sigma(t)))^{(\lambda+1)/\lambda}} \\
 &\leq -L\xi(t)\phi(t) + \left(\phi^\Delta(t) - \frac{b(t)\phi(t)}{A(t)} \right) \frac{W(\sigma(t))}{\phi(\sigma(t))} - \frac{\lambda\tau_0\phi(t)(W(\sigma(t)))^{(\lambda+1)/\lambda}}{(A(\tau(t)))^{1/\lambda}(\phi(\sigma(t)))^{(\lambda+1)/\lambda}} \\
 &\quad + \frac{(b_0\eta)^\lambda}{\tau_0} \left(\phi^\Delta(t) - \frac{\tau_0 b(\tau(t))\phi(t)}{A(\tau(t))} \right) \frac{V(\sigma(t))}{\phi(\sigma(t))} - \frac{(b_0\eta)^\lambda}{\tau_0} \frac{\lambda\tau_0\phi(t)(V(\sigma(t)))^{(\lambda+1)/\lambda}}{(A(\tau(t)))^{1/\lambda}(\phi(\sigma(t)))^{(\lambda+1)/\lambda}},
 \end{aligned}$$

which yields, for some $t_3 \in [t_1, +\infty)_T$,

$$\begin{aligned}
 L\xi(s)\phi(s) &\leq -W^\Delta(s) - \frac{(b_0\eta)^\lambda}{\tau_0} V^\Delta(s) + \left(\phi^\Delta(s) - \frac{b(s)\phi(s)}{A(s)} \right) \frac{W(\sigma(s))}{\phi(\sigma(s))} \\
 &\quad - \frac{\lambda\tau_0\phi(s)(W(\sigma(s)))^{(\lambda+1)/\lambda}}{(A(\tau(s)))^{1/\lambda}(\phi(\sigma(s)))^{(\lambda+1)/\lambda}} + \frac{(b_0\eta)^\lambda}{\tau_0} \left(\phi^\Delta(s) - \frac{\tau_0 b(\tau(s))\phi(s)}{A(\tau(s))} \right) \frac{V(\sigma(s))}{\phi(\sigma(s))} \\
 &\quad - \frac{(b_0\eta)^\lambda}{\tau_0} \frac{\lambda\tau_0\phi(s)(V(\sigma(s)))^{(\lambda+1)/\lambda}}{(A(\tau(s)))^{1/\lambda}(\phi(\sigma(s)))^{(\lambda+1)/\lambda}}.
 \end{aligned}$$

Multiplying both sides of this inequality by $H(t, s)$ and integrating the resulting inequality from t_3 to t , we obtain

$$\begin{aligned}
 &\int_{t_3}^t LH(t, s)\xi(s)\phi(s)\Delta s \\
 &\leq - \int_{t_3}^t H(t, s)W^\Delta(s)\Delta s - \frac{(b_0\eta)^\lambda}{\tau_0} \int_{t_3}^t H(t, s)V^\Delta(s)\Delta s
 \end{aligned}$$

$$\begin{aligned}
 & + \int_{t_3}^t H(t,s) \left(\phi^\Delta(s) - \frac{b(s)\phi(s)}{A(s)} \right) \frac{W(\sigma(s))}{\phi(\sigma(s))} \Delta s \\
 & - \int_{t_3}^t H(t,s) \frac{\lambda \tau_0 \phi(s) (W(\sigma(s)))^{(\lambda+1)/\lambda}}{(A(\tau(s)))^{1/\lambda} (\phi(\sigma(s)))^{(\lambda+1)/\lambda}} \Delta s \\
 & + \frac{(b_0 \eta)^\lambda}{\tau_0} \int_{t_3}^t H(t,s) \left(\phi^\Delta(s) - \frac{\tau_0 b(\tau(s))\phi(s)}{A(\tau(s))} \right) \frac{V(\sigma(s))}{\phi(\sigma(s))} \Delta s \\
 & - \frac{(b_0 \eta)^\lambda}{\tau_0} \int_{t_3}^t H(t,s) \frac{\lambda \tau_0 \phi(s) (V(\sigma(s)))^{(\lambda+1)/\lambda}}{(A(\tau(s)))^{1/\lambda} (\phi(\sigma(s)))^{(\lambda+1)/\lambda}} \Delta s \\
 = & H(t, t_3) W(t_3) + \int_{t_3}^t H^{\Delta s}(t, s) W(\sigma(s)) \Delta s + \frac{(b_0 \eta)^\lambda}{\tau_0} H(t, t_3) V(t_3) \\
 & + \frac{(b_0 \eta)^\lambda}{\tau_0} \int_{t_3}^t H^{\Delta s}(t, s) V(\sigma(s)) \Delta s + \int_{t_3}^t H(t, s) \left(\phi^\Delta(s) - \frac{b(s)\phi(s)}{A(s)} \right) \frac{W(\sigma(s))}{\phi(\sigma(s))} \Delta s \\
 & - \int_{t_3}^t H(t, s) \frac{\lambda \tau_0 \phi(s) (W(\sigma(s)))^{(\lambda+1)/\lambda}}{(A(\tau(s)))^{1/\lambda} (\phi(\sigma(s)))^{(\lambda+1)/\lambda}} \Delta s \\
 & + \frac{(b_0 \eta)^\lambda}{\tau_0} \int_{t_3}^t H(t, s) \left(\phi^\Delta(s) - \frac{\tau_0 b(\tau(s))\phi(s)}{A(\tau(s))} \right) \frac{V(\sigma(s))}{\phi(\sigma(s))} \Delta s \\
 & - \frac{(b_0 \eta)^\lambda}{\tau_0} \int_{t_3}^t H(t, s) \frac{\lambda \tau_0 \phi(s) (V(\sigma(s)))^{(\lambda+1)/\lambda}}{(A(\tau(s)))^{1/\lambda} (\phi(\sigma(s)))^{(\lambda+1)/\lambda}} \Delta s \\
 = & H(t, t_3) W(t_3) + \frac{(b_0 \eta)^\lambda}{\tau_0} H(t, t_3) V(t_3) \\
 & + \int_{t_3}^t \left[\sqrt{H(t,s)h(t,s)} - \frac{H(t,s)b(s)\phi(s)}{\phi(\sigma(s))A(s)} \right] W(\sigma(s)) \Delta s \\
 & - \int_{t_3}^t \frac{\lambda \tau_0 H(t,s)\phi(s)(W(\sigma(s)))^{(\lambda+1)/\lambda}}{(A(\tau(s)))^{1/\lambda} (\phi(\sigma(s)))^{(\lambda+1)/\lambda}} \Delta s \\
 & + \frac{(b_0 \eta)^\lambda}{\tau_0} \int_{t_3}^t \left[\sqrt{H(t,s)h(t,s)} - \frac{\tau_0 H(t,s)b(\tau(s))\phi(s)}{\phi(\sigma(s))A(\tau(s))} \right] V(\sigma(s)) \Delta s \\
 & - \frac{(b_0 \eta)^\lambda}{\tau_0} \int_{t_3}^t \frac{\lambda \tau_0 H(t,s)\phi(s)(V(\sigma(s)))^{(\lambda+1)/\lambda}}{(A(\tau(s)))^{1/\lambda} (\phi(\sigma(s)))^{(\lambda+1)/\lambda}} \Delta s. \tag{2.13}
 \end{aligned}$$

Applying the inequality (see [11])

$$Bu - Au^{\frac{\lambda+1}{\lambda}} \leq \frac{\lambda^\lambda B^{\lambda+1}}{(\lambda+1)^{\lambda+1} A^\lambda}, \quad A > 0,$$

we have

$$\begin{aligned}
 & \left[\sqrt{H(t,s)h(t,s)} - \frac{H(t,s)b(s)\phi(s)}{\phi(\sigma(s))A(s)} \right] W(\sigma(s)) - \frac{\lambda \tau_0 H(t,s)\phi(s)(W(\sigma(s)))^{(\lambda+1)/\lambda}}{(A(\tau(s)))^{1/\lambda} (\phi(\sigma(s)))^{(\lambda+1)/\lambda}} \\
 & \leq h_1(t,s) W(\sigma(s)) - \frac{\lambda \tau_0 H(t,s)\phi(s)(W(\sigma(s)))^{(\lambda+1)/\lambda}}{(A(\tau(s)))^{1/\lambda} (\phi(\sigma(s)))^{(\lambda+1)/\lambda}} \\
 & \leq \frac{A(\tau(s))(\phi(\sigma(s)))^{\lambda+1}}{(\lambda+1)^{\lambda+1} \tau_0^\lambda \phi^\lambda(s)(H(t,s))^\lambda} (h_1(t,s))^{\lambda+1} \tag{2.14}
 \end{aligned}$$

and

$$\begin{aligned}
 & \left[\sqrt{H(t,s)}h(t,s) - \frac{\tau_0 H(t,s)b(\tau(s))\phi(s)}{\phi(\sigma(s))A(\tau(s))} \right] V(\sigma(s)) - \frac{\lambda \tau_0 H(t,s)\phi(s)(V(\sigma(s)))^{(\lambda+1)/\lambda}}{(A(\tau(s)))^{1/\lambda}(\phi(\sigma(s)))^{(\lambda+1)/\lambda}} \\
 & \leq h_2(t,s)V(\sigma(s)) - \frac{\lambda \tau_0 H(t,s)\phi(s)(V(\sigma(s)))^{(\lambda+1)/\lambda}}{(A(\tau(s)))^{1/\lambda}(\phi(\sigma(s)))^{(\lambda+1)/\lambda}} \\
 & \leq \frac{A(\tau(s))(\phi(\sigma(s)))^{\lambda+1}}{(\lambda+1)^{\lambda+1}\tau_0^\lambda \phi^\lambda(s)(H(t,s))^\lambda} (h_2(t,s))^{\lambda+1}. \tag{2.15}
 \end{aligned}$$

Using (2.14) and (2.15) in (2.13), we conclude that

$$\begin{aligned}
 & \int_{t_3}^t LH(t,s)\xi(s)\phi(s)\Delta s \\
 & \leq H(t,t_3)W(t_3) + \frac{(b_0\eta)^\lambda}{\tau_0}H(t,t_3)V(t_3) \\
 & \quad + \int_{t_3}^t \psi(t,s) \left[(h_1(t,s))^{\lambda+1} + \frac{(b_0\eta)^\lambda}{\tau_0}(h_2(t,s))^{\lambda+1} \right] \Delta s \\
 & \leq H(t,t_0) \left[W(t_3) + \frac{(b_0\eta)^\lambda}{\tau_0}V(t_3) \right] \\
 & \quad + \int_{t_3}^t \psi(t,s) \left[(h_1(t,s))^{\lambda+1} + \frac{(b_0\eta)^\lambda}{\tau_0}(h_2(t,s))^{\lambda+1} \right] \Delta s,
 \end{aligned}$$

and hence

$$\begin{aligned}
 & \frac{1}{H(t,t_0)} \int_{t_3}^t \left\{ L\xi(s)\phi(s)H(t,s) - \psi(t,s) \left[(h_1(t,s))^{\lambda+1} + \frac{(b_0\eta)^\lambda}{\tau_0}(h_2(t,s))^{\lambda+1} \right] \right\} \Delta s \\
 & \leq W(t_3) + \frac{(b_0\eta)^\lambda}{\tau_0}V(t_3),
 \end{aligned}$$

which contradicts (2.1). This completes the proof. □

Remark 2.1 One can deduce from Theorem 2.1 a great number of oscillation criteria for (1.1) with different choices of the functions ϕ and H .

Remark 2.2 Results reported in this paper can be easily extended to (1.1) for $\lambda > 1$ when using the inequality

$$a^\lambda + b^\lambda \geq 2^{1-\lambda}(a+b)^\lambda, \quad a, b \in [0, +\infty), \lambda \geq 1.$$

As a matter of fact, we have to replace $\xi(t) := \min\{P(t), P(\tau(t))\}$ with

$$\xi(t) := 2^{1-\lambda} \min\{P(t), P(\tau(t))\}.$$

3 Example

The following example is given to illustrate possible applications of theoretical results obtained in Section 2.

Example 3.1 For $T = \overline{2\mathbb{Z}}$ and $t \in [2, +\infty)_T$, consider the second-order 2-difference equation

$$\left[x(t) + 5x\left(\frac{t}{2}\right) \right]^{\Delta\Delta} + t^{-\frac{5}{2}} \left[x(t) + 5x\left(\frac{t}{2}\right) \right]^{\Delta} + p_0 x\left(\frac{t}{2}\right) = 0, \tag{3.1}$$

where $p_0 > 0$ is a constant. Let $A(t) = 1$, $B(t) = b_0 = 5$, $b(t) = t^{-5/2}$, $P(t) = p_0$, $\tau^\Delta = 1/2$, $\eta = L = 1$, $\phi(t) = 1$, and $H(t, s) = (t - s)^2$. By virtue of ([7], Lemma 2),

$$\begin{aligned} e_{-b/A}(t, t_0) &\geq 1 - \int_2^t \frac{b(s)}{A(s)} \Delta s = 1 - \int_2^t s^{-5/2} \Delta s = 1 - \frac{t^{-3/2} - 2^{-3/2}}{2^{-3/2} - 1} = \frac{t^{-3/2} - 2^{-1/2} + 1}{1 - 2^{-3/2}} \\ &\geq t^{-3/2} - 2^{-1/2} + 1 > 1 - 2^{-1/2} > \frac{1}{4}. \end{aligned}$$

It is not difficult to verify that all assumptions of Theorem 2.1 are satisfied. Therefore, equation (3.1) is oscillatory.

Remark 3.1 Using the double Riccati substitutions (2.9) and (2.11), Theorem 2.1 complements and improves the results reported in [2, 5, 8, 9, 12, 14–16, 22, 25, 26] because our criteria can be applied to the case where $B(t) \neq 0$ and $b(t) \neq 0$. However, as in the paper by Zhang *et al.* [22], such flexibility is achieved at the cost of imposing conditions $\tau([t_0, +\infty)_T) = [\tau(t_0), +\infty)_T$, $\tau^\Delta(t) \geq \tau_0 > 0$, and $\tau \circ \delta = \delta \circ \tau$ on the function τ . It would be of interest to analyze the oscillatory behavior of (1.1) with other methods that do not require these restrictive assumptions.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

Both authors contributed equally to this work. They both read and approved the final version of the manuscript.

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