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Oscillation of solutions for certain fractional partial differential equations

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Abstract

In this paper, sufficient conditions are established for the forced oscillation of fractional partial differential equations of the form

$$\frac{\partial}{\partial t}(D^{\alpha}_{+,t}u(x,t)) = a(t)\Delta u(x,t) - m(x,t,u(x,t)) + f(x,t), \quad (x,t) \in \Omega \times \mathbb{R}_{+} \equiv G,$$

with one of the two following boundary conditions:

$$\frac{\partial u(x,t)}{\partial N} = \psi(x,t), \quad (x,t) \in \partial\Omega \times \mathbb{R}_{+}, \quad \text{or} \quad u(x,t) = 0, \quad (x,t) \in \partial\Omega \times \mathbb{R}_{+},$$

where Ω is a bounded domain in $\mathbb{R}^n$ with a piecewise smooth boundary $\partial\Omega$, $\mathbb{R}_{+} = [0,\infty)$, $\alpha \in (0,1)$ is a constant, $D^{\alpha}_{+,t}u(x,t)$ is the Riemann-Liouville fractional derivative of order α of u with respect to t, Δ is the Laplacian in $\mathbb{R}^n$, N is the unit exterior normal vector to $\partial\Omega$ and $\psi(x,t)$ is a continuous function on $\partial\Omega \times \mathbb{R}_{+}$. The main results are illustrated by two examples.

MSC: 35B05; 35R11; 26A33

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1 Introduction

The fractional calculus may be considered an old and yet novel topic. On 30 September 1695, Leibniz wrote a letter to L'Hôspital to discuss the meaning of the derivative of order $\frac{1}{2}$ [1]. After that in pure mathematics field the foundation of the fractional differential equations had been established. At the same time, many researchers found that the fractional differential equations play increasingly important roles in the modeling of engineering and science problems. It has been established that, in many situations, these models provide more suitable results than analogous models with integer derivatives. In the past few years, the fractional calculus and the theory of fractional differential equations have been investigated extensively. For example, see [2–22] and the references cited therein.

In recent years, the research on the oscillatory behavior of solutions of fractional differential equation has been a hot topic and some results have been established. For example, see [11–19]. However, to the best of the author's knowledge, very little is known regarding the oscillatory behavior of fractional partial differential equations which involve the Riemann-Liouville fractional partial derivative up to now [20–22].

In [20], Prakash *et al.* investigated the oscillation of the fractional partial differential equation

$$\frac{\partial}{\partial t}\left(r(t)D^{\alpha}_{+,t}u(x,t)\right) + q(x,t)f\left(\int_0^t (t-\nu)^{-\alpha}u(x,\nu)\,d\nu\right)$$
$$= a(t)\Delta u(x,t), \quad (x,t) \in \Omega \times \mathbb{R}_+ \equiv G. \tag{E1}$$

In [21], Harikrishnan *et al.* studied the oscillatory behavior of the fractional partial differential equation of the form

$$D^{\alpha}_{+,t}\left(r(t)D^{\alpha}_{+,t}u(x,t)\right) + q(x,t)f\left(u(x,t)\right) = a(t)\Delta u(x,t) + g(x,t), \quad (x,t) \in G. \tag{E2}$$

In [22], Prakash *et al.* established the oscillation of a nonlinear fractional partial differential equation with damping and forced term of the form

$$D^{\alpha}_{+,t}\left(r(t)D^{\alpha}_{+,t}u(x,t)\right) + p(t)D^{\alpha}_{+,t}u(x,t) + q(x,t)f\left(u(x,t)\right)$$
$$= a(t)\Delta u(x,t) + g(x,t), \quad (x,t) \in G. \tag{E3}$$

2 Formulation of the problems

Our aim in this paper is to study the forced oscillation of fractional partial differential equations of the form

$$\frac{\partial}{\partial t}\left(D^{\alpha}_{+,t}u(x,t)\right) = a(t)\Delta u(x,t) - m\left(x,t,u(x,t)\right) + f(x,t), \quad (x,t) \in \Omega \times \mathbb{R}_+ \equiv G, \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with a piecewise smooth boundary $\partial\Omega$, $\mathbb{R}_+ = [0,\infty)$, $\alpha \in (0,1)$ is a constant, $D^{\alpha}_{+,t}u(x,t)$ is the Riemann-Liouville fractional derivative of order α of u with respect to t, and Δ is the Laplacian in $\mathbb{R}^n$.

Throughout this paper, we assume that the following conditions hold:

(A1) $a \in C(\mathbb{R}_+;(0,\infty))$;

(A2) $m \in C(\overline{G} \times \mathbb{R};\mathbb{R})$, and

$$m(x,t,\xi)\begin{cases} \geq 0, & \text{if } \xi \in (0,\infty), \\ \leq 0, & \text{if } \xi \in (-\infty,0); \end{cases}$$

(A3) $f \in C(\overline{G};\mathbb{R})$.

Consider one of the two following boundary conditions:

$$\frac{\partial u(x,t)}{\partial N} = \psi(x,t), \quad (x,t) \in \partial\Omega \times \mathbb{R}_+, \tag{2}$$

or

$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times \mathbb{R}_+, \tag{3}$$

where N is the unit exterior normal vector to $\partial\Omega$ and $\psi(x,t)$ is a continuous function on $\partial\Omega \times \mathbb{R}_+$.

By a solution of the problem (1)-(2) (or (1)-(3)) we mean a function $u(x,t)$ which satisfies (1) on $\overline{G}$ and the boundary condition (2) (or (3)).

A solution $u(x,t)$ of the problem (1)-(2) (or (1)-(3)) is said to be oscillatory in G if it is neither eventually positive nor eventually negative, otherwise it is nonoscillatory.

Definition 2.1 The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $y : \mathbb{R}_+ \to R$ on the half-axis $\mathbb{R}_+$ is given by

$$I_+^\alpha y(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-\xi)^{\alpha-1} y(\xi)\,d\xi \quad \text{for } t > 0 \tag{4}$$

provided the right hand side is pointwise defined on $\mathbb{R}_+$, where Γ is the Gamma function.

Definition 2.2 The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function $y : \mathbb{R}_+ \to R$ on the half-axis $\mathbb{R}_+$ is given by

$$\begin{aligned} D_+^\alpha y(t) &:= \frac{d^{\lceil\alpha\rceil}}{dt^{\lceil\alpha\rceil}} \big(I_+^{\lceil\alpha\rceil-\alpha} y\big)(t) \\ &= \frac{1}{\Gamma(\lceil\alpha\rceil-\alpha)} \frac{d^{\lceil\alpha\rceil}}{dt^{\lceil\alpha\rceil}} \int_0^t (t-\xi)^{\lceil\alpha\rceil-\alpha-1} y(\xi)\,d\xi \quad \text{for } t > 0 \end{aligned} \tag{5}$$

provided the right hand side is pointwise defined on $\mathbb{R}_+$, where $\lceil\alpha\rceil$ is the ceiling function of α.

Definition 2.3 The Riemann-Liouville fractional partial derivative of order $0 < \alpha < 1$ with respect to t of a function $u(x,t)$ is given by

$$D_{+,t}^\alpha u(x,t) := \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial t} \int_0^t (t-\xi)^{-\alpha} u(x,\xi)\,d\xi \tag{6}$$

provided the right hand side is pointwise defined on $\mathbb{R}_+$.

Lemma 2.1 [20] *Let*

$$E(t) =: \int_0^t (t-v)^{-\alpha} y(v)\,dv \quad \textit{for } \alpha \in (0,1) \textit{ and } t > 0. \tag{7}$$

Then $E'(t) = \Gamma(1-\alpha) D_+^\alpha y(t)$.

3 Main results

First, we introduce the following fact [23]:

The smallest eigenvalue β_0 of the Dirichlet problem

$$\begin{cases} \Delta\omega(x) + \beta\omega(x) = 0, & \text{in } \Omega, \\ \omega(x) = 0, & \text{on } \partial\Omega, \end{cases}$$

is positive and the corresponding eigenfunction $\varphi(x)$ is positive in Ω.

The following notations will be used for our convenience:

$$U(t) = \int_\Omega u(x,t)\,dx, \qquad \Psi(t) = \int_{\partial\Omega} \psi(x,t)\,dS, \qquad F(t) = \int_\Omega f(x,t)\,dx,$$

$$H(t) = F(t) + a(t)\Psi(t), \qquad W(t) = \int_\Omega u(x,t)\varphi(x)\,dx,$$
$$Y(t) = \int_\Omega f(x,t)\varphi(x)\,dx, \quad t \geq 0,$$

where dS is the surface element on $\partial\Omega$.

Next, we establish some useful lemmas.

Lemma 3.1 *If $u(x,t) > 0$ is a solution of the problem* (1)-(2) *in G, then $U(t)$ satisfies the differential inequality*

$$\bigl(D_+^\alpha U(t)\bigr)' \leq H(t), \quad t \geq 0. \tag{8}$$

Proof Integrating (1) with respect to x over the domain Ω, we have

$$\frac{d}{dt}\left(\int_\Omega D_{+,t}^\alpha u(x,t)\,dx\right) = a(t)\int_\Omega \Delta u(x,t)\,dx - \int_\Omega m\bigl(x,t,u(x,t)\bigr)\,dx + \int_\Omega f(x,t)\,dx, \quad t \geq 0. \tag{9}$$

Green's formula and (2) yield

$$\int_\Omega \Delta u(x,t)\,dx = \int_{\partial\Omega} \frac{\partial u(x,t)}{\partial N}\,dS = \int_{\partial\Omega} \psi(x,t)\,dS = \Psi(t), \quad t \geq 0. \tag{10}$$

Noting that $u(x,t) > 0$, from (A2), it is easy to see that $m(x,t,u(x,t)) > 0$. Combining (9) and (10), we have

$$\bigl(D_+^\alpha U(t)\bigr)' \leq F(t) + a(t)\Psi(t), \quad t \geq 0,$$

which shows that $U(t) > 0$ is a positive solution of the inequality (8). The proof of Lemma 3.1 is complete. □

The proof of the following lemma is similar to that of Lemma 3.1 and we omit it.

Lemma 3.2 *If $u(x,t) < 0$ is a solution of the problem* (1)-(2) *in G, then $U(t)$ satisfies the differential inequality*

$$\bigl(D_+^\alpha U(t)\bigr)' \geq H(t), \quad t \geq 0. \tag{11}$$

Lemma 3.3 *If $u(x,t) > 0$ is a solution of the problem* (1)-(3) *in G, then $W(t)$ satisfies the differential inequality*

$$\bigl(D_+^\alpha W(t)\bigr)' \leq Y(t), \quad t \geq 0. \tag{12}$$

Proof Multiplying both sides of (1) by $\varphi(x)$ and integrating with respect to x over the domain Ω, we have

$$\begin{aligned}\frac{d}{dt}\left(\int_\Omega D^\alpha_{+,t}u(x,t)\varphi(x)\,dx\right) &= a(t)\int_\Omega \Delta u(x,t)\varphi(x)\,dx \\ &\quad - \int_\Omega m\big(x,t,u(x,t)\big)\varphi(x)\,dx \\ &\quad + \int_\Omega f(x,t)\varphi(x)\,dx, \quad t \geq 0. \end{aligned} \tag{13}$$

Green's formula and (3) yield

$$\int_\Omega \Delta u(x,t)\varphi(x)\,dx = \int_\Omega u(x,t)\Delta\varphi(x)\,dx = -\beta_0\int_\Omega u(x,t)\varphi(x)\,dx \leq 0, \quad t \geq 0. \tag{14}$$

From (A2), it is easy to see that $m(x,t,u(x,t))\varphi(x) > 0$. Combining (13) and (14), we obtain

$$\big(D^\alpha_+ W(t)\big)' \leq Y(t), \quad t \geq 0,$$

which shows that $W(t) > 0$ is a positive solution of the inequality (12). The proof is complete. □

Similarly we can obtain the following lemma.

Lemma 3.4 *If* $u(x,t) < 0$ *is a solution of the problem* (1)-(3) *in* G, *then* $W(t)$ *satisfies the differential inequality*

$$\big(D^\alpha_+ W(t)\big)' \geq Y(t), \quad t \geq 0. \tag{15}$$

Finally, we give our main results.

Theorem 3.1 *If the inequality* (8) *has no eventually positive solutions and the inequality* (11) *has no eventually negative solutions, then every solution of the problem* (1)-(2) *is oscillatory in* G.

Proof Suppose to the contrary that there is a nonoscillatory solution $u(x,t)$ of the problem (1)-(2). It is obvious that there exist $t_0 \geq 0$ such that $|\ (x,t)\ | > 0$ for $t \geq t_0$. Therefore $u(x,t) > 0$ or $u(x,t) < 0$, $t \geq t_0$.

If $u(x,t) > 0$, $t \geq t_0$, using Lemma 3.1 we see that $U(t) > 0$ is a solution of the inequality (8), which is a contradiction.

If $u(x,t) < 0$, $t \geq t_0$, using Lemma 3.2, it is easy to see that $U(t) < 0$ is a solution of the inequality (11), which is a contradiction. This completes the proof. □

Theorem 3.2 *Assume that*

$$\liminf_{t\to\infty}\int_{t_1}^t \left(1-\frac{s}{t}\right)H(s)\,ds = -\infty, \quad t_1 \geq t_0, \tag{16}$$

and

$$\limsup_{t\to\infty}\int_{t_1}^t \left(1-\frac{s}{t}\right)H(s)\,ds = \infty, \quad t_1 \geq t_0, \tag{17}$$

hold. Then every solution of the problem (1)-(2) *is oscillatory in* G.

Proof We prove that the (8) has no eventually positive solutions and the inequality (11) has no eventually negative solutions.

Assume to a contrary that (8) has a positive solution $V(t)$, then there exists $t_1 \geq t_0$ such that $V(t) > 0$, $t \geq t_1$. Therefore, from (8) we have

$$\left(D_+^\alpha V(t)\right)' \leq H(t), \quad t \geq t_1. \tag{18}$$

Integrating (18) from t_1 to t, we obtain

$$D_+^\alpha V(t) \leq D_+^\alpha V(t_1) + \int_{t_1}^t H(s)\,ds. \tag{19}$$

Using Lemma 2.1, it follows from (19) that

$$\frac{E'(t)}{\Gamma(1-\alpha)} = D_+^\alpha V(t) \leq D_+^\alpha V(t_1) + \int_{t_1}^t H(s)\,ds, \quad t \geq t_1. \tag{20}$$

Integrating (20) from t_1 to t, we have

$$\begin{aligned} E(t) &\leq E(t_1) + \Gamma(1-\alpha)\int_{t_1}^t \left(D_+^\alpha V(t_1) + \int_{t_1}^\xi H(s)\,ds\right) d\xi \\ &= E(t_1) + \Gamma(1-\alpha) D_+^\alpha V(t_1)(t-t_1) + \Gamma(1-\alpha)\int_{t_1}^t (t-s)H(s)\,ds. \end{aligned} \tag{21}$$

Therefore,

$$\frac{E(t)}{t} \leq \frac{E(t_1)}{t} + \Gamma(1-\alpha) D_+^\alpha V(t_1)\left(1 - \frac{t_1}{t}\right) + \Gamma(1-\alpha)\int_{t_1}^t \left(1 - \frac{s}{t}\right) H(s)\,ds. \tag{22}$$

Taking $t \to \infty$ in (22) and noting the assumption (16), we have

$$\liminf_{t\to\infty} \frac{E(t)}{t} = -\infty,$$

which contradicts the assumption that $V(t) > 0$.

Assume that (11) has a negative solution $\widetilde{V}(t)$. Noting that condition (17) holds and using the above mentioned method, we easily obtain a contradiction. This completes the proof of Theorem 3.2. □

Using Lemma 3.3 and Lemma 3.4, we easily establish the following theorems.

Theorem 3.3 *If the inequality* (12) *has no eventually positive solutions and the inequality* (15) *has no eventually negative solutions, then every solution of the problem* (1)-(3) *is oscillatory in G.*

Theorem 3.4 *Assume that*

$$\liminf_{t\to\infty} \int_{t_1}^t \left(1 - \frac{s}{t}\right) Y(s)\,ds = -\infty, \quad t_1 \geq t_0, \tag{23}$$

and

$$\limsup_{t\to\infty}\int_{t_1}^{t}\left(1-\frac{s}{t}\right)Y(s)\,ds=\infty,\quad t_1\geq t_0, \tag{24}$$

hold. Then every solution of the problem (1)-(3) *is oscillatory in G.*

4 Examples

In this section, we give two illustrative examples.

Example 4.1 Consider the fractional partial differential equation

$$\frac{\partial}{\partial t}\left(D_{+,t}^{\alpha}u(x,t)\right)=\Delta u(x,t)-\frac{e^t u(x,t)}{2+u^6(x,t)}+\pi e^t\sin t\sin x,\quad (x,t)\in(0,\pi)\times\mathbb{R}_+, \tag{25}$$

with the boundary condition

$$-\frac{\partial u(0,t)}{\partial x}=\frac{\partial u(\pi,t)}{\partial x}=-e^t\sin t,\quad t\geq 0. \tag{26}$$

Here $\Omega=(0,\pi)$, $n=1$, $a(t)=1$, $m(x,t,u(x,t))=\frac{e^t u(x,t)}{2+u^6(x,t)}$, $f(x,t)=\pi e^t\sin t\sin x$. It is obvious that $\Psi(t)=-\pi e^t\sin t$, and

$$F(t)=\int_{\Omega}f(x,t)\,dx=\int_0^{\pi}\pi e^t\sin t\sin x\,dx=2\pi e^t\sin t,$$
$$H(t)=F(t)+a(t)\Psi(t)=\pi e^t\sin t.$$

Hence

$$\int_{t_1}^{t}\left(1-\frac{s}{t}\right)H(s)\,ds=-\frac{\pi e^t\cos t}{2t}+\frac{C_1}{t}+C_2,$$

where C_1 and C_2 are constants.

We easily see that

$$\liminf_{t\to\infty}\int_{t_1}^{t}\left(1-\frac{s}{t}\right)H(s)\,ds=-\infty\quad\text{and}\quad\limsup_{t\to\infty}\int_{t_1}^{t}\left(1-\frac{s}{t}\right)H(s)\,ds=\infty,$$

which shows that all the conditions of Theorem 3.2 are fulfilled. Then every solution of the problem (25)-(26) is oscillatory in $(0,\pi)\times\mathbb{R}_+$.

Example 4.2 Consider the fractional partial differential equation

$$\begin{aligned}&\frac{\partial}{\partial t}\left(D_{+,t}^{\alpha}u(x,t)\right)\\&\quad=e^{-t}\Delta u(x,t)-3te^x u^3(x,t)+e^{t-\pi}\sin t\sin x,\quad (x,t)\in(0,\pi)\times\mathbb{R}_+,\end{aligned} \tag{27}$$

with the boundary condition

$$u(0,t)=u(\pi,t)=0,\quad t\geq 0. \tag{28}$$

Here $\Omega=(0,\pi)$, $n=1$, $a(t)=e^{-t}$, $m(x,t,u(x,t))=3te^x u^3(x,t)$, $f(x,t)=e^{t-\pi}\sin t\sin x$.

It is obvious that $\beta_0 = 1$, $\varphi(x) = \sin x$. Therefore,

$$Y(t) = \int_\Omega f(x,t)\varphi(x)\,dx = \int_0^\pi e^{t-\pi} \sin t \sin^2 x\,dx = \frac{\pi e^{-\pi}}{2} e^t \sin t.$$

Hence

$$\liminf_{t\to\infty} \int_{t_1}^t \left(1 - \frac{s}{t}\right) Y(s)\,ds = -\infty \quad \text{and} \quad \limsup_{t\to\infty} \int_{t_1}^t \left(1 - \frac{s}{t}\right) Y(s)\,ds = \infty.$$

Using Theorem 3.4, it is easy to see that every solution of the problem (27)-(28) oscillates in $(0,\pi) \times \mathbb{R}_+$.

Competing interests
The author declares that there are no competing interests.

Author's contributions
The author contributed to the manuscript. The author read and approved the final manuscript.

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