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# The existence and nonexistence of positive solutions for fractional differential equations with nonhomogeneous boundary conditions

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## Abstract

In this paper, we study the existence and nonexistence of the positive solutions for a class of fractional differential equations with nonhomogeneous boundary conditions and the impact of the disturbance parameters  $a, b$  on the existence of positive solutions. By using the upper and lower solutions method and the Schauder fixed point theorem, we obtain the sufficient conditions for the boundary value problem to have at least one positive solution, two positive solutions, and no positive solution, respectively. Moreover, under certain conditions, we prove that there exists a bounded and continuous curve  $L$  dividing  $[0, +\infty) \times [0, +\infty)$  into two separate subsets  $\Lambda^E$  and  $\Lambda^N$  with  $L \subseteq \Lambda^E$  such that the boundary value problem has at least two positive solutions for each  $(a, b) \in \Lambda^E \setminus L$ , one positive solution for each  $(a, b) \in L$ , and no positive solution for any  $(a, b) \in \Lambda^N$ . Finally, we give some examples to illustrate our main results.

**MSC:** 34B15; 26A33

**Keywords:** fractional differential equations; existence and nonexistence; positive solutions; upper and lower solution method; fixed point theorem

## 1 Introduction

In recent years, since the fractional differential equation has been widely applied in various areas such as mathematics, physics, chemistry, biology, and so forth, lots of books on fractional calculus have appeared, see [1–4] and the references therein. The boundary value problems are a very important part of fractional differential equations theory, and more and more researchers' attention has been attracted and plenty of meaningful results have been obtained; see [5–23]. For example, in [21], the authors studied the existence of multiple positive solutions of systems of the boundary value problems of the Caputo fractional differential equations

$$\begin{cases} -{}^C D^\alpha z_i(t) = f_i(t, z(t)), & \text{for a.e. } t \in [0, 1] \text{ and } i \in I_n, \\ \alpha z_i(0) - \beta z_i'(0) = 0, & \gamma z_i(1) + \delta z_i'(1) = 0, \end{cases}$$

where  $\mathbf{z}(t) = (z_1(t), \dots, z_n(t))$ ,  $q \in (1, 2)$ , the parameters  $\alpha, \beta, \gamma, \delta$  are positive real numbers, and obtained some interesting conclusions. Reference [22] studied the existence of at least one positive solution for the following problem:

$$\begin{cases} D_{0^+}^\alpha u(t) = f(t, u(t)), & t \in (0, \infty), \\ \lim_{t \rightarrow 0^+} t^{2-\alpha} u(t) = a, & \lim_{t \rightarrow 0^+} D_{0^+}^{\alpha-1} u(t) = b, \end{cases}$$

where  $1 < \alpha \leq 2$ ,  $a, b \in \mathbb{R}$ ,  $f : (0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}$  is a continuous function, and  $f$  may be singular at  $t = 0$ , i.e.,  $\lim_{t \rightarrow 0^+} f(t, \cdot) = +\infty$ . And the paper also considered an infinite fractional boundary value problem for singular integro-differential equation of mixed type on the half line

$$\begin{cases} D_{0^+}^\alpha u(t) = f(t, u(t), (Su)(t), (Hu)(t)), & t \in (0, \infty), \\ \lim_{t \rightarrow 0^+} t^{2-\alpha} u(t) = a, & \lim_{t \rightarrow 0^+} D_{0^+}^{\alpha-1} u(t) = b, \end{cases}$$

where  $S, H : (0, \infty) \times (0, \infty) \rightarrow [0, \infty)$ ,  $(Su)(t) = \int_0^\infty K(t, s)u(s) ds$ ,  $(Hu)(t) = \int_0^t H(t, s)u(s) ds$ . The authors proved the existence of at least one positive solution for two problems of the boundary value problem for the fractional differential equation by using a fixed point theorem in partially ordered sets and the contraction mapping principle.

The disturbance parameter in boundary conditions is a very important factor while solving the actual problems with the boundary value problems of differential equations and also it is inevitable that the impact of such a disturbance parameter to the existence of solutions always exists. However, as far as we know, there are only several papers studying this kind of impact, see [23–28]. In [24], the authors studied the second-order differential equation boundary value problem with one-parameter nonhomogeneous boundary conditions

$$\begin{cases} u'' + p(t)u' + g(t) = 0, & t \in (0, 1), \\ u(0) = 0, & u(1) = \lambda. \end{cases}$$

And the multi-point boundary value problem

$$\begin{cases} u''(t) + a(t)f(u) = 0, & t \in (0, 1), \\ u'(0) = 0, & u(1) - \sum_{i=1}^{n-2} k_i u(\xi_i) = b \end{cases}$$

was considered in [25]. References [26, 27] studied the two-parameter nonhomogeneous boundary value problem of the second-order differential equation

$$\begin{cases} u''(t) = f(t, u, u'), & t \in (0, 1), \\ u(0) - \sum_{i=1}^m a_i u(t_i) = \lambda_1, & u(1) - \sum_{i=1}^m b_i u(t_i) = \lambda_2. \end{cases}$$

It was shown that under some assumptions there exists a continuous curve  $\Gamma$  separating the plane  $\mathbb{R}^2$  into disjoint regions  $\Lambda^E$  and  $\Lambda^N$  with  $\Gamma \subseteq \Lambda^E$  such that this boundary value

problem has at least two solutions on  $\Lambda^E \setminus \Gamma$ , one solution on  $\Gamma$ , and no solution on  $\Lambda^N$ . The authors also studied the higher order boundary value problems with nonhomogeneous boundary conditions in [28] and get the some conclusion. Reference [23] studied the existence and nonexistence of positive solutions for the boundary value problems of the fractional differential equations

$$\begin{cases} -{}^C D^\delta u(t) = f(t, u(t)), & t \in (0, 1), \\ m_1 u(0) - n_1 u'(0) = 0, \\ m_2 u(1) + n_2 u'(1) = \int_0^1 g(s)u(s) ds + a, \end{cases}$$

where  ${}^C D^\delta$  is the Caputo fractional derivative of order  $\delta$ . And one studied the impact of the disturbance parameter  $a$  on the existence of positive solutions. Under certain conditions, the authors also obtained the demarcation point which divides the disturbance parameters into two subintervals such that the boundary value problem has positive solutions for the disturbance parameter in one subinterval while there are no positive solutions in the other.

In this paper, we study the existence and nonexistence of the positive solutions of the fractional order the boundary value problem which is composed of the equation

$$D_{0+}^\delta u(t) + f(t, u(t)) = 0, \quad \text{for a.e. } t \in (0, 1), \tag{1.1}$$

and nonhomogeneous boundary conditions with the disturbance parameters  $a, b$

$$\lim_{t \rightarrow 0^+} t^{2-\delta} u(t) = a, \quad u(1) = b, \tag{1.2}$$

where  $D_{0+}^\delta$  is the standard Riemann-Liouville fractional derivative of order  $\delta, 1 < \delta < 2, J = (0, 1]$ , disturbance parameters  $a \geq 0, b \geq 0, f : [0, 1] \times [0, +\infty) \rightarrow [0, +\infty)$ , and  $f$  satisfies the  $L_1$ -Caratheódory conditions.

We say that  $f$  satisfies the  $L_1$ -Caratheódory conditions on  $[0, 1] \times [0, +\infty)$ , if

- (i)  $f(\cdot, u)$  is measurable for all  $u \in [0, +\infty)$ ;
- (ii)  $f(t, \cdot)$  is continuous for a.e.  $t \in [0, 1]$ ;
- (iii) for every  $\gamma > 0$ , there exists  $\varphi_\gamma \in L_1[0, 1]$  such that  $|f(t, t^{\delta-2}u)| \leq \varphi_\gamma(t)$  for all  $u \in [0, \gamma]$  and a.e.  $t \in [0, 1]$ .

In the paper, we have the following.

- (H) For any  $t \in J, f(t, u)$  is monotone increasing with respect to  $u$ .

We focus on studying the impact of the disturbance parameters  $a, b$  on the existence of positive solutions. We use the upper and lower solutions method and the Schauder fixed point theorem, and we obtain sufficient conditions so that the problem has at least one positive solution, two positive solutions, and no positive solution, respectively. In addition, we also prove that there exists a bounded and continuous curve  $L$  dividing  $[0, +\infty) \times [0, +\infty)$  into two separate subsets  $\Lambda^E$  and  $\Lambda^N$  with  $L \subseteq \Lambda^E$  such that the boundary value problem has at least two positive solutions for each  $(a, b) \in \Lambda^E \setminus L$ , one positive solution for each  $(a, b) \in L$ , and no positive solution for any  $(a, b) \in \Lambda^N$  under certain conditions.

## 2 Preliminaries

In this section, we present some useful definitions and the related theorems.

**Definition 2.1** (See [4]) Let  $u \in L_1[0, 1]$ . The integrals

$$I_{0^+}^p u(t) = \frac{1}{\Gamma(p)} \int_0^t \frac{u(s)}{(t-s)^{1-p}} ds, \quad t > 0,$$

where  $p > 0$ , is called the Riemann-Liouville fractional integral of order  $p$ .

**Definition 2.2** (See [4]) The Riemann-Liouville derivative of order  $p$  for a continuous function  $u : (0, +\infty) \rightarrow \mathbb{R}$  is given by

$$D_{0^+}^p u(t) = \frac{1}{\Gamma(n-p)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{u(s)}{(t-s)^{p+1-n}} ds = \left(\frac{d}{dt}\right)^n I_{0^+}^{n-p} u(t),$$

provided the right side is pointwise defined on  $(0, +\infty)$ , where  $n = [p] + 1$  and  $n - 1 < p < n$ .

**Lemma 2.1** (See [3], Lemma 1.1) *The space  $AC^n[0, 1]$  ( $n \in \mathbb{N}_+$ ) consists of those and only those functions  $u(t)$  which can be represented in the form*

$$u(t) = (I_{0^+}^n y)(t) + \sum_{k=0}^{n-1} c_k t^k,$$

where  $y(t) \in L_1[0, 1]$ ,  $c_k$  ( $k = 0, 1, \dots, n - 1$ ) are arbitrary constants, and

$$(I_{0^+}^n y)(t) = \frac{1}{(n-1)!} \int_0^t (t-s)^{n-1} y(s) ds.$$

**Lemma 2.2** (See [3], Lemma 2.4) *Let  $p > 0$ . If  $u \in L_1[0, 1]$ , then the equality*

$$(D_{0^+}^p I_{0^+}^p u)(t) = u(t)$$

holds a.e. on  $[0, 1]$ .

**Lemma 2.3** (See [3], Corollary 2.1) *Let  $p > 0$  and  $n = [p] + 1$ , the equality  $(D_{0^+}^p u)(t) = 0$  is valid if, and only if,*

$$u(t) = \sum_{k=1}^n c_k t^{p-k},$$

where  $c_k \in \mathbb{R}$  ( $k = 1, 2, \dots$ ) are arbitrary constants.

**Lemma 2.4** (See [3], Lemma 2.5) *Let  $p > 0$  and  $n = [p] + 1$ . If  $u \in L_1[0, 1]$  and  $I_{0^+}^{n-p} u \in AC^n[0, 1]$ , then the equality*

$$I_{0^+}^p (D_{0^+}^p u)(t) = u(t) - \sum_{k=1}^n c_k t^{p-k}$$

holds a.e. on  $[0, 1]$ .

Let

$$E := C_{2-\delta}^0[0,1] = \left\{ u \in C(0,1) : \lim_{t \rightarrow 0^+} t^{2-\delta} u(t) < +\infty \right\},$$

then  $E$  is a Banach space with the norm  $\|u\| = \sup_{t \in [0,1]} t^{2-\delta} |u(t)|$ .

**Lemma 2.5** *If  $D_{0^+}^\delta u \in L_1[0,1]$ , then  $I_{0^+}^{2-\delta} u \in AC^2[0,1]$ .*

*Proof* Since  $D_{0^+}^\delta u \in L_1[0,1]$  and  $(D_{0^+}^\delta u)(t) = (\frac{d}{dt})^2 I_{0^+}^{2-\delta} u(t)$ , it follows that  $I_{0^+}^{2-\delta} u \in AC^2[0,1]$  from Lemma 2.1. □

**Definition 2.3** If  $u \in E$ ,  $D_{0^+}^\delta u \in L_1[0,1]$  and satisfies (1.1) and (1.2),  $u$  is called a solution of the fractional boundary value problem (1.1) and (1.2). Furthermore, if  $u(t) > 0$ ,  $t \in (0,1)$ ,  $u$  is called a positive solution of the fractional boundary value problem (1.1) and (1.2).

**Lemma 2.6** *For any  $y \in L_1[0,1]$ , the boundary value problem*

$$D_{0^+}^\delta u(t) + y(t) = 0, \quad \text{for a.e. } t \in (0,1), \tag{2.1}$$

$$\lim_{t \rightarrow 0^+} t^{2-\delta} u(t) = a, \quad u(1) = b, \tag{2.2}$$

*has a unique solution*

$$u(t) = \int_0^t G(t,s)y(s) ds + (b-a)t^{\delta-1} + at^{\delta-2}, \tag{2.3}$$

where

$$G(t,s) = \frac{1}{\Gamma(\delta)} \begin{cases} (t(1-s))^{\delta-1} - (t-s)^{\delta-1}, & 0 \leq s \leq t \leq 1, \\ (t(1-s))^{\delta-1}, & 0 \leq t \leq s \leq 1. \end{cases} \tag{2.4}$$

*Proof* Suppose  $u$  is a solution of the boundary value problem (2.1) and (2.2). Since  $y \in L_1[0,1]$ , (2.1) means that there exists a.e. on  $[0,1]$  the fractional derivative  $D_{0^+}^\delta u \in L_1[0,1]$ . By Lemma 2.5,  $I_{0^+}^{2-\delta} u(t) \in AC^2[0,1]$ . Thus we can apply Lemma 2.4, we have

$$(I_{0^+}^\delta D_{0^+}^\delta u)(t) = u(t) - \sum_{j=1}^2 c_j t^{\delta-j},$$

that is,

$$u(t) = -I^\delta y(t) + c_0 t^{\delta-1} + c_1 t^{\delta-2}. \tag{2.5}$$

By the boundary conditions  $\lim_{t \rightarrow 0^+} t^{2-\delta} u(t) = a$  and  $u(1) = b$ , we can show

$$c_0 = \frac{1}{\Gamma(\delta)} \int_0^1 (1-s)^{\delta-1} y(s) ds + (b-a), \quad c_1 = a.$$

Then

$$\begin{aligned} u(t) &= -\frac{1}{\Gamma(\delta)} \int_0^t (t-s)^{\delta-1} y(s) \, ds + \frac{1}{\Gamma(\delta)} \int_0^1 (t(1-s))^{\delta-1} y(s) \, ds + (b-a)t^{\delta-1} + at^{\delta-2} \\ &= \frac{1}{\Gamma(\delta)} \int_0^t ((t(1-s))^{\delta-1} - (t-s)^{\delta-1}) y(s) \, ds + \frac{1}{\Gamma(\delta)} \int_t^1 (t(1-s))^{\delta-1} y(s) \, ds \\ &\quad + at^{\delta-2} - at^{\delta-1} + bt^{\delta-1} \\ &= \int_0^1 G(t,s)y(s) \, ds + (b-a)t^{\delta-1} + at^{\delta-2}. \end{aligned}$$

On the other hand, if  $u$  satisfies the (2.3), then  $u$  satisfies the (2.5), too. Applying the operator  $D_{0+}^\delta$  to both sides of (2.5), we have

$$(D_{0+}^\delta u)(t) = \left(\frac{d}{dt}\right)^2 I_{0+}^{2-\delta} u(t) = -\left(\frac{d}{dt}\right)^2 (I_{0+}^{2-\delta} I_{0+}^\delta y)(t) = -\left(\frac{d}{dt}\right)^2 I_{0+}^2 y(t) = -y(t),$$

which implies (2.1) is satisfied.

We can easily show that  $u$  satisfies (2.2). □

**Lemma 2.7** *Let  $G$  be defined by (2.4). Then,*

- (1) *for any  $t, s \in (0, 1)$ ,  $G(t, s) > 0$ ;*
- (2) *for any  $t, s \in [0, 1]$ ,  $G(t, s)$  is continuous and  $G(t, s) \leq \frac{1}{\Gamma(\delta)}$ ;*
- (3) *for  $0 \leq s < t \leq 1$ , the  $G(t, s)$  is monotone increasing with respect to  $t$  and for  $0 \leq t < s \leq 1$ , the  $G(t, s)$  is monotone decreasing with respect to  $t$ ;*
- (4)  $\frac{1}{\Gamma(\delta)}(\delta - 1)t(1 - t)s(1 - s)^{\delta-1} \leq t^{2-\delta}G(t, s) \leq \frac{1}{\Gamma(\delta)}s(1 - s)^{\delta-1} = G(s, s)$ ,  $t, s \in (0, 1)$ .

*Proof* (1) By the expression of  $G(t, s)$ , it is easy to see  $G(t, s) > 0$ ,  $t, s \in (0, 1)$ .

(2) By the expression of  $G(t, s)$ , we can get results easily.

(3) According to the definition of  $G(t, s)$ , for  $0 < s < t \leq 1$ , we have

$$\begin{aligned} \frac{\partial G(t, s)}{\partial t} &= \frac{\delta - 1}{\Gamma(\delta)} (t^{\delta-2}(1 - s)^{\delta-1} - (t - s)^{\delta-2}) \\ &\leq \frac{\delta - 1}{\Gamma(\delta)} ((t - s)^{\delta-2}(1 - s)^{\delta-1} - (t - s)^{\delta-2}) \\ &\leq 0. \end{aligned}$$

Therefore,  $G(t, s)$  is monotone decreasing with respect to  $t$  for  $0 \leq s < t \leq 1$ .

It is obvious  $G(t, s)$  is monotone increasing with respect to  $t$  for  $0 \leq t < s \leq 1$ .

(4) The result can be found in [13]. □

**Lemma 2.8** *If  $u \in C_{2-\delta}^0[0, 1]$ ,  $D_{0+}^\delta u \in L_1[0, 1]$ , then the boundary value problem (1.1) and (1.2) is equivalent to the integral equation*

$$u(t) = \int_0^1 G(t, s)f(s, u(s)) \, ds + (b-a)t^{\delta-1} + at^{\delta-2}, \quad t \in J.$$

*Proof* By Lemma 2.6, we can easily see that Lemma 2.8 holds. □

Let

$$P = \{u \in E : u(t) \geq 0, t \in J\},$$

then  $P$  is a cone in  $E$ .

For each  $u \in P$ , we have

$$\lim_{t \rightarrow 0^+} \left[ t^{2-\delta} \int_0^1 G(t,s)f(s,u(s)) \, ds + (b-a)t + a \right] = a.$$

Let

$$Tu(t) = \int_0^1 G(t,s)f(s,u(s)) \, ds + (b-a)t^{\delta-1} + at^{\delta-2} \quad \text{for } u \in P.$$

Hence,  $T : P \rightarrow E$ .

**Lemma 2.9** *The operator  $T : P \rightarrow P$  is completely continuous.*

*Proof* It is easy to show  $T : P \rightarrow P$ .

(1)  $T$  is a continuous operator.

If  $\{u_n\} \subset P$ ,  $u \in P$ , and  $\|u_n - u\| \rightarrow 0$  as  $n \rightarrow \infty$ , there exists a constant  $\gamma > 0$ , such that  $\|u_n\| \leq \gamma$  and  $\|u\| \leq \gamma$ , that is, for any  $t \in J$ ,  $|t^{2-\delta}u_n(t)| \leq \gamma$ ,  $|t^{2-\delta}u(t)| \leq \gamma$ .

Therefore, for a.e.  $s \in J$ , we can show

$$\lim_{n \rightarrow \infty} f(s, u_n(s)) = \lim_{n \rightarrow \infty} f(s, s^{\delta-2}s^{2-\delta}u_n(s)) = f(s, s^{\delta-2}s^{2-\delta}u(s)) = f(s, u(s)),$$

and there exists  $\varphi_\gamma \in L_1[0, 1]$ , for a.e.  $s \in J$ ,

$$|f(s, u_n(s)) - f(s, u(s))| \leq 2\varphi_\gamma(s).$$

Then by the Lebesgue dominated convergence theorem, we can get

$$\begin{aligned} \lim_{n \rightarrow \infty} \|Tu_n - Tu\| &= \lim_{n \rightarrow \infty} \sup_{t \in [0,1]} |t^{2-\delta}Tu_n(t) - t^{2-\delta}Tu(t)| \\ &\leq \lim_{n \rightarrow \infty} \sup_{t \in [0,1]} t^{2-\delta} \int_0^1 G(t,s)|f(s, u_n(s)) - f(s, u(s))| \, ds \\ &\leq \frac{1}{\Gamma(\delta)} \int_0^1 \lim_{n \rightarrow \infty} |f(s, u_n(s)) - f(s, u(s))| \, ds \\ &= 0. \end{aligned}$$

So  $\lim_{n \rightarrow \infty} \|Tu_n - Tu\| = 0$ .

By Lemma 2.7,  $a \geq 0$ , and  $b \geq 0$ , we have  $Tu(t) \geq 0$ ,  $t \in J$ . Hence,  $T : P \rightarrow P$  is continuous.

(2)  $T : P \rightarrow P$  is relatively compact.

Let  $A_1$  be any bounded set, then there exists a constant  $\gamma > 0$  such that  $\|u\| \leq \gamma$  for each  $u \in A_1$ , that is,  $\sup_{t \in J} t^{2-\delta}|u(t)| \leq \gamma$ .

There exists  $\varphi_\gamma \in L_1[0, 1]$  for any  $u \in A_1$ , we have

$$|f(s, u(s))| = |f(s, s^{\delta-2} s^{2-\delta} u(s))| \leq \varphi_\gamma(s), \quad s \in J,$$

$$|t^{2-\delta} Tu(t)| = \left| \int_0^1 t^{2-\delta} G(t, s) f(s, u(s)) \, ds + (b-a)t + a \right|$$

$$\leq \frac{1}{\Gamma(\delta)} \int_0^1 \varphi_\gamma(s) \, ds + b + 2a.$$

It shows that  $T(A_1)$  is uniformly bounded.

In addition, for any given  $u \in A_1$ , because the  $G(t, s)$  is continuous for  $(t, s) \in [0, 1] \times [0, 1]$ , then it must be uniformly continuous. So for any  $\varepsilon > 0$ , there exists a constant  $\sigma \in (0, \frac{\varepsilon}{2(b+a)+1})$  such that for any  $t_1, t_2, s_1, s_2 \in [0, 1]$ , as  $|t_1 - t_2| < \sigma$  and  $|s_1 - s_2| < \sigma$ , we can get

$$|t_1^{2-\delta} G(t_1, s_1) - t_2^{2-\delta} G(t_2, s_2)| < \frac{\varepsilon}{2 \int_0^1 \varphi_\gamma(s) \, ds + 1}.$$

Then

$$|t_2^{2-\delta} Tu(t_2) - t_1^{2-\delta} Tu(t_1)|$$

$$\leq \int_0^1 |t_2^{2-\delta} G(t_2, s) - t_1^{2-\delta} G(t_1, s)| |f(s, u(s))| \, ds + |b-a||t_2 - t_1|$$

$$\leq \int_0^1 |t_2^{2-\delta} G(t_2, s) - t_1^{2-\delta} G(t_1, s)| \varphi_\gamma(s) \, ds + |b-a||t_2 - t_1|$$

$$< \frac{\varepsilon}{2 \int_0^1 \varphi_\gamma(s) \, ds + 1} \int_0^1 \varphi_\gamma(s) \, ds + \frac{\varepsilon}{2(b+a)+1} (b+a)$$

$$= \varepsilon.$$

Thus, we have proved  $T(A_1)$  is equicontinuous.

By the Arzela-Ascoli theorem, we know that  $T$  is relatively compact.

Therefore,  $T$  is completely continuous. □

### 3 Comparison principle and the existence of solutions

**Definition 3.1** Let  $\alpha \in E, D_{0+}^\delta \alpha \in L_1[0, 1]$ , we say that  $\alpha$  is a lower solution of the boundary value problem (1.1) and (1.2), if

$$-D_{0+}^\delta \alpha(t) \leq f(t, \alpha(t)), \quad \text{for a.e. } t \in (0, 1), \tag{3.1}$$

$$\lim_{t \rightarrow 0^+} t^{2-\delta} \alpha(t) \leq a, \quad \alpha(1) \leq b. \tag{3.2}$$

Let  $\beta \in E, D_{0+}^\delta \beta \in L_1[0, 1]$ , we say that  $\beta$  is an upper solution of the boundary value problem (1.1) and (1.2), if

$$-D_{0+}^\delta \beta(t) \geq f(t, \beta(t)), \quad \text{for a.e. } t \in (0, 1), \tag{3.3}$$

$$\lim_{t \rightarrow 0^+} t^{2-\delta} \beta(t) \geq a, \quad \beta(1) \geq b. \tag{3.4}$$

**Lemma 3.1** *If  $u \in E, D_{0^+}^\delta u \in L_1[0,1]$  and it satisfies*

$$D_{0^+}^\delta u(t) \leq 0, \quad \text{for a.e. } t \in (0,1),$$

$$\lim_{t \rightarrow 0^+} t^{2-\delta} u(t) \geq 0, \quad u(1) \geq 0,$$

*then  $u(t) \geq 0$ , for  $t \in J$ .*

*Proof* Denote  $-D_{0^+}^\delta u(t) = y(t)$ , then  $y(t) \geq 0$ , for a.e.  $t \in (0,1)$ .

Let  $\lim_{t \rightarrow 0^+} t^{2-\delta} u(t) = a, u(1) = b$ , then  $a \geq 0, b \geq 0$ .

Now we consider the boundary value problem

$$-D_{0^+}^\delta u(t) = y(t), \quad \text{for a.e. } t \in (0,1),$$

$$\lim_{t \rightarrow 0^+} t^{2-\delta} u(t) = a, \quad u(1) = b.$$

By Lemma 2.6, we have

$$u(t) = \int_0^1 G(t,s)y(s) \, ds + (b-a)t^{\delta-1} + at^{\delta-2}$$

$$= \int_0^1 G(t,s)y(s) \, ds + at^{\delta-2}(1-t) + bt^{\delta-1}.$$

It follows that  $u(t) \geq 0$  for  $t \in J$  from Lemma 2.7. □

**Theorem 3.1** *Suppose (H) holds, there exist a nonnegative lower solution  $\alpha$  and an upper solution  $\beta$  of the boundary value problem (1.1) and (1.2) such that  $\alpha(t) \leq \beta(t)$  for  $t \in J$ . Then the boundary value problem (1.1) and (1.2) has at least one positive solution  $u$  and it satisfies  $\alpha(t) \leq u(t) \leq \beta(t)$  for  $t \in J$ .*

*Proof* Let

$$f_1(t, u) = \begin{cases} f(t, \beta(t)), & u > \beta(t), \\ f(t, u), & \alpha(t) \leq u \leq \beta(t), \\ f(t, \alpha(t)), & u < \alpha(t). \end{cases}$$

Since  $f$  satisfies the  $L_1$ -Caratheódory conditions, we have  $f_1$  satisfies  $L_1$ -Caratheódory conditions, too.

We consider the boundary value problem

$$D_{0^+}^\delta u(t) + f_1(t, u(t)) = 0, \quad \text{for a.e. } t \in (0,1), \tag{3.5}$$

$$\lim_{t \rightarrow 0^+} t^{2-\delta} u(t) = a, \quad u(1) = b. \tag{3.6}$$

We define  $T_1 : P \rightarrow P$  by

$$T_1 u(t) = \int_0^1 G(t,s)f_1(s, u(s)) \, ds + (b-a)t^{\delta-1} + at^{\delta-2}.$$

By (H), for any  $u \in P$  and  $t \in J$ , we have

$$\begin{aligned} 0 &\leq f_1(t, u(t)) = f_1(t, t^{\delta-2}t^{2-\delta}u(t)) \leq f_1(t, t^{\delta-2}t^{2-\delta}\beta(t)) \\ &\leq f_1(t, t^{\delta-2}\|\beta\|) \leq \varphi_{\|\beta\|}(t). \end{aligned}$$

Define

$$A_2 = \left\{ u \in P : \|u\| \leq \frac{1}{\Gamma(\delta)} \int_0^1 \varphi_{\|\beta\|}(s) ds + |b - a| + a \right\}.$$

It is easy to see that  $A_2$  is a closed and convex set.

By (H), we can get

$$\begin{aligned} |t^{2-\delta}T_1u(t)| &= \left| \int_0^1 t^{2-\delta}G(t,s)f_1(s, u(s)) ds + (b - a)t + a \right| \\ &\leq \int_0^1 G(t,s)f_1(s, u(s)) ds + |b - a| + a \\ &\leq \int_0^1 G(t,s)f(s, s^{\delta-2}\|\beta\|) ds + |b - a| + a \\ &\leq \frac{1}{\Gamma(\delta)} \int_0^1 \varphi_{\|\beta\|}(s) ds + |b - a| + a. \end{aligned}$$

So

$$\|T_1u\| \leq \frac{1}{\Gamma(\delta)} \int_0^1 \varphi_{\|\beta\|}(s) ds + |b - a| + a,$$

that is,  $T_1 : A_2 \rightarrow A_2$ .

Similar to the proof of Lemma 2.9, we can prove  $T_1$  is completely continuous.

By the Schauder fixed point theorem, we can see that  $T_1$  has at least one fixed point  $u$ , that is, there exists a positive solution  $u$  of the boundary value problem (3.5) and (3.6).

Finally, we prove  $\alpha(t) \leq u(t) \leq \beta(t)$ , for  $t \in J$ .

Let  $v(t) = u(t) - \alpha(t)$ , by (H), we have

$$\begin{aligned} D_{0^+}^\delta v(t) &= D_{0^+}^\delta u(t) - D_{0^+}^\delta \alpha(t) \\ &= -f_1(t, u(t)) - D_{0^+}^\delta \alpha(t) \\ &\leq -f_1(t, u(t)) + f_1(t, \alpha(t)) \\ &\leq 0, \\ \lim_{t \rightarrow 0^+} t^{2-\delta}v(t) &= \lim_{t \rightarrow 0^+} t^{2-\delta}u(t) - \lim_{t \rightarrow 0^+} t^{2-\delta}\alpha(t) \geq a - a = 0, \end{aligned}$$

and

$$v(1) = u(1) - \alpha(1) \geq b - b = 0.$$

It follows that  $v(t) \geq 0$  for  $t \in J$  from Lemma 3.1.

Hence, we can show  $u(t) \geq \alpha(t)$  for  $t \in J$ .

Similarly, we can get  $u(t) \leq \beta(t)$  for  $t \in J$ .

So each solution of the boundary value problem (3.5) and (3.6) satisfies  $\alpha(t) \leq u(t) \leq \beta(t)$  for  $t \in J$ . That is,  $f_1(t, u(t)) = f(t, u(t))$  and  $u$  is a positive solution of the boundary value problem (1.1) and (1.2). □

#### 4 Impact of disturbance parameters $a, b$ on the existence of solutions

**Theorem 4.1** *Let (H) hold.*

- (1) *If there exist parameters  $a = \bar{a} \geq 0$  and  $b = \bar{b} \geq 0$  such that the boundary value problem (1.1) and (1.2) has a positive solution  $\bar{u}$ , then for any  $0 \leq a \leq \bar{a}$  and  $0 \leq b \leq \bar{b}$ , the boundary value problem (1.1) and (1.2) has a positive solution  $u$  and  $(b - a)t^{\delta-1} + at^{\delta-2} \leq u \leq \bar{u}$  for  $t \in J$ .*
- (2) *If there exist parameters  $a = \bar{a} \geq 0$  and  $b = \bar{b} \geq 0$  such that the boundary value problem (1.1) and (1.2) does not have positive solutions, then for any  $a \geq \bar{a}$  and  $b \geq \bar{b}$ , the boundary value problem (1.1) and (1.2) does not have positive solutions.*

*Proof* (1) Since there exist parameters  $a = \bar{a} \geq 0$  and  $b = \bar{b} \geq 0$  such that the boundary value problem (1.1) and (1.2) has a positive solution  $\bar{u}$ , by Lemma 2.8, we have

$$\bar{u}(t) = \int_0^1 G(t, s)f(s, \bar{u}(s)) ds + (\bar{b} - \bar{a})t^{\delta-1} + \bar{a}t^{\delta-2}.$$

Hence, for any  $0 \leq a \leq \bar{a}$  and  $0 \leq b \leq \bar{b}$ ,

$$\bar{u}(t) \geq (\bar{b} - \bar{a})t^{\delta-1} + \bar{a}t^{\delta-2} \geq (b - a)t^{\delta-1} + at^{\delta-2}, \quad t \in J.$$

We take  $\alpha = (b - a)t^{\delta-1} + at^{\delta-2}$  and  $\beta = \bar{u}$ , then

$$\begin{aligned} -D_{0^+}^\delta \alpha(t) &= -D_{0^+}^\delta ((b - a)t^{\delta-1} + at^{\delta-2}) \leq f(t, \alpha(t)), \quad t \in (0, 1), \\ \lim_{t \rightarrow 0^+} t^{2-\delta} \alpha(t) &= \lim_{t \rightarrow 0^+} t^{2-\delta} ((b - a)t^{\delta-1} + at^{\delta-2}) = a, \quad \alpha(1) = b. \end{aligned}$$

We see that  $\alpha$  and  $\beta$  satisfy (3.1), (3.2), (3.3), and (3.4).

Therefore,  $\alpha$  and  $\beta$  are the lower solution and the upper solution of the boundary value problem (1.1) and (1.2), respectively.

For the parameters  $0 \leq a \leq \bar{a}$  and  $0 \leq b \leq \bar{b}$ , by Theorem 3.1, we see that the boundary value problem (1.1) and (1.2) has at least one positive solution  $u$  and  $\alpha(t) \leq u(t) \leq \beta(t)$  for  $t \in J$ .

(2) If there exist constants  $a_0 \geq \bar{a}$  and  $b_0 \geq \bar{b}$  such that the boundary value problem (1.1) and (1.2) has a positive solution for the parameters  $a_0$  and  $b_0$ , by (1), we can show that for each  $0 \leq a \leq a_0$  and  $0 \leq b \leq b_0$ , the boundary value problem (1.1) and (1.2) has a positive solution. Therefore, the boundary value problem (1.1) and (1.2) has a positive solution for the parameters  $\bar{a}$  and  $\bar{b}$ , which is a contradiction. □

For convenience, we denote

$$\begin{aligned} N &= \frac{\Gamma(\delta + 2)}{\Gamma(\delta + 2) + 1}, \quad M = \frac{2,048\Gamma(\delta - 1)}{11(\delta - 1)}, \\ f^0 &= \limsup_{u \rightarrow 0^+} \sup_{t \in J} \frac{f(t, t^{\delta-2}u)}{u}, \quad f_\infty = \liminf_{u \rightarrow +\infty} \inf_{t \in [\frac{1}{4}, \frac{3}{4}]} \frac{f(t, u)}{u}. \end{aligned}$$

Let

$$\hat{P} = \{u \in E : t^{2-\delta}u(t) \geq (\delta - 1)t(1 - t)\|u\|, t \in J\}.$$

Then  $\hat{P}$  is a cone in  $E$  and  $\hat{P} \subset P$ .

**Lemma 4.1** (See [29]) *Let  $E$  be a Banach space and  $\hat{P} \subseteq E$  be a cone in  $E$ . Assume that  $\Omega$  is a bounded open subset of  $E$  and  $\theta \in \Omega$  and that  $T : \hat{P} \cap \overline{\Omega} \rightarrow \hat{P}$  is compact. If*

$$u \neq \tau Tu \quad \text{for all } u \in \hat{P} \cap \partial\Omega \text{ and } \tau \in [0, 1],$$

*then the fixed point index  $i(T, \hat{P} \cap \Omega, \hat{P}) = 1$ .*

**Lemma 4.2** (See [29]) *Let  $E$  be a Banach space and  $\hat{P} \subseteq E$  be a cone in  $E$ . Assume that  $\Omega$  is a bounded open subset of  $E$  and that  $T : \hat{P} \cap \overline{\Omega} \rightarrow \hat{P}$  is compact. If there exists  $u_0 \in \hat{P} \setminus \{\theta\}$  such that*

$$u - Tu \neq \tau u_0 \quad \text{for all } u \in \hat{P} \cap \partial\Omega \text{ and } \tau \geq 0,$$

*then the fixed point index  $i(T, \hat{P} \cap \Omega, \hat{P}) = 0$ .*

**Lemma 4.3** *If  $u$  is a positive solution of the boundary value problem (1.1) and (1.2), then for  $t \in J$ ,*

$$t^{2-\delta}u(t) \geq (\delta - 1)t(1 - t)\|u\|.$$

*Proof* By Lemma 2.8, we can show

$$u(t) = \int_0^1 G(t, s)f(s, u(s)) \, ds + (b - a)t^{\delta-1} + at^{\delta-2}.$$

It follows from Lemma 2.7, for  $t \in J$ , that

$$\begin{aligned} t^{2-\delta}u(t) &= \int_0^1 t^{2-\delta}G(t, s)f(s, u(s)) \, ds + (b - a)t + a \\ &\geq \int_0^1 \frac{1}{\Gamma(\delta)}(\delta - 1)t(1 - t)s(1 - s)^{\delta-1}f(s, u(s)) \, ds + (1 - t)a + bt \\ &\geq (\delta - 1)t(1 - t)\left(\frac{1}{\Gamma(\delta)} \int_0^1 s(1 - s)^{\delta-1}f(s, u(s)) \, ds + b + a\right). \end{aligned}$$

On the other hand,

$$t^{2-\delta}u(t) \leq \frac{1}{\Gamma(\delta)} \int_0^1 s(1 - s)^{\delta-1}f(s, u(s)) \, ds + b + a,$$

we have

$$\|u\| \leq \frac{1}{\Gamma(\delta)} \int_0^1 s(1 - s)^{\delta-1}f(s, u(s)) \, ds + b + a.$$

Hence,

$$t^{2-\delta}u(t) \geq (\delta - 1)t(1 - t)\|u\|. \quad \square$$

**Theorem 4.2** *Let (H) hold,  $f^0 = 0$ , and  $f_\infty = +\infty$ .*

- (1) *If  $a = 0$ , the boundary value problem (1.1) and (1.2) has at least one positive solution for small enough  $b$ ; the boundary value problem (1.1) and (1.2) has no positive solution for large enough  $b$ .*
- (2) *If  $b = 0$ , the boundary value problem (1.1) and (1.2) has at least one positive solution for small enough  $a$ ; the boundary value problem (1.1) and (1.2) has no positive solution for large enough  $a$ .*
- (3) *If  $b = ka$  (for any  $k > 0$ ), the boundary value problem (1.1) and (1.2) has at least one positive solution for small enough  $a$  and  $b$ ; the boundary value problem (1.1) and (1.2) has no positive solution for large enough  $a$  and  $b$ .*

*Proof* (1) By  $f^0 = 0$ , for given  $N > 0$ , there exists a constant  $r_1 > 0$  such that

$$f(t, t^{\delta-2}u) \leq Nu \leq Nr_1, \quad \text{for } t \in J \text{ and } u \in (0, r_1].$$

Let

$$\Omega_1 = \{u \in E : \|u\| < r_1\}.$$

When  $b < Nr_1$ , for any  $u \in \hat{P} \cap \partial\Omega_1$ , we have  $|t^{2-\delta}u(t)| \leq r_1, t \in J$ . By Lemma 2.7, we can get

$$\begin{aligned} t^{2-\delta}(Tu)(t) &= \int_0^1 t^{2-\delta}G(t,s)f(s,u(s)) \, ds + (b - a)t + a \\ &= \int_0^1 t^{2-\delta}G(t,s)f(s,u(s)) \, ds + bt \\ &\leq \frac{1}{\Gamma(\delta)} \int_0^1 s(1 - s)^{\delta-1}f(s, s^{\delta-2}s^{2-\delta}u(s)) \, ds + b \\ &< \frac{Nr_1}{\Gamma(\delta)} \cdot \frac{\Gamma(\delta)}{\Gamma(\delta + 1)} + Nr_1 \\ &= r_1 \\ &= \|u\|, \end{aligned}$$

that is,

$$\|Tu\| < \|u\|, \quad u \in \hat{P} \cap \partial\Omega_1. \tag{4.1}$$

We can show that

$$u \neq \tau Tu \quad \text{for all } u \in \hat{P} \cap \partial\Omega_1 \text{ and } \tau \in [0, 1].$$

Otherwise, there exist  $u^* \in \hat{P} \cap \partial\Omega_1$  and  $\tau^* \in [0, 1]$  such that  $u^* = \tau^* Tu^*$ , then

$$\|u^*\| = \tau \|Tu^*\| \leq \|Tu^*\|,$$

which contradicts to (4.1).

In view of Lemma 4.1, we can get

$$i(T, \hat{P} \cap \Omega_1, \hat{P}) = 1. \tag{4.2}$$

On the other hand, by  $f_\infty = +\infty$ , for given  $M > 0$ , there exists a constant  $R > 0$  such that

$$f(t, u) > Mu, \quad \text{for } t \in \left[\frac{1}{4}, \frac{3}{4}\right] \text{ and } u \in [R, +\infty). \tag{4.3}$$

Let  $R_1 > \frac{16}{3(\delta-1)}R$  and

$$\Omega_2 = \{u \in E : \|u\| < R_1\}.$$

For any  $u \in \hat{P} \cap \partial\Omega_2$ , we have  $u \in \hat{P}$  and  $\|u\| = R_1$ , so

$$\min_{t \in [\frac{1}{4}, \frac{3}{4}]} u(t) \geq \min_{t \in [\frac{1}{4}, \frac{3}{4}]} t^{2-\delta} u(t) \geq \frac{3}{16}(\delta-1)\|u\| = \frac{3(\delta-1)}{16}R_1 > R. \tag{4.4}$$

By Lemma 2.7, we can get

$$\begin{aligned} t^{2-\delta}(Tu)(t) &= \int_0^1 t^{2-\delta} G(t, s) f(s, u(s)) \, ds + (b-a)t + a \\ &= \int_0^1 t^{2-\delta} G(t, s) f(s, u(s)) \, ds + bt \\ &\geq \frac{1}{\Gamma(\delta)} \int_0^1 (\delta-1)t(1-t)s(1-s)^{\delta-1} f(s, u(s)) \, ds + bt. \end{aligned}$$

When  $t = \frac{1}{2}$ , by (4.3) and (4.4), we can show

$$\begin{aligned} \left(\frac{1}{2}\right)^{2-\delta} (Tu)\left(\frac{1}{2}\right) &> \frac{1}{4\Gamma(\delta)} \int_{\frac{1}{4}}^{\frac{3}{4}} (\delta-1)s(1-s)^{\delta-1} Mu(s) \, ds \\ &\geq \left(\frac{3M(\delta-1)^2}{64\Gamma(\delta)} \int_{\frac{1}{4}}^{\frac{3}{4}} s(1-s)^{\delta-1} \, ds\right) \|u\| \\ &\geq \left(\frac{3M(\delta-1)^2}{64\Gamma(\delta)} \int_{\frac{1}{4}}^{\frac{3}{4}} s(1-s) \, ds\right) \|u\| \\ &= \frac{11M(\delta-1)}{2,048\Gamma(\delta-1)} \|u\| \\ &= \|u\|. \end{aligned}$$

Hence,

$$\|Tu\| > \|u\|, \quad u \in \hat{P} \cap \partial\Omega_2. \tag{4.5}$$

In the following, we show that there exists  $u_0 \in \hat{P} \setminus \{\theta\}$  such that

$$u - Tu \neq \tau Tu_0 \quad \text{for all } u \in \hat{P} \cap \partial\Omega_2 \text{ and } \tau \geq 0.$$

If this is not true, then there exist  $u^* \in \hat{P} \cap \partial\Omega_2$  and  $\tau^* \geq 0$  such that  $u^* - Tu^* = \tau^* u_0$ , so

$$u^* = Tu^* + \tau^* u_0 \geq Tu^*,$$

and

$$\|u^*\| \geq \|Tu^*\|,$$

which contradicts (4.5).

By Lemma 4.2, we have

$$i(T, \hat{P} \cap \Omega_2, \hat{P}) = 0. \tag{4.6}$$

By (4.2), (4.6), and the additivity property of the fixed point index, we obtain

$$i(T, \hat{P} \cap (\Omega_2 \setminus \bar{\Omega}_1), \hat{P}) = i(T, \hat{P} \cap \Omega_2, \hat{P}) - i(T, \hat{P} \cap \bar{\Omega}_1, \hat{P}) = 0 - 1 = -1 \neq 0.$$

Then  $T$  has at least one fixed point  $u \in \hat{P} \cap (\Omega_2 \setminus \bar{\Omega}_1)$ , which implies the boundary value problem (1.1) and (1.2) has at least one positive solution.

In the following, we prove that there exists large enough  $b$  such that the boundary value problem (1.1) and (1.2) has no positive solution.

Otherwise, for any large enough  $b_n$ , the boundary value problem (1.1) and (1.2) has solutions, then there exists  $\{b_n\}$  such that, for  $\lim_{n \rightarrow +\infty} b_n = +\infty$ , the boundary value problem

$$\begin{aligned} D_{0^+}^\delta u(t) + f(t, u(t)) &= 0, \quad \text{for a.e. } t \in (0, 1), \\ \lim_{t \rightarrow 0^+} t^{2-\delta} u(t) &= 0, \quad u(1) = b_n \end{aligned}$$

has a positive solution, which we denote  $u_n$ .

Due to this,

$$\begin{aligned} u_n(1) &= \int_0^1 G(1, s) f(s, u_n(s)) \, ds + b_n \\ &\geq b_n \rightarrow +\infty \quad (n \rightarrow +\infty), \end{aligned}$$

so  $\|u_n\| \rightarrow +\infty$ .

In addition, by  $f_\infty = +\infty$  and given  $M > 0$ , there exists a constant  $\bar{R} > 0$  such that  $f(t, u) > 2Mu$ ,  $t \in [\frac{1}{4}, \frac{3}{4}]$ , and  $u \in [\bar{R}_1, +\infty)$ .

Let  $\bar{R}_1 > \frac{16}{3(\delta-1)} \bar{R}$ . For any  $u \in \partial \hat{P}_{\bar{R}_1} = \{u | u \in \hat{P}, \|u\| = \bar{R}_1\}$ , we have  $|t^{2-\delta} u| \leq \bar{R}_1$ , so

$$\min_{t \in [\frac{1}{4}, \frac{3}{4}]} u(t) \geq \min_{t \in [\frac{1}{4}, \frac{3}{4}]} t^{2-\delta} u(t) \geq \frac{3}{16} (\delta - 1) \|u\| = \frac{3(\delta - 1)}{16} \bar{R}_1 > \bar{R}.$$

Similar to the proof above, we can see that, for large integer  $n$ ,

$$\left(\frac{1}{2}\right)^{2-\delta} u_n \left(\frac{1}{2}\right) > 2 \cdot \frac{11M(\delta-1)}{2,048\Gamma(\delta-1)} \|u_n\| = 2\|u_n\|.$$

So  $\|u_n\| > 2\|u_n\|$ , which is a contradiction.

(2) Similar to the proof of (1), we can prove (2) holds.

(3) By  $f^0 = 0$  and given  $N > 0$ , there exists  $\tilde{r}_1 > 0$  such that

$$f(t, t^{\delta-2}u) \leq Nu < N\tilde{r}_1, \quad \text{for } t \in J \text{ and } u \in (0, \tilde{r}_1].$$

Let

$$\Omega_3 = \{u \in E : \|u\| < \tilde{r}_1\}.$$

Since  $b = ka$  and  $k > 0$ , let  $\max\{a, b\} = a \max\{1, k\} < N\tilde{r}_1$ , we have  $0 \leq a((k-1)t + 1) \leq a \max\{1, k\} < N\tilde{r}_1$ .

Hence, for any  $u \in \hat{P} \cap \partial\Omega_3$ , by Lemma 2.7, we can get

$$\begin{aligned} t^{2-\delta}(Tu)t &= \int_0^1 t^{2-\delta}G(t,s)f(s,u(s)) \, ds + (b-a)t + a \\ &= \int_0^1 t^{2-\delta}G(t,s)f(s,u(s)) \, ds + (ka-a)t + a \\ &\leq \frac{1}{\Gamma(\delta)} \int_0^1 s(1-s)^{\delta-1}f(s, s^{\delta-2}s^{2-\delta}u(s)) \, ds + a((k-1)t + 1) \\ &< N\left(\frac{1}{\Gamma(\delta+2)} + 1\right)\tilde{r}_1 \\ &= \tilde{r}_1 \\ &= \|u\| \end{aligned} \tag{4.7}$$

and

$$\lim_{t \rightarrow 0^+} t^{2-\delta}(Tu)t = a < N\tilde{r}_1 < \tilde{r}_1.$$

This implies  $\|Tu\| < \|u\|$  for  $u \in \hat{P} \cap \partial\Omega_3$ .

It is similar to the proof of (1), by Lemma 4.1, we have

$$i(T, \hat{P} \cap \Omega_3, \hat{P}) = 1. \tag{4.8}$$

On the other hand, by  $f_\infty = +\infty$  and given  $M > 0$ , similar to the proof of (1), we can see that there exists a constant  $\tilde{R}_1$  such that

$$\Omega_4 = \{u \in E : \|u\| < \tilde{R}_1\}$$

and

$$i(T, \hat{P} \cap \Omega_4, \hat{P}) = 0. \tag{4.9}$$

By (4.8) and (4.9), and the additivity property of the fixed point index, we obtain

$$i(T, \hat{P} \cap (\Omega_4 \setminus \overline{\Omega}_3), \hat{P}) = i(T, \hat{P} \cap \Omega_4, \hat{P}) - i(T, \hat{P} \cap \overline{\Omega}_3, \hat{P}) = 0 - 1 = -1 \neq 0,$$

then  $T$  has at least one fixed point  $u \in \Omega_4 \setminus \overline{\Omega}_3$ , that is, the boundary value problem (1.1) and (1.2) has a positive solution.

Similar to the proof of (1), we can prove that the boundary value problem (1.1) and (1.2) has no positive solution for large enough  $a$  and  $b$ .

In summary, the boundary value problem (1.1) and (1.2) has at least one positive solution for small enough  $a$  and  $b$ ; the boundary value problem (1.1) and (1.2) has no solution for large enough  $a$  and  $b$ . □

**Theorem 4.3** *Let  $f_\infty = +\infty$  and  $\Lambda \subset \mathbb{R}_+^2$  be a bounded set. Then for each parameters  $(a, b) \in \Lambda$ , there exists a constant  $\eta(\Lambda)$  such that the positive solution  $u$  of the boundary value problem (1.1) and (1.2) satisfies  $\|u\| \leq \eta(\Lambda)$ .*

*Proof* Because  $\Lambda$  is a bounded set, there are constants  $\eta_1, \eta_2$ , such that for each parameter  $(a, b) \in \Lambda, 0 \leq a \leq \eta_1, 0 \leq b \leq \eta_2$ .

Since  $f_\infty = +\infty$ , for  $M > 0$ , there exists a constant  $R > 0$  such that

$$f(t, u) > Mu, \quad u \in [R, +\infty), t \in J.$$

Let  $\eta = \frac{16R}{3(\delta-1)}$ , we can show  $\|u\| \leq \eta$ .

Otherwise,  $\|u\| > \eta$ . Hence,

$$\min_{t \in [\frac{1}{4}, \frac{3}{4}]} u(t) \geq \min_{t \in [\frac{1}{4}, \frac{3}{4}]} t^{2-\delta} u(t) \geq \frac{3(\delta-1)}{16} \|u\| > \frac{3(\delta-1)}{16} \eta = R.$$

By Lemma 2.7, we have

$$\begin{aligned} t^{2-\delta} u(t) &= \int_0^1 t^{2-\delta} G(t, s) f(s, u(s)) \, ds + (b-a)t + a \\ &\geq \frac{1}{\Gamma(\delta)} \int_{\frac{1}{4}}^{\frac{3}{4}} (\delta-1)t(1-t)s(1-s)^{\delta-1} f(s, u(s)) \, ds + (b-a)t + a. \end{aligned}$$

When  $t = \frac{1}{2}$ , we can show

$$\begin{aligned} \left(\frac{1}{2}\right)^{2-\delta} u\left(\frac{1}{2}\right) &> \frac{1}{4\Gamma(\delta)} \int_{\frac{1}{4}}^{\frac{3}{4}} (\delta-1)s(1-s)^{\delta-1} Mu(s) \, ds \\ &\geq \frac{M(\delta-1)}{4\Gamma(\delta)} \int_{\frac{1}{4}}^{\frac{3}{4}} s(1-s)^{\delta-1} s^{2-\delta} u(s) \, ds \\ &\geq \left(\frac{3M(\delta-1)^2}{64\Gamma(\delta)} \int_{\frac{1}{4}}^{\frac{3}{4}} s(1-s)^{\delta-1} \, ds\right) \|u\| \\ &\geq \left(\frac{3M(\delta-1)^2}{64\Gamma(\delta)} \int_{\frac{1}{4}}^{\frac{3}{4}} s(1-s) \, ds\right) \|u\| \end{aligned}$$

$$\begin{aligned}
 &= \frac{11M(\delta - 1)}{2,048\Gamma(\delta - 1)} \|u\| \\
 &= \|u\|,
 \end{aligned}$$

which is a contradiction. □

**Theorem 4.4** *Let (H) hold. If the  $f^0 = 0$  and  $f_\infty = +\infty$ , then there exists a continuous curve  $L \subseteq \mathbb{R}_+^2 = [0, +\infty) \times [0, +\infty)$  such that the following conclusions hold:*

- (1)  $\mathbb{R}_+^2$  is divided into two separate sub-regions by  $L$ :  $\Lambda^E, \Lambda^N$ , and  $L \subseteq \Lambda^E$ .
- (2) The boundary value problem (1.1) and (1.2) has at least two positive solutions for each parameters  $(a, b) \in \Lambda^E \setminus L$ ; the boundary value problem (1.1) and (1.2) has at least one positive solution for  $(a, b) \in L$ ; the boundary value problem (1.1) and (1.2) does not have a positive solution for any  $(a, b) \in \Lambda^N$ .

*Proof* Let

$$\Lambda^E = \{(a, b) \in \mathbb{R}_+^2 : \text{the boundary value problem (1.1) and (1.2) has at least one positive solution}\}.$$

By Theorem 4.2,  $\Lambda^E \neq \emptyset$ . For any  $(\tilde{a}, \tilde{b}) \in \Lambda^E$ , we define

$$S(\tilde{a}, \tilde{b}) = \{(a, b) \in \mathbb{R}_+^2 : 0 \leq a \leq \tilde{a}, 0 \leq b \leq \tilde{b}\}.$$

By Theorem 4.1,  $S \subseteq \Lambda^E$  if and only if  $(\tilde{a}, \tilde{b}) \in \Lambda^E$ . So

$$\Lambda^E = \bigcup_{(\tilde{a}, \tilde{b}) \in \Lambda^E} S.$$

Let  $\Lambda^N = \mathbb{R}_+^2 \setminus \Lambda^E$ , by Theorem 4.2,  $\Lambda^N \neq \emptyset$ , and  $\Lambda^E \cup \Lambda^N = \mathbb{R}_+^2$ .

- (i) It follows from Theorem 4.2 that the  $\Lambda^E$  is a bounded set.
- (ii) Let  $L$  be the boundary of  $\Lambda^E$ . We can prove that the boundary value problem (1.1) and (1.2) has at least one positive solution for  $(a, b) \in L$ . That is  $L \subseteq \Lambda^E$ .

Take  $(a, b) \in L$ , then there exists a sequence  $\{(a_m, b_m)\}_{m=1}^\infty \subseteq \Lambda^E$ , such that  $a_m \rightarrow a, b_m \rightarrow b, (m \rightarrow \infty)$ . Let  $u_m(t)$  be the solution of the boundary value problem (1.1) and (1.2) for the parameters  $a = a_m$  and  $b = b_m$ . By Theorem 4.3, there exists a constant  $\eta = \eta(\Lambda^E)$  such that  $\|u_m\| \leq \eta$ . Then  $\{u_m(t)\}_{m=1}^\infty$  is uniformly bounded and we can prove  $\{u_m(t)\}_{m=1}^\infty$  is equicontinuous, similar to the proof of Lemma 2.9.

It follows that  $\{u_m(t)\}_{m=1}^\infty$  has a subsequence which is uniformly convergent on  $J$  from the Ascoli-Arzelà theorem. Without loss of generality, we can assume that  $\{u_m(t)\}_{m=1}^\infty$  itself converges uniformly to  $u$  on  $J$ , then  $u \in P$ . Since  $f$  satisfies the  $L_1$ -Carathéodory conditions and

$$u_m(t) = \int_0^1 G(t, s)f(s, u_m(s)) \, ds + (b_m - a_m)t^{\delta-1} + a_mt^{\delta-2},$$

letting  $m \rightarrow \infty$ , by the Lebesgue dominated convergence theorem, we can get

$$u(t) = \int_0^1 G(t, s)f(s, u(s)) \, ds + (b - a)t^{\delta-1} + at^{\delta-2}.$$

So the boundary value problem (1.1) and (1.2) has at least one positive solution for the parameters  $(a, b) \in L$  and  $L \subseteq \Lambda^E$ . By Lemma 4.3, we have  $u \in \hat{P}$ .

(iii) By Theorem 4.1, the boundary value problem (1.1) and (1.2) has at least one positive solution for  $(a, b) \in \Lambda^E$ , has no positive solution for any  $(a, b) \in \Lambda^N$ , and  $L$  is the boundary of  $\Lambda^E$ .

(iv) We can prove that the boundary value problem (1.1) and (1.2) has at least two positive solutions for  $(a, b) \in \Lambda^E \setminus L$ .

Let  $(a, b) = (a_0, b_0) \in \Lambda^E \setminus L$  be fixed, then there exist  $(\bar{a}, \bar{b})$  and  $(\underline{a}, \underline{b}) \in \Lambda^E \setminus L$  such that  $\underline{a} < a_0 < \bar{a}$  and  $\underline{b} < b_0 < \bar{b}$ , the boundary value problem (1.1) and (1.2) has a solution  $\bar{u}$  for  $(a, b) = (\bar{a}, \bar{b})$ . In view of Theorem 4.1, the boundary value problem (1.1) and (1.2) has at least one positive solution  $u_1$ , and  $u_1 \leq \bar{u}$  for  $(a, b) = (a_0, b_0)$ . Similarly, the boundary value problem (1.1) and (1.2) has a solution  $\underline{u}$  for  $(a, b) = (\underline{a}, \underline{b})$  and  $\underline{u} \leq u_1$ .

Hence,  $\underline{u} < u_1 < \bar{u}$ .

Let  $\alpha = \underline{u}$ ,  $\beta = \bar{u}$ , Obviously,  $\alpha, \beta$  are the lower solution and the upper solution of the boundary value problem (1.1) and (1.2) for  $(a, b) = (a_0, b_0)$ , respectively.

Let  $(\hat{a}, \hat{b}) \in \Lambda^N$  such that  $a_0 < \hat{a}$  and  $b_0 < \hat{b}$ .

Define  $K : [a_0, \hat{a}] \times [b_0, \hat{b}] \times \hat{P} \rightarrow \hat{P}$ ,

$$K(k_1, k_2, u) = \int_0^1 G(t, s)f(s, u(s)) \, ds + (k_2 - k_1)t^{\delta-1} + k_1t^{\delta-2}.$$

Let

$$F(t, u) = \begin{cases} f(t, \beta(t)), & u > \beta(t), \\ f(t, u), & \alpha(t) \leq u \leq \beta(t), \\ f(t, \alpha(t)), & u < \alpha(t), \end{cases}$$

define  $\hat{K} : [a_0, \hat{a}] \times [b_0, \hat{b}] \times \hat{P} \rightarrow \hat{P}$ ,

$$\hat{K}(k_1, k_2, u) = \int_0^1 G(t, s)F(s, u(s)) \, ds + (k_2 - k_1)t^{\delta-1} + k_1t^{\delta-2}.$$

Similar to the proof of Lemma 2.9, for each  $(k_1, k_2) \in [a_0, \hat{a}] \times [b_0, \hat{b}]$ , we can show  $K$  and  $\hat{K}$  are completely continuous operators.

By Lemma 2.8,  $u$  is a positive solution of the boundary value problem (1.1) and (1.2) if and only if  $u = K(a, b, u)$ .

Define a linear segmentation

$$S_0 = \{(k_1, k_2) \in \mathbb{R}_+^2 : k_1 = (1 - \tau)a_0 + \tau\hat{a}, k_2 = (1 - \tau)b_0 + \tau\hat{b}, \tau \in [0, 1]\}.$$

Let  $\Lambda \subseteq \mathbb{R}_+^2$  be a bounded set such that  $S_0 \cap \Lambda^E \subseteq \Lambda$ ,  $(\underline{a}, \underline{b}) \in \Lambda$ ,  $(\bar{a}, \bar{b}) \in \Lambda$ , and  $\eta = \eta(\Lambda)$  given by Theorem 4.3.

Let

$$\Omega = \{u \in \hat{P} : \|u\| < \eta, \alpha(t) < u(t) < \beta(t), t \in J\},$$

then  $\Omega$  is a nonempty bounded open set of  $\hat{P}$  and  $u_1 \in \Omega$ .

Since  $F$  is bounded, there exists a constant  $R > \eta > 0$  such that for each  $(k_1, k_2, u) \in [a_0, \hat{a}] \times [b_0, \hat{b}] \times \hat{P}$ , we have  $|t^{2-\delta} \hat{K}(k_1, k_2, u)| < R$ .

Let

$$B(\theta, R) = \{u \in E : \|u\| < R\}.$$

It is easy to show  $\Omega \subset \hat{P} \cap B(\theta, R)$  and  $u \neq \mu \hat{K}u$  when  $u \in \hat{P} \cap \partial B(\theta, R)$  and  $\mu \in [0, 1]$ .

By Lemma 4.1, for each  $a \leq k_1 \leq \hat{a}$ ,  $b \leq k_2 \leq \hat{b}$ , we have

$$i(\hat{K}(k_1, k_2, u), \hat{P} \cap B(\theta, R), \hat{P}) = 1.$$

Due to  $\hat{K}$  having no fixed point on  $\hat{P} \cap (\overline{B(\theta, R)} \setminus \Omega)$ , for each  $a_0 \leq k_1 \leq \hat{a}$ ,  $b_0 \leq k_2 \leq \hat{b}$ ,

$$i(\hat{K}(k_1, k_2, u), \hat{P} \cap (B(\theta, R) \setminus \overline{\Omega}), \hat{P}) = 0.$$

We notice  $\hat{K}|_{\Omega} = K$ , by the excision property of the fixed point index, for each  $a_0 \leq k_1 \leq \hat{a}$ ,  $b_0 \leq k_2 \leq \hat{b}$ , we have

$$\begin{aligned} & i(K(k_1, k_2, u), \hat{P} \cap \Omega, \hat{P}) \\ &= i(\hat{K}(k_1, k_2, u), \hat{P} \cap \Omega, \hat{P}) \\ &= i(\hat{K}(k_1, k_2, u), \hat{P} \cap B(\theta, R), \hat{P}) - i(\hat{K}(k_1, k_2, u), \hat{P} \cap (B(\theta, R) \setminus \overline{\Omega}), \hat{P}) \\ &= 1. \end{aligned} \tag{4.10}$$

Because  $(\hat{a}, \hat{b}) \in \Lambda^N$ , for each  $u \in \hat{P}$ , we have  $K(\hat{a}, \hat{b}, u) \neq u$  and

$$i(\hat{K}(\hat{a}, \hat{b}, u), \hat{P} \cap B(\theta, R), \hat{P}) = 0. \tag{4.11}$$

Define  $H : [0, 1] \times (\hat{P} \cap \overline{B(\theta, R)}) \rightarrow C_{2-\delta}^0[0, 1]$ ,

$$H(\tau, u) = K((1 - \tau)a_0 + \tau \hat{a}, (1 - \tau)b_0 + \tau \hat{b}, u),$$

then  $H$  is completely continuous.

We can prove  $H(\tau, u) \neq u$  for  $(\tau, u) \in [0, 1] \times (\hat{P} \cap \partial B(\theta, R))$ .

Otherwise, there exists  $(\tau_0, u_0) \in [0, 1] \times (\hat{P} \cap \partial B(\theta, R))$  such that  $H(\tau_0, u_0) = u_0$ , that is

$$K((1 - \tau_0)a_0 + \tau_0 \hat{a}, (1 - \tau_0)b_0 + \tau_0 \hat{b}, u_0) = u_0, \quad u_0 \in \hat{P}, \|u_0\| = R.$$

So for  $\tilde{k}_1 = (1 - \tau_0)a_0 + \tau_0 \hat{a}$ ,  $\tilde{k}_2 = (1 - \tau_0)b_0 + \tau_0 \hat{b}$ , and  $(\tilde{k}_1, \tilde{k}_2) \in \Lambda$ ,  $u_0$  is a solution of the boundary value problem (1.1) and (1.2), by Theorem 4.3,  $\|u_0\| < \eta$ , which is a contradiction.

According to homotopy invariance of the fixed point index and (4.10), it follows that

$$\begin{aligned} i(K(a_0, b_0, u), \hat{P} \cap B(\theta, R), \hat{P}) &= i(H(0, u), \hat{P} \cap B(\theta, R), \hat{P}) \\ &= i(H(1, u), \hat{P} \cap B(\theta, R), \hat{P}) \\ &= i(\hat{K}(\hat{a}, \hat{b}, u), \hat{P} \cap B(\theta, R), \hat{P}) \\ &= 0. \end{aligned} \tag{4.12}$$

By (4.11) and (4.12),

$$i(K(a_0, b_0, u), \hat{P} \cap (B(\theta, R) \setminus \overline{\Omega}), \hat{P}) = -1.$$

Hence, the boundary value problem (1.1) and (1.2) has a solution  $u_2 \in \hat{P} \cap (B(\theta, R) \setminus \overline{\Omega})$  for  $(a, b) = (a_0, b_0)$ .

Since  $u_1 \in \Omega$ , then  $u_1 \neq u_2$ , that is, the boundary value problem (1.1) and (1.2) has at least two positive solutions for  $(a, b) = (a_0, b_0)$ . □

### 5 Illustration

To illustrate our main results, we present the following examples.

**Example 5.1** We consider the boundary value problem which is composed of the equation

$$D_{0^+}^{\frac{3}{2}} u(t) + e^{t^{\frac{q}{2}} u^q} - 1 = 0, \quad \text{for a.e. } t \in (0, 1), \tag{5.1}$$

and nonhomogeneous boundary conditions

$$\lim_{t \rightarrow 0^+} t^{\frac{1}{2}} u(t) = a, \quad u(1) = b, \tag{5.2}$$

where  $q > 1$ .

Let  $f(t, u) = e^{t^{\frac{q}{2}} u^q} - 1$ . Clearly,  $\delta = \frac{3}{2}$ , it is easy to see that (H) holds,

$$f^0 = \limsup_{u \rightarrow 0^+} \sup_{t \in J} \frac{f(t, t^{-\frac{1}{2}} u)}{u} = \limsup_{u \rightarrow 0^+} \frac{e^{u^q} - 1}{u} = 0,$$

and

$$f_\infty = \liminf_{u \rightarrow +\infty} \inf_{t \in [\frac{1}{4}, \frac{3}{4}]} \frac{f(t, u)}{u} = \liminf_{u \rightarrow +\infty} \frac{e^{(\frac{u}{2})^q} - 1}{u} = +\infty.$$

By Theorem 4.4, we see that there exists a bounded and continuous curve  $L$  in  $\mathbb{R}_+^2$  separating  $\mathbb{R}_+^2$  into two disjoint subsets  $\Lambda^E$  and  $\Lambda^N$  with  $L \subseteq \Lambda^E$  such that boundary value problem (5.1) and (5.2) has at least two positive solutions for each for each parameters  $(a, b) \in \Lambda^E \setminus L$ , one positive solution for  $(a, b) \in L$  and no positive solution for any  $(a, b) \in \Lambda^N$ .

**Example 5.2** We consider the boundary value problem which is composed of the equation

$$D_{0^+}^{\frac{7}{4}} u(t) + (t \sin t + t)u^2 + t^{\frac{3}{4}} u^{\frac{3}{2}} \arctan tu = 0, \quad \text{for a.e. } t \in (0, 1), \tag{5.3}$$

and nonhomogeneous boundary conditions

$$\lim_{t \rightarrow 0^+} t^{\frac{1}{4}} u(t) = a, \quad u(1) = b. \tag{5.4}$$

Let  $f(t, u) = t(\sin t + 1)u^2 + t^{\frac{3}{4}}u^{\frac{3}{2}} \arctan tu$ . Clearly,  $\delta = \frac{7}{4}$ , it is easy to see that (H) holds,

$$f^0 = \limsup_{u \rightarrow 0^+} \sup_{t \in J} \frac{f(t, t^{-\frac{1}{2}}u)}{u} = \limsup_{u \rightarrow 0^+} ((\sin 1 + 1)u + u^{\frac{1}{2}} \arctan u) = 0,$$

and

$$f_\infty = \liminf_{u \rightarrow +\infty} \inf_{t \in [\frac{1}{4}, \frac{3}{4}]} \frac{f(t, u)}{u} = \liminf_{u \rightarrow +\infty} \left( \left( \sin \frac{1}{4} + 1 \right) \frac{1}{4}u + \left( \frac{1}{4} \right)^{\frac{3}{4}} u^{\frac{1}{2}} \arctan \frac{1}{4}u \right) = +\infty.$$

By Theorem 4.4, we see that there exists a bounded and continuous curve  $L$  in  $\mathbb{R}_+^2$  separating  $\mathbb{R}_+^2$  into two disjoint subsets  $\Lambda^E$  and  $\Lambda^N$  with  $L \subseteq \Lambda^E$  such that the boundary value problem (5.3) and (5.4) has at least two positive solutions for each for each parameters  $(a, b) \in \Lambda^E \setminus L$ , one positive solution for  $(a, b) \in L$  and no positive solution for any  $(a, b) \in \Lambda^N$ .

**Example 5.3** We consider the boundary value problem which is composed of the equation

$$D_{0+}^{\frac{5}{3}}u(t) + (1 + a(t))t^{\frac{p}{2}}u^p = 0, \quad \text{for a.e. } t \in (0, 1), \tag{5.5}$$

and nonhomogeneous boundary conditions

$$\lim_{t \rightarrow 0^+} t^{\frac{1}{3}}u(t) = a, \quad u(1) = b, \tag{5.6}$$

where  $p > 1$ ,  $a(t)$  is a continuous function for  $t \in [0, 1]$  and  $a(t) > 0$ ,  $t \in [\frac{1}{4}, \frac{3}{4}]$ .

Let  $f(t, u) = (1 + a(t))t^{\frac{p}{2}}u^p$ . Clearly,  $\delta = \frac{5}{3}$ , it is easy to see that (H) holds,

$$f^0 = \limsup_{u \rightarrow 0^+} \sup_{t \in J} \frac{f(t, t^{-\frac{1}{2}}u)}{u} = \sup_{t \in J} (1 + a(t)) \limsup_{u \rightarrow 0^+} u^{p-1} = 0,$$

and

$$f_\infty = \liminf_{u \rightarrow +\infty} \inf_{t \in [\frac{1}{4}, \frac{3}{4}]} \frac{f(t, u)}{u} = \inf_{t \in [\frac{1}{4}, \frac{3}{4}]} (1 + a(t))t^{\frac{p}{2}} \liminf_{u \rightarrow +\infty} u^{p-1} = +\infty.$$

By Theorem 4.4, we see that there exists a bounded and continuous curve  $L$  in  $\mathbb{R}_+^2$  separating  $\mathbb{R}_+^2$  into two disjoint subsets  $\Lambda^E$  and  $\Lambda^N$  with  $L \subseteq \Lambda^E$  such that the boundary value problem (5.5) and (5.6) has at least two positive solutions for each for each parameters  $(a, b) \in \Lambda^E \setminus L$ , one positive solution for  $(a, b) \in L$ , and no positive solution for any  $(a, b) \in \Lambda^N$ .

### 6 Conclusions

The existence and nonexistence of positive solutions for fractional differential equations with nonhomogeneous boundary conditions is an important topic in the area of fractional calculus. When solving the actual problems with the boundary value problems of differential equations, the disturbance parameter under boundary conditions is a very important

factor. Moreover, it is always inevitable that the impact of such a disturbance parameter to the existence of solutions always exists.

In this paper, by using the upper and lower solution method and fixed point theorem, we obtain the sufficient conditions for the boundary value problem to have at least one positive solution, two positive solutions, and no positive solution, respectively. Moreover, under certain conditions, we prove that there exists a bounded and continuous curve  $L$  dividing  $[0, +\infty) \times [0, +\infty)$  into two separate subsets  $\Lambda^E$  and  $\Lambda^N$  with  $L \subseteq \Lambda^E$  such that the boundary value problem has at least two positive solutions for each  $(a, b) \in \Lambda^E \setminus L$ , one positive solution for each  $(a, b) \in L$ , and no positive solution for any  $(a, b) \in \Lambda^N$ .

#### Competing interests

The authors declare that they have no competing interests.

#### Authors' contributions

The authors declare that the work was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

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