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Some remarks on an exact and dynamically consistent scheme for the Burgers-Huxley equation in higher dimensions

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Abstract

Departing from a finite-difference scheme to approximate the solutions of the Burgers-Huxley equation, the present manuscript extends that technique to higher dimensions. We show that our methodology possesses the same numerical properties of the one-dimensional version (exactness, positivity, boundedness, monotonicity, *etc.*). Moreover, helped by a recent theorem on the existence and uniqueness of positive and bounded solutions of the Burgers-Huxley equation, we establish that the present method is a convergent scheme.

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1 Background

Let $\mathbb{R}^+$ represent the set of positive numbers, let $\overline{\mathbb{R}^+} = \mathbb{R}^+ \cup \{0\}$ and suppose that α and γ are real numbers such that $0 < \gamma < 1$. Throughout, we assume that $u = u(x, y, t)$ is a real function on $(x, y, t) \in \mathbb{R} \times \mathbb{R} \times \overline{\mathbb{R}^+}$ which satisfies

$$\frac{\partial u}{\partial t} + \alpha u\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\right) - \Delta u - f(u) = 0. \tag{1}$$

Here, Δ denotes the two-dimensional Laplacian operator, and

$$f(u) = u(1 - u)(u - \gamma). \tag{2}$$

This model is the Burgers-Huxley equation that generalizes the classical Burgers equation [1] and the Hodgkin-Huxley model [2]. Many applications of this equation and its variations may be found in the sciences and in engineering [3–5].

Several discretizations of (1) and related equations have been proposed in the literature using different approaches [6–10]. Recently, the first author derived an exact finite-difference scheme for the one-dimensional Burgers-Huxley equation, which preserved positivity, boundedness and monotonicity [11]. However, questions like the following were left unanswered (the present note intends to provide affirmative responses to all of them).

Problem 1 Are there analytical results that guarantee the existence and uniqueness of positive and bounded solutions of the Burgers-Huxley equation?

Problem 2 Can the scheme of [11] be dimensionally generalized in such way that the resulting method preserves the dynamical properties derived therein?

Problem 3 Is the scheme convergent? In the affirmative case, under which conditions? What is the order of convergence?

The following results are the most important properties on the existence and uniqueness of positive and bounded solutions of the Burgers-Huxley equation. They provide affirmative responses to Problem 1.

Theorem 1 (See [12], p.206) *Let $\Omega \subseteq \mathbb{R}^2$ be open, bounded and connected, let $T \in \mathbb{R}^+$ and let A be an elliptic operator of order 2. Consider the parabolic initial-boundary-value problem*

$$\begin{cases} \frac{\partial u}{\partial t} = Au + F(\mathbf{x}, t, u, \nabla u), \quad \textit{subject to} \\ u(\mathbf{x}, t) = \widehat{u}(\mathbf{x}, t), \quad \textit{for } (\mathbf{x}, t) \in \overline{\Omega} \times \{0\} \cup \partial\Omega \times [0, T], \end{cases} \tag{3}$$

and suppose that the following hypotheses are satisfied:

1. $\widehat{u}$ *is smooth*,
2. $F(\mathbf{x}, t, u, \nabla u)$ *is locally Hölder continuous with respect to* $(\mathbf{x}, t)$,
3. $F(\mathbf{x}, t, u, \nabla u)$ *is Lipschitz continuous in* u, *uniformly for bounded subsets of* $\overline{\Omega} \times [0, T] \times \mathbb{R} \times \mathbb{R}^2$, *and*
4. $\widehat{u}_t = A\widehat{u} + F(\mathbf{x}, 0, \widehat{u}, \nabla\widehat{u})$ *holds for* $(\mathbf{x}, t) \in \partial\Omega \times \{0\}$.

Then there exists a unique solution of (3) *in* $\Omega \times [0, T_0]$, *for some* $0 < T_0 \leq T$.

As a consequence, if the conditions of Theorem 1 are satisfied for the initial-boundary conditions of the partial differential equation (1) then there exist $T_0 > 0$ and a unique solution of (1) in $\overline{\Omega} \times [0, T_0]$. The proof readily follows from the theorem with $F(\mathbf{x}, t, u, \nabla u) = -\alpha u \cdot \nabla u \cdot (1, 1)^t + f(u)$ and f given by (2).

The next result establishes the positivity and the boundedness of classical solutions of (1). It will be a crucial tool in the derivation of the convergent character of our finite-difference method.

Theorem 2 (Ervin *et al.* [13]) *If $\mu \in \{1, \gamma\}$ and if $u(\mathbf{x}, t)$ is a classical solution of* (3) *satisfying $0 < u(\mathbf{x}, t) < \mu$ for each $(\mathbf{x}, t) \in \overline{\Omega} \times \{0\} \cup \partial\Omega \times [0, T]$, then $0 < u(\mathbf{x}, t) < \mu$ for each $(\mathbf{x}, t) \in \overline{\Omega} \times \overline{\mathbb{R}^+}$.*

This work is divided as follows. In Section 2, we introduce the finite-difference scheme to approximate solutions of (1) and derive some technical results. In turn, Section 3 shows that our numerical method has the same dynamical properties as its one-dimensional counterpart, thus providing and affirmative answer to Problem 2. Section 4 is devoted to a proof of the convergence property of our technique. Finally, we close this work with a brief section of concluding remarks.

2 Preliminaries

Mimicking the approach followed in [11], we suppose that a, b, c, d, and T are real numbers which satisfy $a < b$, $c < d$, and $T > 0$. Let us fix a rectangular spatial domain $\Omega = [a,b] \times [c,d]$ of $\subseteq \mathbb{R}^2$, and assume that K, M, and N are positive integers. In this work, we consider regular partitions

$$\begin{aligned} &a = x_0 < x_1 < \cdots < x_m < \cdots < x_M = b, \quad m \in \{0,1,\ldots,M\}, \\ &c = y_0 < y_1 < \cdots < y_n < \cdots < y_N = d, \quad n \in \{0,1,\ldots,N\}, \end{aligned}$$

of the intervals $[a,b]$ and $[c,d]$, respectively, each of them with norm given by Δx and Δy. Also, let us consider a (not necessarily uniform) partition

$$0 = t_0 < t_1 < \cdots < t_k < \cdots < t_K = T, \quad k \in \{0,1,\ldots,K\},$$

of the temporal interval $[0,T]$. We convey that $u_{m,n}^k$ will represent an approximation to the exact value of the solution of (1) at the point (x_m, y_n) and the time t_k.

Let $\Delta t_k = t_{k+1} - t_k$ for each $k \in \{0,1,\ldots,K-1\}$. For each $m \in \{1,\ldots,M-1\}$, $n \in \{1,\ldots,N-1\}$, and $z = x, y$ we define the constants

$$\begin{aligned} r_z^k &= \frac{\Delta t_k}{2\Delta z}, \\ R_z^k &= \frac{\Delta t_k}{(\Delta z)^2}, \\ C_{m,n}^k &= R_x^k\bigl(u_{m+1,n}^k + u_{m-1,n}^k\bigr) + \bigl(1 - 2R_x^k - 2R_y^k\bigr)u_{m,n}^k + R_y^k\bigl(u_{m,n+1}^k + u_{m,n-1}^k\bigr), \\ D_{m,n}^k &= 1 + \alpha r_x^k\bigl(u_{m+1,n}^k - u_{m-1,n}^k\bigr) + \alpha r_y^k\bigl(u_{m,n+1}^k - u_{m,n-1}^k\bigr). \end{aligned}$$

For the sake of convenience, we introduce the following discrete operators (which are already standard in the literature of the area):

$$\begin{aligned} \delta_t u_{m,n}^k &= \frac{u_{m,n}^{k+1} - u_{m,n}^k}{\Delta t}, \\ \delta_x^{(1)} u_{m,n}^k &= \frac{(u_{m+1,n}^k - u_{m-1,n}^k)}{2\Delta x}, \\ \delta_y^{(1)} u_{m,n}^k &= \frac{(u_{m,n+1}^k - u_{m,n-1}^k)}{2\Delta y}, \\ \delta_x^{(2)} u_{m,n}^k &= \frac{u_{m+1,n}^k - 2u_{m,n}^k + u_{m-1,n}^k}{(\Delta x)^2}, \\ \delta_y^{(2)} u_{m,n}^k &= \frac{u_{m,n+1}^k - 2u_{m,n}^k + u_{m,n-1}^k}{(\Delta y)^2}. \end{aligned}$$

With this nomenclature at hand, the following formula provides a discretization of (1) at the point (x_m, y_n, t_k), for each $m \in \{1, \ldots, M-1\}$, $n \in \{1, \ldots, N-1\}$, and $k \in \{0, 1, \ldots, K-1\}$:

$$\delta_t u_{m,n}^k + \alpha u_{m,n}^{k+1}\big(\delta_x^{(1)} + \delta_y^{(1)}\big)u_{m,n}^k - \big(\delta_x^{(2)} + \delta_y^{(2)}\big)u_{m,n}^k - f\big(u_{m,n}^{k+1}\big) = 0. \tag{4}$$

For convenience, we use $Lu_{m,n}^k$ to denote the left-hand side of this identity. One may readily check that (4) can be expressed as $F_{m,n}^k(u_{m,n}^{k+1}) = 0$, where

$$F_{m,n}^k(u) = -u(1-u)(u-\gamma)\Delta t_k + D_{m,n}^k u - C_{m,n}^k.$$

A simplification of the notation may prove convenient here [11]. More precisely, when no danger of confusion is present, we will denote the function $F_{m,n}^k$ by F. It is obvious that F is a cubic polynomial on u which may be rewritten as

$$F(u) = \Delta t_k u^3 - (1+\gamma)\Delta t_k u^2 + \big[\gamma \Delta t_k + D_{m,n}^k\big]u - C_{m,n}^k.$$

As expected, the calculation of the roots of F will be carried out exactly using Cardano's method which, in spite of the fact that it is an elementary technique, has been used as an auxiliary tool in various mathematical problems [11].

For the remainder of this work, we let $\mu \in \{1, \gamma\}$ and assume that the following inequality holds at time t_k:

$$2\big(R_x^k + R_y^k\big) < 1.$$

Lemma 1 *Suppose that* $\mathbf{u}^k > 0$ *for some* $k \in \{0, 1, \ldots, K\}$. *Then* $C_{m,n}^k > 0$ *for each* $m \in \{1, \ldots, M-1\}$ *and* $n \in \{1, \ldots, N-1\}$.

Proof Clearly $1 - 2R_x^k - 2R_y^k > 0$, so all the terms in the right-hand side of $C_{m,n}^k = R_x^k(u_{m+1,n}^k + u_{m-1,n}^k) + (1 - 2R_x^k - 2R_y^k)u_{m,n}^k + R_y^k(u_{m,n+1}^k + u_{m,n-1}^k)$ are positive. □

Lemma 2 *If* $|\alpha|\mu r_z^k < R_z^k$ *and* $\mathbf{u}^k < \mu$ *for* $z = x, y$ *and some* $k \in \{0, 1, \ldots, K\}$ *then* $C_{m,n}^k - \alpha\mu r_x^k(u_{m+1,n}^k - u_{m-1,n}^k) - \alpha\mu r_y^k(u_{m,n+1}^k - u_{m,n-1}^k) < \mu$.

Proof Note that $1 - 2R_x^k - 2R_y^k, R_z^k + \alpha\mu r_z^k, R_z^k - \alpha\mu r_z^k \in \mathbb{R}^+$ for each $z = x, y$. On the other hand, the fact that $\mathbf{u}^k < \mu$ holds yields

$$\begin{aligned}
\mu &= \big(R_x^k - \alpha\mu r_x^k\big)\mu + \big(R_y^k - \alpha\mu r_y^k\big)\mu + \big(R_y^k + \alpha\mu r_y^k\big)\mu \\
&\quad + \big(R_x^k + \alpha\mu r_x^k\big)\mu + \big(1 - 2R_x^k - 2R_y^k\big)\mu \\
&> \big(R_x^k - \alpha\mu r_x^k\big)u_{m+1,n}^k + \big(R_y^k - \alpha\mu r_y^k\big)u_{m,n+1}^k + \big(R_y^k + \alpha\mu r_y^k\big)u_{m,n-1}^k \\
&\quad + \big(R_x^k + \alpha\mu r_x^k\big)u_{m-1,n}^k + \big(1 - 2R_x^k - 2R_y^k\big)u_{m,n}^k \\
&= C_{m,n}^k - \alpha\mu r_x^k\big(u_{m+1,n}^k - u_{m-1,n}^k\big) - \alpha\mu r_y^k\big(u_{m,n+1}^k - u_{m,n-1}^k\big),
\end{aligned}$$

as desired. □

From these lemmas and the intermediate value theorem, F has a root in $(0, \mu)$ when $|\alpha|\mu r_z^k < R_z^k$ and $0 < \mathbf{u}^k$ for $z = x, y$ and $k \in \{0, 1, \ldots, K-1\}$.

Lemma 3 *If* $\Delta t_k < 1$, $|\alpha|\mu r_z^k < \frac{1}{3}$ *for* $z = x, y$, *and* $u_{m-1,n}^k$, $u_{m+1,n}^k$, $u_{m,n-1}^k$, $u_{m,n+1}^k$ *are in* $(0, \mu)$ *then* F *is strictly increasing in* $(0, \mu)$.

Proof From the hypotheses,

$$\pm 3\alpha r_x^k \left(u_{m+1,n}^k - u_{m-1,n}^k\right) < 3|\alpha|\mu r_x^k < 1,$$
$$\pm 3\alpha r_y^k \left(u_{m,n+1}^k - u_{m,n-1}^k\right) < 3|\alpha|\mu r_y^k < 1.$$

Note that $F'(u) = 3\Delta t_k u^2 - 2(1+\gamma)\Delta t_k u + \gamma\Delta t_k + D_{m,n}^k$ is a parabola for which the minimum value v of F' satisfies

$$\begin{aligned}
v &= 1 + \gamma\Delta t_k + \alpha r_x^k \left(u_{m+1,n}^k - u_{m-1,n}^k\right) + \alpha r_y^k \left(u_{m,n+1}^k - u_{m,n-1}^k\right) - \frac{(1+\gamma)^2 \Delta t_k}{3} \\
&= \frac{3 + (\gamma - \gamma^2)\Delta t_k + 3\alpha[r_x^k(u_{m+1,n}^k - u_{m-1,n}^k) + r_y^k(u_{m,n+1}^k - u_{m,n-1}^k)] - \Delta t_k}{3} \\
&> \frac{3 + 3\alpha[r_x^k(u_{m+1,n}^k - u_{m-1,n}^k) + r_y^k(u_{m,n+1}^k - u_{m,n-1}^k)] - \Delta t_k}{3} \\
&= \frac{(1 - \Delta t_k) + [1 + 3\alpha r_x^k(u_{m+1,n}^k - u_{m-1,n}^k)] + [1 + 3\alpha r_y^k(u_{m,n+1}^k - u_{m,n-1}^k)]}{3}.
\end{aligned}$$

So, the ordinate of the vertex of F' is positive. This means that $F' > 0$ in all $\mathbb{R}$. □

Let $\Delta t_k < 1$ and $|\alpha|\mu r_z^k < \frac{2R_z^k}{3}$ for $z = x, y$ and $k \in \{0, 1, \ldots, K\}$. Note that F has a unique root in $(0, \mu)$ if $\mathbf{u}^k \in (0, \mu)$. Indeed, F has roots in $(0, \mu)$ since $|\alpha|\mu r_z^k < \frac{2R_z^k}{3} < R_z^k$ for $z = x, y$. On the other hand $2R_z^k < 1$ implies that $|\alpha|\mu r_z^k < \frac{2R_z^k}{3} < \frac{1}{3}$. The assumptions of the previous lemma are satisfied, and we conclude that F has exactly one zero in $(0, \mu)$, as desired.

3 Dynamical consistency

The dynamical properties of the method (4) are noticed in this section. Following the nomenclature in [11], for each $k \in \{0, 1, \ldots, K\}$ we let

$$\mathbf{u}^k = \left(u_{0,0}^k, u_{0,1}^k, \ldots, u_{0,N}^k, u_{1,0}^k, u_{1,1}^k, \ldots, u_{1,N}^k, \ldots, u_{M,0}^k, u_{M,1}^k, \ldots, u_{M,N}^k\right).$$

Theorem 3 *Let* $0 < \mathbf{u}^0 < \mu$ *and suppose that for each* $k \in \{0, 1, \ldots, K\}$:

(1) $\Delta t_k < 1$,
(2) $2(R_x^k + R_y^k) < 1$,
(3) $|\alpha|\mu r_z^k < 2R_z^k/3$ *for* $z = x, y$, *and*
(4) $u_{0,n}^k$, $u_{M,n}^k$, $u_{m,0}^k$, $u_{m,N}^k \in (0, \mu)$ *for* $m \in \{1, \ldots, M-1\}$, $n \in \{1, \ldots, N-1\}$.

Then (4) *has a unique solution* $(\mathbf{u}^k)_{k=0}^K$ *with each* $0 < \mathbf{u}^k < \mu$.

Proof Suppose that $0 < \mathbf{u}^k < \mu$ and $m \in \{1, \ldots, M-1\}$, $n \in \{1, \ldots, N-1\}$. Then $u_{m,n}^{k+1} \in (0, \mu)$ by discussions in the previous section, whence the result follows. □

The next result establishes that the method (1) preserves the monotonicity. We employ here ideas of [11] again.

Theorem 4 *Let* $0 < \mathbf{u}^0 < \mathbf{v}^0 < \mu$. *Suppose that* (1), (2) *and* (3) *of Theorem* 3 *as well as the condition*

- $0 < u^k_{0,n} < v^k_{0,n} < \mu$, $0 < u^k_{M,n} < v^k_{M,n} < \mu$, $0 < u^k_{m,0} < v^k_{m,0} < \mu$, *and* $0 < u^k_{m,N} < v^k_{m,N} < \mu$ *for each* $m \in \{1,\dots,M\}$ *and* $n \in \{1,\dots,N\}$,

hold for each $k \in \{0,1,\dots,K\}$. *Then* $0 < \mathbf{u}^k < \mathbf{v}^k < \mu$ *for each such* k.

Proof Suppose that $\mathbf{u}^k < \mathbf{v}^k$, let $m \in \{1,\dots,M-1\}$ and $n \in \{1,\dots,N-1\}$, and observe that $v^{k+1}_{m,n}$ is the unique solution of $G(v) = -v(1-v)(v-\gamma)\Delta t_k + E^k_{m,n} - J^k_{m,n}$ in $(0,\mu)$, where

$$\begin{aligned}
J^k_{m,n} &= R^k_x\big(v^k_{m+1,n} + v^k_{m-1,n}\big) + \big(1 - 2R^k_x - 2R^k_y\big)v^k_{m,n} + R^k_y\big(v^k_{m,n+1} + v^k_{m,n-1}\big),\\
E^k_{m,n} &= 1 + \alpha r^k_x\big(v^k_{m+1,n} - v^k_{m-1,n}\big) + \alpha r^k_y\big(v^k_{m,n+1} - v^k_{m,n-1}\big).
\end{aligned}$$

If $H = F - G$ then

$$\begin{aligned}
H(w) = {}& \big[1 - 2R^k_x - 2R^k_y\big]\big(v^k_{m,n} - u^k_{m,n}\big) + \big[R^k_x - \alpha r^k_x\big]\big(v^k_{m+1,n} - u^k_{m+1,n}\big)\\
&+ \big[R^k_x + \alpha r^k_x\big]\big(v^k_{m-1,n} - u^k_{m-1,n}\big) + \big[R^k_y - \alpha r^k_y\big]\big(v^k_{m,n+1} - u^k_{m,n+1}\big)\\
&+ \big[R^k_y + \alpha r^k_y\big]\big(v^k_{m,n-1} - u^k_{m,n-1}\big),
\end{aligned}$$

which is positive for each $w \in [0,\mu]$, so $F(w) > G(w)$. It follows that $u^{k+1}_{m,n} < v^{k+1}_{m,n}$, and the conclusion is reached by recursion. □

Corollary 1 *Let* $0 < \mathbf{u}^0 < \mathbf{u}^1 < \mu$, *and suppose that* (1), (2), *and* (3) *of Theorem* 3 *together with the following condition hold for each* $k \in \{0,1,\dots,K\}$:

- *If* $m \in \{1,\dots,M-1\}$ *and* $n \in \{1,\dots,N-1\}$ *then* $0 < u^k_{0,n} < u^{k+1}_{0,n} < \mu$, $0 < u^k_{M,n} < u^{k+1}_{M,n} < \mu$, $0 < u^k_{m,0} < u^{k+1}_{m,0} < \mu$, *and* $0 < u^k_{m,N} < u^{k+1}_{m,N} < \mu$.

Then $0 < \mathbf{u}^k < \mathbf{u}^{k+1} < \mu$ *for each* $k \in \mathbb{Z}^+ \cup \{0\}$.

This corollary still holds if we reverse the corresponding inequalities between successive approximations.

4 Convergence

We follow now the approach of [14] to show that the method (4) is a convergent technique. First, we quote or prove some technical lemmas.

Lemma 4 (Discrete Gronwall's inequality [15]) *Let* $K > 1$, *and let* $A, B, C_k \in \overline{\mathbb{R}^+}$ *for each* $k \in \{0,1,\dots,K\}$. *If* $(A+B)\Delta t \le \frac{K-1}{2K}$, *and if* $\{w^k\}_{k=0}^K$ *satisfies* $w^k - w^{k-1} \le A\Delta t w^k + B\Delta t w^{k-1} + C_k\Delta t$ *for each* $k = 1,\dots,K$, *then*

$$\max_{1\le k\le K} \big|w^k\big| \le \left(w^0 + \Delta t\sum_{l=1}^{K} C_l\right)e^{2(A+B)T}.$$

For simplicity, we consider now a regular partition of $[0,T]$ consisting of K subintervals. This implies that $\Delta t_k = \Delta t$ for each $k \in \{0,1,\dots,K-1\}$ where $\Delta t = T/K$, and that $R^k_z = R_z$ and $r^k_z = r_z$ for $z = x, y$. We use u to denote an exact solution of the continuous model while

w will represent a solution of the numerical method. Note that u satisfies (4) at the point (x_m, y_n, t_k), with truncation error

$$R^k_{m,n} = \mathcal{L}u(x_m, y_n, t_k) - Lu^k_{m,n} = -Lu^k_{m,n}. \tag{5}$$

Here, $\mathcal{L}u$ represents the left-hand side of equation (1).

Lemma 5 *If $u \in C^{4,4,2}_{x,y,t}(\Omega \times [0,T])$ is a solution of* (1) *bounded in* $(0,1)$ *then there exists $C \in \mathbb{R}^+$ independent of Δt, Δx, and Δy with $|R^k_{m,n}| \leq C(\Delta t + (\Delta x)^2 + (\Delta y)^2)$ for each $m \in \{1,\ldots,M-1\}$ and $n \in \{1,\ldots,N-1\}$.*

Proof Fix a common bound $C_1 \in \mathbb{R}^+$ for the functions $\partial_{tt}u$, $\partial_x u$, $\partial_{xxx}u$, $\partial_{xxxx}u$, $\partial_y u$, $\partial_{yyy}u$, and $\partial_{yyyy}u$ in the interior of Ω, and such that the following holds:

$$\begin{cases} |u^{k+1}_{m,n} - u^k_{m,n}| \leq C_1\Delta t, \\ |\delta_t u^k_{m,n} - \partial_t u^k_{m,n}| \leq \frac{1}{2}C_1\Delta t, \\ |\delta^{(1)}_x u^k_{m,n} - \partial_x u^k_{m,n}| \leq \frac{1}{6}C_1(\Delta x)^2, \\ |\delta^{(1)}_y u^k_{m,n} - \partial_y u^k_{m,n}| \leq \frac{1}{6}C_1(\Delta y)^2, \\ |\delta^{(2)}_x u^k_{m,n} - \partial_{xx} u^k_{m,n}| \leq \frac{1}{12}C_1(\Delta x)^2, \\ |\delta^{(2)}_y u^k_{m,n} - \partial_{yy} u^k_{m,n}| \leq \frac{1}{12}C_1(\Delta y)^2. \end{cases}$$

Use these inequalities to obtain

$$\begin{aligned} \frac{1}{6}C_1(\Delta z)^2 + C_1^2\Delta t &\geq u^{k+1}_{m,n}\left|\delta^{(1)}_z u^k_{m,n} - \partial_z u^k_{m,n}\right| + \left|u^{k+1}_{m,n} - u^k_{m,n}\right|\left|\partial_z u^k_{m,n}\right| \\ &\geq \left|u^{k+1}_{m,n}\delta^{(1)}_z u^k_{m,n} - u^{k+1}_{m,n}\partial_z u^k_{m,n} + u^{k+1}_{m,n}\partial_z u^k_{m,n} - u^k_{m,n}\partial_z u^k_{m,n}\right| \\ &\geq \left|u^{k+1}_{m,n}\delta^{(1)}_z u^k_{m,n} - u^k_{m,n}\partial_z u^k_{m,n}\right| \end{aligned}$$

for $z = x, y$. The fact that u is bounded in $(0,1)$ yields

$$\begin{aligned} (3\gamma + 5)C_1\Delta t &= 2\gamma C_1\Delta t + 2C_1\Delta t + \gamma C_1\Delta t + 3C_1\Delta t \\ &\geq \gamma\left|u^{k+1}_{m,n} - u^k_{m,n}\right|\left|u^{k+1}_{m,n} + u^k_{m,n}\right| \\ &\quad + \left|u^{k+1}_{m,n} - u^k_{m,n}\right|\left|u^{k+1}_{m,n} + u^k_{m,n}\right| + \gamma\left|u^{k+1}_{m,n} - u^k_{m,n}\right| \\ &\quad + \left|u^{k+1}_{m,n} - u^k_{m,n}\right|\left|\left(u^{k+1}_{m,n}\right)^2 + u^{k+1}_{m,n}u^k_{m,n} + \left(u^k_{m,n}\right)^2\right| \\ &\geq \left|u^{k+1}_{m,n}\left(1 - u^{k+1}_{m,n}\right)\left(u^{k+1}_{m,n} - \gamma\right) - u^k_{m,n}\left(1 - u^k_{m,n}\right)\left(u^k_{m,n} - \gamma\right)\right|. \end{aligned}$$

As a consequence,

$$\begin{aligned} \left|R^k_{m,n}\right| &\leq \left|\delta_t u^k_{m,n} - \partial_t u^k_{m,n}\right| + |\alpha|\left\{\left|u^{k+1}_{m,n}\delta^{(1)}_x u^k_{m,n} - u^k_{m,n}\partial_x u^k_{m,n}\right|\right. \\ &\quad \left. + \left|u^{k+1}_{m,n}\delta^{(1)}_y u^k_{m,n} - u^k_{m,n}\partial_y u^k_{m,n}\right|\right\} + \left|\delta^{(2)}_x u^k_{m,n} - \partial_{xx} u^k_{m,n}\right| \\ &\quad + \left|u^{k+1}_{m,n}\left(1 - u^{k+1}_{m,n}\right)\left(u^{k+1}_{m,n} - \gamma\right) - u^k_{m,n}\left(1 - u^k_{m,n}\right)\left(u^k_{m,n} - \gamma\right)\right| \\ &\quad + \left|\delta^{(2)}_y u^k_{m,n} - \partial_{yy} u^k_{m,n}\right| \\ &\leq \frac{1}{2}C_1\Delta t + |\alpha|\left\{\frac{1}{6}C_1\left[(\Delta x)^2 + (\Delta y)^2\right] + 2C_1^2\Delta t\right\} \end{aligned}$$

$$+\frac{1}{12}C_1\big[(\Delta x)^2+(\Delta y)^2\big]+3\gamma C_1\Delta t+5C_1\Delta t$$
$$=C_1\left(\frac{11}{2}+2|\alpha|C_1+3\gamma\right)\Delta t+C_1\left(\frac{|\alpha|}{6}+\frac{1}{12}\right)\big[(\Delta x)^2+(\Delta y)^2\big].$$

If we define $D_1=C_1(\frac{11}{2}+2|\alpha|C_1+3\gamma)$ and $D_2=C_1(\frac{|\alpha|}{6}+\frac{1}{12})$ then the number $C=\max\{D_1,D_2\}$ has the desired property. □

Lemma 6 *Let $\varepsilon^k_{m,n}=w^k_{m,n}-u^k_{m,n}$ for each $m\in\{0,1,\ldots,M\}$, $n\in\{0,1,\ldots,N\}$, and $k\in\{0,1,\ldots,K\}$. For each $m\in\{1,\ldots,M-1\}$, $n\in\{1,\ldots,N-1\}$, and $k\in\{0,1,\ldots,K-1\}$,*

$$\begin{aligned}\big|\varepsilon^{k+1}_{m,n}\big|\le{}&\big|\varepsilon^k_{m,n}\big||1-2R_x-2R_y|+\big|\varepsilon^k_{m+1,n}\big|\big|R_x-\alpha w^{k+1}_{m,n}r_x\big|\\&+\big|\varepsilon^k_{m-1,n}\big|\big|R_x+\alpha w^{k+1}_{m,n}r_x\big|+\big|\varepsilon^k_{m,n+1}\big|\big|R_y-\alpha w^{k+1}_{m,n}r_y\big|\\&+\big|\varepsilon^k_{m,n-1}\big|\big|R_y+\alpha w^{k+1}_{m,n}r_y\big|+\big|\Delta tR^k_{m,n}\big|\\&+\big|\Delta t\varepsilon^{k+1}_{m,n-1}\big|\big|\big(1-u^{k+1}_{m,n}\big)\big(u^{k+1}_{m,n}+\gamma\big)\\&+w^{k+1}_{m,n}\big(1-w^{k+1}_{m,n}-u^{k+1}_{m,n}-\gamma\big)-\alpha\big(\delta^{(1)}_x+\delta^{(1)}_y\big)u^k_{m,n}\big|.\end{aligned}$$

Proof Departing from (5), using the fact that w is a solution for the finite-difference scheme and simplifying, we reach the following expression for each $m\in\{1,\ldots,M-1\}$, each $n\in\{1,\ldots,N-1\}$, and $k\in\{0,1,\ldots,K-1\}$:

$$\begin{aligned}R^k_{m,n}={}&\delta_t\varepsilon^k_{m,n}+\alpha w^{k+1}_{m,n}\big(\delta^{(1)}_x+\delta^{(1)}_y\big)\varepsilon^k_{m,n}+\alpha\varepsilon^{k+1}_{m,n}\big(\delta^{(1)}_x+\delta^{(1)}_y\big)u^k_{m,n}\\&-\varepsilon^{k+1}_{m,n}\big[\big(1-u^{k+1}_{m,n}\big)\big(u^{k+1}_{m,n}+\gamma\big)+w^{k+1}_{m,n}\big(1-w^{k+1}_{m,n}-u^{k+1}_{m,n}-\gamma\big)\big]\\&-\big(\delta^{(2)}_x+\delta^{(2)}_y\big)\varepsilon^k_{m,n}.\end{aligned}$$

Using the definitions of the discrete operators and rearranging terms, we obtain

$$\begin{aligned}\varepsilon^{k+1}_{m,n}={}&\varepsilon^k_{m,n}[1-2R_x-2R_y]+\varepsilon^k_{m+1,n}\big[R_x-\alpha w^{k+1}_{m,n}r_x\big]\\&+\varepsilon^k_{m-1,n}\big[R_x+\alpha w^{k+1}_{m,n}r_x\big]+\varepsilon^k_{m,n+1}\big[R_y-\alpha w^{k+1}_{m,n}r_y\big]\\&+\varepsilon^k_{m,n-1}\big[R_y+\alpha w^{k+1}_{m,n}r_y\big]+\Delta t\varepsilon^{k+1}_{m,n-1}\big[\big(1-u^{k+1}_{m,n}\big)\big(u^{k+1}_{m,n}+\gamma\big)\\&+w^{k+1}_{m,n}\big(1-w^{k+1}_{m,n}-u^{k+1}_{m,n}-\gamma\big)-\alpha\big(\delta^{(1)}_x+\delta^{(1)}_y\big)u^k_{m,n}\big]+\Delta tR^k_{m,n},\end{aligned}$$

and the result follows after taking the absolute values on both sides of this identity. □

Theorem 5 *Let $u\in C^{4,4,2}_{x,y,t}(\Omega\times[0,T])$ be a solution of* (1) *such that $0<u(\mathbf{x},t)<\mu$ for $(\mathbf{x},t)\in\Omega\times[0,T]$, and suppose that the following conditions hold*:

(1) $\Delta t<1$,
(2) $2(R_x+R_y)<1$,
(3) $|\alpha|\mu r_z<\frac{2R_z}{3}$ *for $z=x,y$, and*
(4) $(\Delta x)^2+(\Delta y)^2<6$.

If $(\mathbf{w})^K_{k=0}$ is the unique solution of (4) *in $(0,\mu)$ then there exists a constant $\widetilde{C}\in\mathbb{R}^+$ independent of Δt, Δx, and Δy such that*

$$\max_{0\le k\le K}\big|\mathbf{u}^k-\mathbf{w}^k\big|\le\widetilde{C}\big(\Delta t+(\Delta x)^2+(\Delta y)^2\big).$$

Proof Following Lemma 6, let $\xi^k = \max\{|\varepsilon^k_{m,n}| : m = 0,1,\dots,M; n = 0,1,\dots,N\}$ for each $k \in \{0,1,\dots,K\}$. Obviously $|\varepsilon^k_{m,n}| \le \xi^k$ for each $m \in \{0,1,\dots,M-1\}$, $n \in \{0,1,\dots,N-1\}$, and $k \in \{0,1,\dots,K\}$, so

$$\begin{aligned}
\xi^{k+1} \le{} & \xi^k|1 - 2R_x - 2R_y| + \xi^k\left|R_x - \alpha w^{k+1}_{m,n} r_x\right| + \xi^k\left|R_x + \alpha w^{k+1}_{m,n} r_x\right| \\
& + \xi^k\left|R_y - \alpha w^{k+1}_{m,n} r_y\right| + \xi^k\left|R_y + \alpha w^{k+1}_{m,n} r_y\right| + \left|\Delta t R^k_{m,n}\right| \\
& + \Delta t\xi^{k+1}\left|\left(1 - u^{k+1}_{m,n}\right)\left(u^{k+1}_{m,n} + \gamma\right) + w^{k+1}_{m,n}\left(1 - w^{k+1}_{m,n} - u^{k+1}_{m,n} - \gamma\right)\right. \\
& \left. - \alpha\left(\delta^{(1)}_x + \delta^{(1)}_y\right)u^k_{m,n}\right|.
\end{aligned}$$

Using the fact that the continuous and the discrete solutions are bounded in $(0,1)$ as well as Lemma 5 and the positivity of the coefficients in the above inequality, one readily obtains $\xi^{k+1} - \xi^k \le \Delta t\xi^{k+1}|Q^k_{m,n}| + \Delta tC(\Delta t + (\Delta x)^2 + (\Delta y)^2)$, where

$$Q^k_{m,n} = \left(1 - u^{k+1}_{m,n}\right)\left(u^{k+1}_{m,n} + \gamma\right) + w^{k+1}_{m,n}\left(1 - w^{k+1}_{m,n} - u^{k+1}_{m,n} - \gamma\right) - \alpha\left(\delta^{(1)}_x + \delta^{(1)}_y\right)u^k_{m,n}.$$

Observe on the other hand that

$$\begin{aligned}
\left|Q^k_{m,n}\right| \le{} & \left|1 - u^{k+1}_{m,n}\right|\left(\left|u^{k+1}_{m,n}\right| + \gamma\right) + \left|w^{k+1}_{m,n}\right|\left(\left|1 - w^{k+1}_{m,n}\right| + \left|u^{k+1}_{m,n}\right| + \gamma\right) \\
& + |\alpha|\left|\left(\delta^{(1)}_x + \delta^{(1)}_y\right)u^k_{m,n}\right| \\
\le{} & 5 + |\alpha|\left|\left(\delta^{(1)}_x + \delta^{(1)}_y\right)u^k_{m,n}\right| \le 5 + 3|\alpha|\left[C_1\left(2 + \frac{(\Delta_x)^2 + (\Delta_y)^2}{6}\right)\right] \\
={} & 5 + 3|\alpha|C_1.
\end{aligned}$$

As a consequence, we obtain

$$\xi^{k+1} - \xi^k \le \Delta t\xi^{k+1}(5 + 3|\alpha|C_1) + \Delta tC\left(\Delta t + (\Delta x)^2 + (\Delta y)^2\right).$$

Suppose that $\Delta t \le \frac{K-1}{2K(5+3|\alpha|C_1)}$. An application of Lemma 4 yields

$$\max_{1\le k\le K}\left|\xi^k\right| \le e^{2(5+3|\alpha|C_1)T}\left[\xi^0 + K\Delta tC\left(\Delta t + (\Delta x)^2 + (\Delta y)^2\right)\right].$$

Finally, the exactness of the initial-boundary conditions guarantees that $\xi^0 = 0$. The conclusion follows with $\widetilde{C} = e^{2(5+3|\alpha|C_1)T}TC$. □

5 Conclusions and perspectives

In this note, we extended dimensionally a numerical technique to approximate the solution of the well-known Burgers-Huxley equation, using a finite-difference perspective. The method is an exact technique that requires one to solve a cubic polynomial at each temporal step and each spatial node via the Cardano formulas. The method is an exact technique which preserves positivity, boundedness and monotonicity, resembling thus the features of many classical solutions of the model under investigation. Finally, we also established that the method is convergent of first order in time and second order in space.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions

The authors declare that both contributed equally in the elaboration of this manuscript.

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