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Periodic-wave solutions of the two-dimensional Toda lattice equation by a direct method

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Abstract

Hirota bilinear method is proposed to directly construct periodic wave solutions in terms of Riemann theta functions for (2 + 1)-dimensional Toda lattice equations. The asymptotic properties of the periodic waves are analyzed in detail, including one-periodic and two-periodic solutions. Furthermore, the curves of the solutions are plotted to analyze the solutions. It is shown that well-known soliton solutions can be reduced from the periodic wave solutions.

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Keywords: Riemann theta functions; periodic wave solutions; a direct method

1 Introduction

It is well known that there are many successful methods to construct explicit solutions for differential equations, such as the scattering transform [1], the Darboux transformation [2], Hirota direct method [3–6], algebra-geometrical approach [7–10], *etc.* Quasi-periodic or algebraic-geometric solutions can be obtained by using algebraic-geometric approach; however, the forms of their solutions are complicated with the help of a Riemann surface and the Abel-Jacobi function. The Hirota direct method provides a powerful approach to construct exact solutions of nonlinear equations. Once nonlinear equations are written in bilinear forms by a dependent variable transformations, then multisoliton solutions and rational solutions can be obtained. Nakamura [11, 12] in 1979 and 1980 presented one-periodic wave solutions and two-periodic wave solutions based on the Hirota method with the help of the Riemann theta function, where the periodic solutions of the KdV and Boussinesq equations were derived. The important advantages of this approach, as first demonstrated in Dai *et al.* [13] for the KP equation, are that the solution profiles can be explicitly plotted and by using suitable asymptotic limits multisoliton solutions can be deduced from the quasi-periodic solutions. The procedures introduced in Dai *et al.* [13] are adopted by other authors to study a number of soliton equations for constructing quasi-periodic solutions (see [14–18]).

The problems of the Toda lattice have been subjected to a number of investigations. Nakamura [19] studied the (3 + 1)-dimensional Toda equation, and its solutions are expressed by the series expansions of the Bessel functions. Krichever and Vaninsky [20] ob-

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tained the relation between the periodic and open Toda lattice. Furthermore, algebraic-geometric approach for the open Toda lattice was developed. Based on Lie superalgebraic approach, it was found in [21] that super Toda lattice and super-KdV equation have certain relations. Baleanu and Baskal [22] discussed the tensorial form of the Lax pair equations in a compact and geometrically transparent form in the presence of Cartan torsion tensor. Moreover, solutions to the Lax tensor equations of the Toda lattice were given. Baleanu *et al.* [23] presented a connection between Killing tensors and Lax operators and analyzed an application the Toda lattice equation in detail. Ito and Locke [24] studied affine Toda field equations and derived some interesting solutions. Mahmood [25] derived quasi-determinant solutions of the NC Painlevé equation with the Toda solution at $n = 1$ by using the Darboux transformation. Klein and Roidot [26] presented a numerical study of the $(2 + 1)$-dimensional Toda in the limit of the wavelengths for both hyperbolic and elliptic cases. Wu *et al.* [27] introduced the tool of discrete fractional calculus to discrete modeling of diffusion problem and presented a model of a fractional time discretization diffusion in the Caputo sense. Li [28] constructed the Sato theory of the bilinear equations and tau-function of a new q-deformed Toda hierarchy. Furthermore, the multicomponent extension was studied in detail. In [29], the authors studied the asymptotics of the dynamics of periodic Toda chains with a large number of particles of equal mass for initial data close to the equilibrium. Wu *et al.* [30] proposed a lattice fractional diffusion equation and, as applications, discussed various difference orders.

For the 2D Toda lattice equation

$$\alpha u_{xx}(x,y,n) + \beta u_{yy}(x,y,n) + e^{-u(x,y,n+1)} + e^{-u(x,y,n-1)} - 2e^{-u(x,y,n)} = 0, \tag{1.1}$$

Nakamura [31] found new type exact solutions (ripplon solutions); the new solutions reflect the effect of essential multidimensionality of the system. In fact, equation (1.1) is a discretization of the modified Laplace equation (see [31])

$$\alpha u_{xx} + \beta u_{yy} - u_{zz} = 0. \tag{1.2}$$

In this paper, we adopt the approach proposed in Dai *et al.* [13] to directly construct periodic-wave solutions in Riemann theta functions of equation (1.1). Both one-periodic and two-periodic solutions are obtained and derived by conducting a suitable asymptotic analysis; furthermore, we plot some solution curves to analyze the solutions in detail.

The paper is organized as follows. In Section 2, we derive a bilinear form of the 2D Toda lattice equation. In Section 3, one-periodic wave solutions and asymptotic behaviors are given; moreover, some solution curves are given. In Section 4, we obtain two-periodic wave solutions and their asymptotic behaviors; similarly to Section 3, some solution curves for imaginary parts are dropped.

2 Bilinear form of the 2D Toda lattice equation

We consider the equation

$$\alpha u_{xx}(x,y,n) + \beta u_{yy}(x,y,n) + e^{-u(x,y,n+1)} + e^{-u(x,y,n-1)} - 2e^{-u(x,y,n)} = 0. \tag{2.1}$$

Under the transformation

$$e^{-u(x,y,n)} - 1 = \left(\alpha \partial_x^2 + \beta \partial_y^2\right) \ln f(x,y,n), \tag{2.2}$$

equation (2.1) has the bilinear form

$$\begin{aligned} &G(D_x, D_y, \cosh D_n) f(x,y,n) \cdot f(x,y,n) \\ &\quad \equiv \left[\alpha D_x^2 + \beta D_y^2 - 2\cosh D_n + 2 + c\right] f(x,y,n) \cdot f(x,y,n) = 0, \end{aligned} \tag{2.3}$$

where $c = c_1(n)x + c_2(n)y + c_3(n)$, which arises as a result of integration. The Hirota bilinear differential operator is defined as [4]

$$D_x^m D_y^n a(x,y) \cdot b(x,y) \equiv (\partial_x - \partial_{x'})^m (\partial_y - \partial_{y'})^n a(x,y) \times b(x',y')|x' = x, y' = y,$$

and the difference operator is defined as

$$\begin{aligned} &e^{D_n} a_n \cdot b_n = a_{n+1} b_{n-1}; \qquad e^{-D_n} a_n \cdot b_n = a_{n-1} b_{n+1}, \\ &\cosh D_n a_n \cdot b_n = \frac{1}{2}\left(e^{D_n} + e^{-D_n}\right) a_n \cdot b_n = \frac{1}{2}(a_{n+1} b_{n-1} + a_{n-1} b_{n+1}). \end{aligned}$$

From the definition of the Hirota operator we have the relations

$$D_x^m D_t^l e^{\xi_1} \cdot e^{\xi_2} = (k_1 - k_2)^m (\omega_1 - \omega_2)^l e^{\xi_1 + \xi_2},$$

where $\xi_j = k_j x + \omega_j t + \mu_j n$, $j = 1, 2$. Moreover, it is easy to deduce the relations

$$\cosh D_n e^{\xi_1} \cdot e^{\xi_2} = \cosh(\mu_1 - \mu_2) e^{\xi_1 + \xi_2}, \tag{2.4}$$

$$G(D_x, D_t, \cosh D_n) e^{\xi_1} \cdot e^{\xi_2} = G(k_1 - k_2, \omega_1 - \omega_2, \mu_1 - \mu_2) e^{\xi_1 + \xi_2}. \tag{2.5}$$

3 One-periodic wave solution and asymptotic behavior

3.1 One-periodic wave solution

We consider the Riemann theta function solution of the bilinear form of the 2D-Toda lattice equation

$$f = \sum_{k \in Z^N} e^{\pi i \langle \tau k, k \rangle + 2\pi i \langle \xi, k \rangle}, \tag{3.1}$$

where $k = (k_1, \dots, k_N)$, $\xi = (\xi_1, \dots, \xi_N)$, τ is a symmetric matrix with $\operatorname{Im} \tau > 0$, and $\xi_j = p_j x + l_j y + \mu_j m + \xi_0$, $j = 1, \dots, N$.

We consider the case where $N = 1$. Then (3.1) becomes

$$f = \sum_{k=-\infty}^{\infty} e^{2\pi i k \xi + \pi i k^2 \tau}. \tag{3.2}$$

In order that the above form can be a solution, p, l, and μ cannot be independent, and we proceed to find their relations. Substituting (3.2) into (2.3) and using property (2.4)-(2.5), we have

$$\begin{aligned} Gf \cdot f &= \sum_{k,k'=-\infty}^{\infty} G(D_x, D_y, \cosh D_n) \exp\left(2\pi i k \xi + \pi i k^2 \tau\right) \cdot \exp\left(2\pi i k' \xi + \pi i k'^2 \tau\right) \\ &= \sum_{k,m=-\infty}^{\infty} G(D_x, D_y, \cosh D_n) \exp\left(2\pi i k \xi + \pi i k^2 \tau\right) \end{aligned}$$

$$
\begin{aligned}
&\quad \times \exp\bigl(2\pi i(m-k)\xi + \pi i(m-k)^2\tau\bigr) \\
&= \sum_{k,m=-\infty}^{\infty} G\bigl(2\pi i(2k-m)p, 2\pi i(2k-m)l, \cosh\bigl[2\pi i(2k-m)\mu\bigr]\bigr) \\
&\quad \times \exp\bigl(2\pi i m\eta + \pi i\bigl[k^2 + (k-m)^2\bigr]\tau\bigr) \\
&= \sum_{m=-\infty}^{\infty} \tilde{G}(m)\exp(2\pi i m\eta) = 0,
\end{aligned}
$$

where the new summation index $m = k + k'$ is introduced, and $\tilde{G}(m)$ is defined by

$$
\begin{aligned}
\tilde{G}(m) = {}& \sum_{k=-\infty}^{\infty} G\bigl(2\pi i(2k-m)p, 2\pi i(2k-m)l, \cosh\bigl[2\pi i(2k-m)\mu\bigr]\bigr) \\
& \times \exp \pi i\bigl[k^2 + (k-m)^2\bigr]\tau.
\end{aligned}
\tag{3.3}
$$

In equation (3.3), letting $k = k' + 1$, we have the relation

$$
\begin{aligned}
\tilde{G}(m) &= \sum_{k=-\infty}^{\infty} G\bigl(2\pi i\bigl(2k' - (m-2)\bigr)p, 2\pi i\bigl(2k' - (m-2)\bigr)l, \cosh\bigl[2\pi i\bigl(2k' - (m-2)\bigr)\mu\bigr]\bigr) \\
&\quad \times \exp \pi i\bigl[k'^2 + \bigl(k' - (m-2)\bigr)^2\bigr]\tau \exp\bigl[2\pi i(m-1)\tau\bigr] \\
&= \tilde{G}(m-2)\exp\bigl[2\pi i(m-1)\tau\bigr] = \cdots \\
&= \begin{cases} \tilde{G}(0)e^{\pi i m'(m'-1)\tau}, & m' \text{ is even}, \\ \tilde{G}(1)e^{\pi i(m'+1)(m'+2)\tau}, & m' \text{ is odd}. \end{cases}
\end{aligned}
\tag{3.4}
$$

This relation implies that if $\tilde{G}(0) = \tilde{G}(1) = 0$, then $\tilde{G}(m') = 0$, $m' \in Z$.

In this way, we may let

$$
\tilde{G}(0) = \sum_{k=-\infty}^{\infty} \bigl[16\pi^2k^2\bigl(\alpha p^2 + \beta l^2\bigr) + 4\sinh^2(2\pi i\mu k) + c\bigr]\exp\bigl(2\pi i k^2\tau\bigr) = 0,
\tag{3.5}
$$

$$
\begin{aligned}
\tilde{G}(1) = {}& \sum_{k=-\infty}^{\infty} \bigl[4\pi^2(2k-1)^2\bigl(\alpha p^2 + \beta l^2\bigr) + 4\sinh^2(2\pi i\mu(2k-1) + c\bigr] \\
& \times \exp\bigl(\pi i\bigl(k^2 + (k-1)^2\bigr)\tau\bigr) = 0.
\end{aligned}
\tag{3.6}
$$

Denote

$$
\begin{aligned}
&\delta_1(k) = \exp\bigl(2\pi i k^2\tau\bigr), \qquad \delta_2(k) = \exp\bigl(\pi i\bigl(k^2 + (k-1)^2\bigr)\tau\bigr), \\
&a_{11} = \sum_{k=-\infty}^{\infty} 16\pi^2k^2\delta_1(k), \qquad a_{12} = \sum_{k=-\infty}^{\infty} \delta_1(k), \\
&b_1 = \sum_{k=-\infty}^{\infty} 4\sinh^2(2\pi i\mu k)\delta_1(k), \qquad a_{21} = \sum_{k=-\infty}^{\infty} 4\pi^2(2k-1)^2\delta_2(k), \\
&a_{22} = \sum_{k=-\infty}^{\infty} \delta_2(k), \qquad b_2 = \sum_{k=-\infty}^{\infty} 4\sinh^2\bigl(2\pi i(2k-1)\mu\bigr)\delta_2(k).
\end{aligned}
$$

Then, equations (3.5)-(3.6) are reduced to

$$a_{11}\left(\alpha p^2+\beta l^2\right)+a_{12}c+b_1=0, \tag{3.7}$$

$$a_{21}\left(\alpha p^2+\beta l^2\right)+a_{22}c+b_2=0. \tag{3.8}$$

Solving the system, we have

$$\alpha p^2+\beta l^2=\frac{b_1a_{22}-b_2a_{12}}{a_{21}a_{12}-a_{11}a_{22}}, \tag{3.9}$$

$$c=\frac{b_2a_{11}-b_1a_{21}}{a_{21}a_{12}-a_{11}a_{22}}. \tag{3.10}$$

The coefficients p, l, and μ need to satisfy (3.8), and correspondingly (3.2) and (2.2) give a periodic solution.

3.2 Asymptotic behavior of the one-periodic wave solution

The well-known soliton solution of the 2D Toda lattice equation can be obtained as the limit of the periodic solution. For this purpose, we write $q=\exp\pi i\tau$ and take the limit as $q\to 0$ (or $\operatorname{Im}\tau\to\infty$).

Theorem 1 *As* $q\to 0$ (*or* $\operatorname{Im}\tau\to\infty$), *the periodic solution* (3.1) *of* (2.1) *tends to the soliton solution via* (2.2)

$$e^{-u(x,y,n)}-1=\left(\alpha\partial_x^2+\beta\partial_y^2\right)\ln f=-4\pi^2\left(\alpha p^2+\beta l^2\right)\frac{4+2\cos 2\pi\eta}{(1+2\cos 2\pi\eta)^2}, \tag{3.11}$$

where $\alpha p^2+\beta l^2=-\frac{\sin^2(2\pi\mu)}{\pi^2}$ *and* $\eta=px+ly+\mu n+\eta_0$.

Proof Denoting $q=\exp\pi i\tau$, the quantifies defined are then expanded in powers of q as

$$\begin{aligned}
&a_{11}=16\pi^2\left(2q^2+8q^8+\cdots\right), \qquad a_{12}=1+2q^2+2q^8+\cdots,\\
&b_1=8q^2\sinh^2(2\pi i\mu)+8q^8\sinh^2(4\pi i\mu)+\cdots,\\
&a_{21}=8\pi^2q+72\pi^2q^5+\cdots, \qquad a_{22}=2q+2q^5+2q^{13}+\cdots,\\
&b_2=8\sinh^2(2\pi i\mu)q+4\sinh^2(6\pi i\mu)q^5+\cdots,\\
&a_{12}a_{21}-a_{22}a_{11}=8\pi^2q-48\pi^2q^3+o\left(q^3\right),\\
&b_1a_{22}-b_2a_{12}=-8q\sinh^2(2\pi i\mu)+o(q),\\
&b_2a_{11}-b_1a_{21}=192q^3\sinh^2(2\pi i\mu)+o\left(q^3\right),
\end{aligned}$$

Therefore, as $q\to 0$, we have $c\to 0$, and thus $\alpha p^2+\beta l^2=-\frac{-\sinh^2(2\pi i\mu)}{\pi^2}=-\frac{\sin^2(2\pi\mu)}{\pi^2}$.

The one-periodic wave solution (2.1) converges, as $q\to 0$, to

$$\begin{aligned}
&f_n=1+\exp^{-2\pi i\eta+\pi i\tau}+\exp^{2\pi i\eta+\pi i\tau}+\cdots,\\
&f_{nx}=2\pi ip\left(\exp^{-2\pi i\eta+\pi i\tau}-\exp^{2\pi i\eta+\pi i\tau}\right),\\
&f_{nxx}=-4\pi^2p^2\left(\exp^{-2\pi i\eta+\pi i\tau}+\exp^{2\pi i\eta+\pi i\tau}\right),
\end{aligned}$$

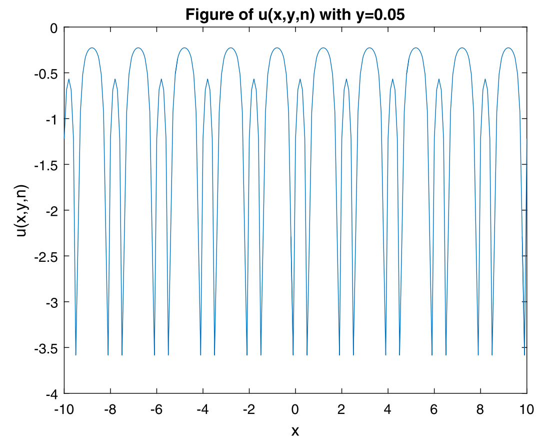

Figure 1 **The soliton solution curve of $u(x,y,n)$ for $\mu = 0.05$, $p = 0.5$, $l = 3$, $n = 5$, $\eta_0 = 2$, $\alpha = 1.5$, $\beta = 2$, $y = 0.05$, $x \in [-10, 10]$.**

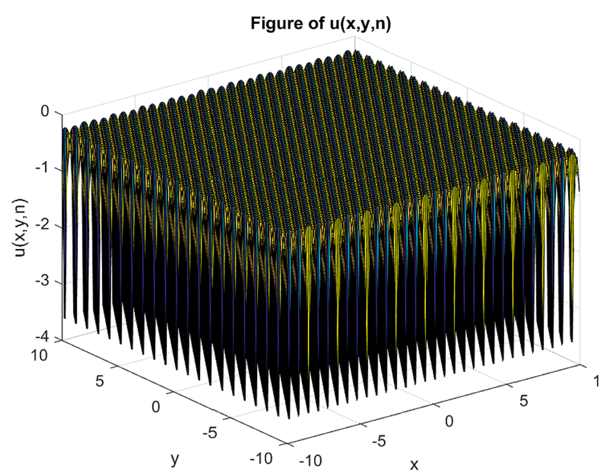

Figure 2 **The soliton solution of $u(x,y,n)$ for $\mu = 0.05$, $p = 0.5$, $l = 3$, $n = 5$, $\eta_0 = 2$, $\alpha = 1.5$, $\beta = 2$, $x \in [-10, 10]$.**

$$\begin{aligned} f_{ny} &= 2\pi il\left(\exp^{-2\pi i\eta+\pi i\tau} - \exp^{2\pi i\eta+\pi i\tau}\right), \\ f_{nyy} &= -4\pi^2 l^2\left(\exp^{-2\pi i\eta+\pi i\tau} + \exp^{2\pi i\eta+\pi i\tau}\right). \end{aligned} \tag{3.12}$$

After some tedious calculations, we derive (3.11).

A solution curve of $u(x,y,n)$ for a fixed y and $\mu = 0.05$ is presented in Figure 1, and the corresponding solution for varying y is presented in Figure 2, from which we see that this solution is periodic in the space coordinate.

A solution curve of $u(x,y,n)$ for a fixed y and $\mu = 0.05i$ is presented in Figure 3, and the corresponding solution for varying y is presented in Figure 4, from which we see that this solution is periodic in the space coordinate. However, the shapes of the solutions between $\mu = 0.05$ and $\mu = 0.05i$ are different. This shown μ is affect to the one-periodic solution. □

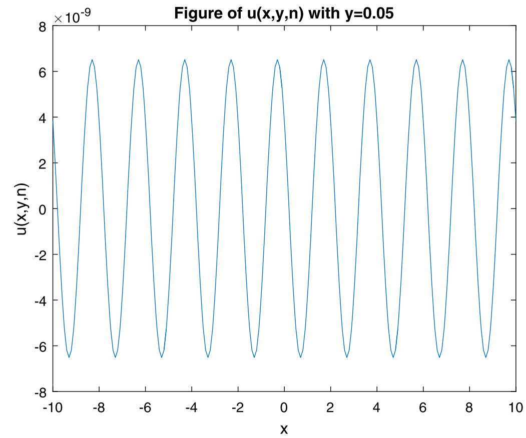

Figure 3 The soliton solution curve of $u(x,y,n)$ for $\mu = 0.05i$, $p = 0.5$, $l = 3$, $n = 5$, $\eta_0 = 2$, $\alpha = 1.5$, $\beta = 2$, $y = 0.05$, $x \in [-10, 10]$.

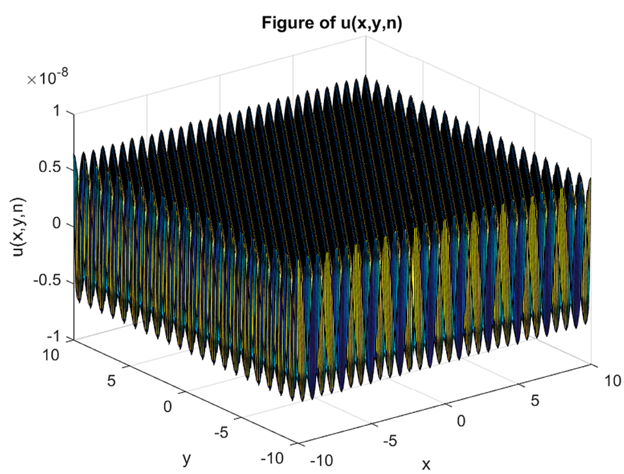

Figure 4 The soliton solution of $u(x,y,n)$ for $\mu = 0.05i$, $p = 0.5$, $l = 3$, $n = 5$, $\eta_0 = 2$, $\alpha = 1.5$, $\beta = 2$, $x \in [-10, 10]$.

4 Two-periodic wave solution and asymptotic behavior

In what follows, we consider the two-periodic wave solution to the (2 + 1)-dimensional Toda lattice equation (2.1), which is a two-dimensional generalization of a one-periodic wave solution.

4.1 Construction of the two-periodic wave solution

Now we consider the two-periodic wave solution of the 2D Toda lattice equation. By setting $N = 2$ in equation (3.1) and substituting it into (2.3), we have

$$\begin{aligned} G(f_n \cdot f_n) &= \sum_{k_1,k_2\in Z^2} G(D_x, D_y, \cosh D_n) e^{2\pi i\langle\xi,k_1\rangle+\pi i\langle\tau k_1,k_1\rangle} \cdot e^{2\pi i\langle\xi,k_2\rangle+\pi i\langle\tau k_2,k_2\rangle} \\ &= \sum_{k_1,k_2\in Z^2} G\big(2\pi i\langle k_1-k_2,p\rangle, 2\pi i\langle k_1-k_2,l\rangle\big) \exp\big(2\pi i\langle\xi,k_1+k_2\rangle + \pi i\langle\tau k_1,k_2\rangle\big) \\ &\quad \times \exp(2\pi i\langle\xi,k_2\rangle + \pi i\big(\langle\tau k_2,k_2\rangle + \langle\tau k_1,k_1\rangle\big) \end{aligned}$$

$$= \sum_{s' \in Z^2} \sum_{s'=-\infty}^{\infty} G\bigl(2\pi i\langle 2k_1 - s', p\rangle, 2\pi i\langle 2k_1 - s', l\rangle\bigr)$$

$$\times \exp \pi i\bigl(\bigl\langle \eta\bigl(k_1 - s'\bigr), k_1 - s'\bigr\rangle + \langle \tau k_1, k_1\rangle\bigr) \exp\bigl(2\pi i\langle \xi, s'\rangle\bigr)$$

$$\equiv \sum_{s' \in Z^2} \tilde{G}\bigl(s_1', s_2'\bigr) \exp 2\pi i\langle \xi, s'\rangle = 0, \qquad (4.1)$$

where the new summation index $k_1 + k_2 = s'$ is introduced, and $\tilde{G}(s_1', s_2')$ is defined by

$$G\bigl(\tilde{s_1'}, s_2'\bigr) = \sum_{k_1,k_2=-\infty}^{\infty} G\bigl(2\pi i\langle 2k_1 - s', p\rangle, 2\pi i\langle 2k_1' - s', l\rangle\bigr)$$

$$\times \exp \pi i\bigl(\bigl\langle \tau\bigl(k_1 - s'\bigr), k_1 - s'\bigr\rangle, \langle \tau k_1, k_1\rangle\bigr)$$

$$= \sum_{k_1,k_2=-\infty}^{\infty} G\left(2\pi i \sum_{j=1}^{2} \bigl(2k_j' - \bigl(s_j' - 2\delta_{jl}\bigr)\bigr)p_j, 2\pi i \sum_{j=1}^{2} \bigl(2k_j' - \bigl(s_j' - 2\delta_{jl}\bigr)\bigr)l_j\right)$$

$$\times \exp \pi i \sum_{j,l=1}^{2} \bigl[\bigl(k_j' + \delta_{jl}\bigr)\tau_{jl}\bigl(k_j' + \delta_{jl}\bigr) + \bigl(\bigl(s_j - 2\delta_{jl} - k_j'\bigr) + \delta_{jl}\bigr)$$

$$\times \tau_{jk}\bigl(s_j - 2\delta_{jl} - k_j'\bigr) + \delta_{kl}\bigr)\bigr]$$

$$= \begin{cases} G(s_1' - 2, s_2')e^{2\pi i(s_1'-1)\tau_{11} + 2\pi i s_2' \tau_{12}}, & l \text{ is even}, \\ G(s_1', s_2' - 2)e^{2\pi i(s_2'-1)\tau_{22} + 2\pi i s_1' \tau_{12}}, & l \text{ is even}. \end{cases} \qquad (4.2)$$

This relation implies that if $\tilde{G}(0,0) = \tilde{G}(0,1) = \tilde{G}(1,0) = \tilde{G}(1,1) = 0$, then $\tilde{G}(s_1', s_2') = 0$, $s_1', s_2' \in Z$.

Denote

$$\delta_j(n) = e^{\pi i\langle \tau n - m^j, n - m^j\rangle + \pi i\langle \tau m, n\rangle}$$

and

$$A \begin{pmatrix} \alpha p_1^2 + \beta l_1^2 \\ \alpha p_2^2 + \beta l_2^2 \\ \alpha p_1 p_2 + \beta l_1 l_2 c \end{pmatrix} = -b,$$

where $m^{(1)} = (0,0)$, $m^{(2)} = (1,0)$, $m^{(1)} = (0,1)$, $m^{(4)} = (1,1)$, and the elements of the matrix A and vector b are

$$a_{j1} = \sum_{n_1,n_2=-\infty}^{\infty} \bigl[2\pi i\bigl(2n_1 - m_1^j\bigr)\bigr]^2 \delta_j(n) = -\sum_{n_1,n_2=-\infty}^{\infty} 4\pi^2\bigl(2n_1 - m_1^j\bigr)^2 \delta_j(n),$$

$$a_{j2} = \sum_{n_1,n_2=-\infty}^{\infty} \bigl[2\pi i\bigl(2n_2 - m_2^j\bigr)\bigr]^2 \delta_j(n) = -\sum_{n_1,n_2=-\infty}^{\infty} 4\pi^2\bigl(2n_2 - m_2^j\bigr)^2 \delta_j(n),$$

$$a_{j3} = \sum_{n_1,n_2=-\infty}^{\infty} (2\pi i)^2\bigl(2n_1 - m_1^j\bigr)\bigl(2n_2 - m_2^j\bigr)\delta_j(n)$$

$$= -\sum_{n_1,n_2=-\infty}^{\infty} 4\pi^2\bigl(2n_1 - m_1^j\bigr)\bigl(2n_2 - m_2^j\bigr)\delta_j(n),$$

$$a_{j4} = \sum_{n_1,n_2=-\infty}^{\infty} \delta_j(n),$$

$$b_j = -4 \sum_{n_1,n_2=-\infty}^{\infty} \big(\sinh \pi i \langle 2n - m^j, \mu\rangle\big)\delta_j(n), \quad \mu = \langle \mu^1, \mu^2\rangle.$$

Then we have

$$\alpha p_1^2 + \beta l_1^2 = \frac{\triangle_1}{\triangle}, \qquad \alpha p_2^2 + \beta l_2^2 = \frac{\triangle_2}{\triangle}, \qquad \alpha p_1 p_2 + \beta l_1 l_2 = \frac{\triangle_3}{\triangle}, \qquad c = \frac{\triangle_4}{\triangle}, \tag{4.3}$$

where $\triangle = \det A$, and $\triangle_1$, $\triangle_2$, $\triangle_3$, $\triangle_4$ are from $\triangle$ by replacing columns 1-4 with b.

4.2 Asymptotic behavior of the two-periodic wave solution

The two-soliton solution of the 2D Toda lattice equation can be obtained as a limit of the two-periodic solution.

Theorem 2 *Suppose that* $1 < r_1 < 2$ *and* $1 < r_2 < 2$ *are constants satisfying* $|\lambda_1|^{r_1} \to 0$ *and* $|\lambda_2|^{r_2} \to 0$ (*the definitions of* λ_1 *and* λ_2 *are given below*). *Then the periodic solution* (3.1) *of equation* (2.1) *tends to the soliton solution via equation* (2.2)

$$\begin{aligned} e^{-u_n} - 1 = \big(\alpha\partial_x^2 + \beta\partial_y^2\big)\ln f_n &= \frac{(\alpha\tilde{p}_1^2 + \beta\tilde{l}_1^2)e^{\tilde{\eta}_1}(1 + e^{\tilde{\eta}_2} + e^{2\tilde{\eta}_2+A_{12}} + e^{\tilde{\eta}_2+A_{12}})}{(1 + e^{\tilde{\eta}_1} + e^{\tilde{\eta}_2} + e^{\tilde{\eta}_1+\tilde{\eta}_2+A_{12}})^2} \\ &+ \frac{(\alpha\tilde{p}_2^2 + \beta\tilde{l}_2^2)e^{\tilde{\eta}_2}(1 + e^{\tilde{\eta}_1} + e^{2\tilde{\eta}_1+A_{12}} + e^{\tilde{\eta}_1+A_{12}})}{(1 + e^{\tilde{\eta}_1} + e^{\tilde{\eta}_2} + e^{\tilde{\eta}_1+\tilde{\eta}_2+A_{12}})^2} \\ &+ \frac{2(\alpha\tilde{p}_1\tilde{p}_2 + \beta\tilde{l}_1\tilde{l}_2)e^{\tilde{\eta}_1+\tilde{\eta}_2}(e^{a_{12}} - 1)}{(1 + e^{\tilde{\eta}_1} + e^{\tilde{\eta}_2} + e^{\tilde{\eta}_1+\tilde{\eta}_2+A_{12}})^2} \end{aligned} \tag{4.4}$$

with the constraints

$$\alpha\tilde{p_1}^2 + \beta\tilde{l_1}^2 = \sinh^2\tilde{\mu_1}, \qquad \alpha\tilde{p_1}^2 + \beta\tilde{l_1}^2 = \sinh^2\tilde{\mu_2}, \tag{4.5}$$

$$\exp(A_{12}) = \frac{-\alpha(\tilde{p_1^2} + \tilde{p_2^2}) - \beta(\tilde{l_1^2} + \tilde{l_2^2}) + 2(\alpha\tilde{p}_1\tilde{p}_2 + \beta\tilde{l}_1\tilde{l}_2) + 2\sinh(\tilde{\mu_1} - \tilde{\mu_2}) - 2}{-\alpha(\tilde{p_1^2} + \tilde{p_2^2}) - \beta(\tilde{l_1^2} + \tilde{l_2^2}) - 2(\alpha\tilde{p}_1\tilde{p}_2 + \beta\tilde{l}_1\tilde{l}_2) + 2\sinh(\tilde{\mu_1} + \tilde{\mu_2}) - 2}, \tag{4.6}$$

where $A_{12} = 2\pi i\tau_{12}$.

Proof Using the quantities

$$\tilde{p}_j = 2\pi i p_j, \qquad \tilde{l}_j = 2\pi i l_j, \qquad \tilde{\mu}_j = 2\pi i \mu_j, \qquad \tilde{\eta}_j = \tilde{p}_j x + \tilde{l}_j y + \tilde{\mu}_j n + \tilde{\eta}_{0j}$$

$$\tilde{\eta}_{0j} = 2\pi i \eta_{oj} + \pi\tau_{jj}, \quad j = 1, 2, \qquad \lambda_1 = e^{\pi i\tau_{11}}, \qquad \lambda_2 = e^{\pi i\tau_{22}}, \qquad \lambda_3 = e^{2\pi i\tau_{12}},$$

we expand the two-periodic wave solution (3.1) ($N = 2$) in the following form:

$$\begin{aligned} f_n \;=\; & 1 + \exp(2\pi i\eta_1 + \pi i\tau_{11}) + \exp(-2\pi i\eta_1 + \pi i\tau_{11}) + \exp(2\pi i\eta_2 + \pi i\tau_{22}) \\ & + \exp(-2\pi i\eta_2 + \pi i\tau_{11}) + \exp\big(2\pi i(\eta_1 + \eta_2) + \pi i(\tau_{11} + 2\tau_{12} + \tau_{22})\big) \\ & + \exp\big(-2\pi i(\eta_1 + \eta_2) + \pi i(\tau_{11} + 2\tau_{12} + \tau_{22})\big) + \cdots \end{aligned}$$

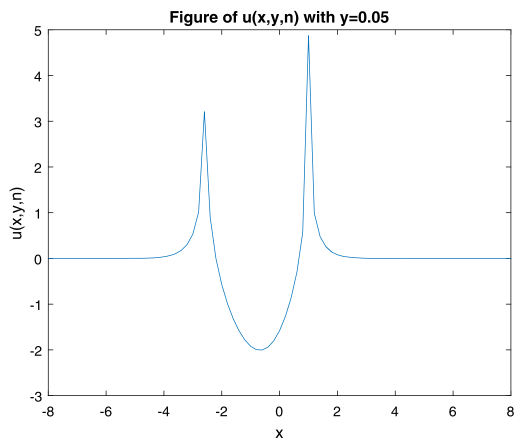

Figure 5 The soliton solution curve of $u(x,y,n)$ for $\mu_1 = 0.05$, $p_1 = 1$, $\mu_2 = 0.02$, $p_2 = 2$, $l_1 = 3$, $l_2 = 6$, $n = 5$, $\eta_{01} = 2$, $\eta_{02} = 2$, $\alpha = 1.5$, $\beta = 2$, $y = 0.05$, $x \in [-10, 10]$.

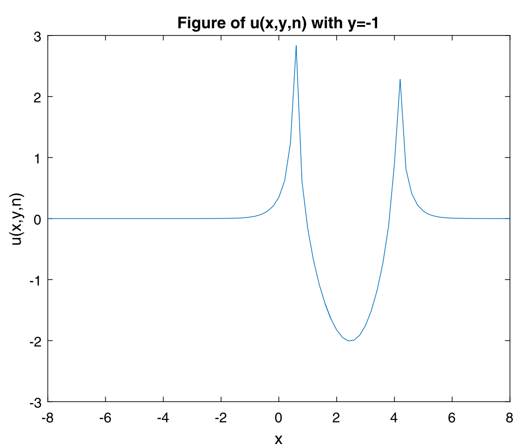

Figure 6 The soliton solution curve of $u(x,y,n)$ for $\mu_1 = 0.05$, $p_1 = 1$, $\mu_2 = 0.02$, $p_2 = 2$, $l_1 = 3$, $l_2 = 6$, $n = 5$, $\eta_{01} = 2$, $\eta_{02} = 2$, $\alpha = 1.5$, $\beta = 2$, $y = -1$, $x \in [-10, 10]$.

$$\begin{aligned} &= 1 + \exp\tilde{\eta}_1 + \exp\tilde{\eta}_2 + \exp(\tilde{\eta}_1 + \tilde{\eta}_2 + 2\pi i\tau_{12}) \\ &\quad + \lambda_1^2\exp(-\tilde{\eta}_1) + \lambda_2^2\exp(-\tilde{\eta}_2) + \lambda_1^2\lambda_2^2\exp(-\tilde{\eta}_1 - \tilde{\eta}_2 + 2\pi i\tau_{12}) + \cdots \\ &\longrightarrow 1 + \exp\tilde{\eta}_1 + \exp\tilde{\eta}_2 + \exp(\tilde{\eta}_1 + \tilde{\eta}_2 + A_{12}). \end{aligned} \tag{4.7}$$

We now verify formulas (4.5) and (4.6). To this end, we expand each function in $\tilde{G}(0,0) = \tilde{G}(0,1) = \tilde{G}(1,0) = \tilde{G}(1,1) = 0$ into series of λ_1 and λ_2. We only need to make the first-order expansions with λ_1 and λ_2 to show the asymptotic relations (4.5) and (4.6). Here we keep the second-order terms in order to see deeper relations among the parameters of the two-periodic solution and two-soliton solution.

From

$$\begin{aligned} \tilde{G}(0,0) = {}& \bigl(-16\pi^2\alpha p_1^2 - 16\pi^2\beta l_1^2 - 4\sinh^2(2\pi i\mu_1) + c\bigr)\lambda_1^2 \\ & + \bigl(-16\pi^2\alpha p_2^2 - 16\pi^2\beta l_2^2 - 4\sinh^2(2\pi i\mu_2) + c\bigr)\lambda_2^2 \end{aligned}$$

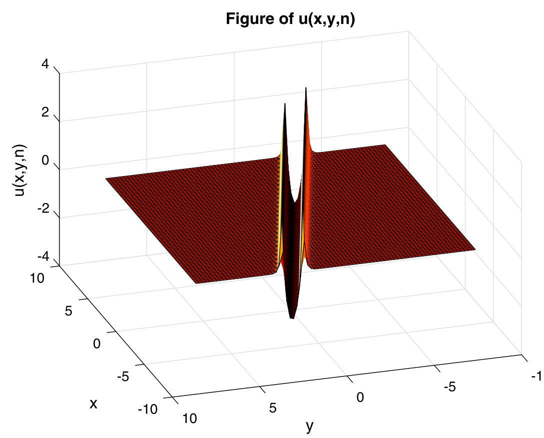

Figure 7 **The soliton solution of $u(x,y,n)$ for $\mu_1 = 0.05$, $p_1 = 1$, $\mu_2 = 0.02$, $p_2 = 2$, $l_1 = 3$, $l_2 = 6$, $n = 5$, $\eta_{01} = 2$, $\eta_{02} = 2$, $\alpha = 1.5$, $\beta = 2$, $x \in [-10, 10]$.**

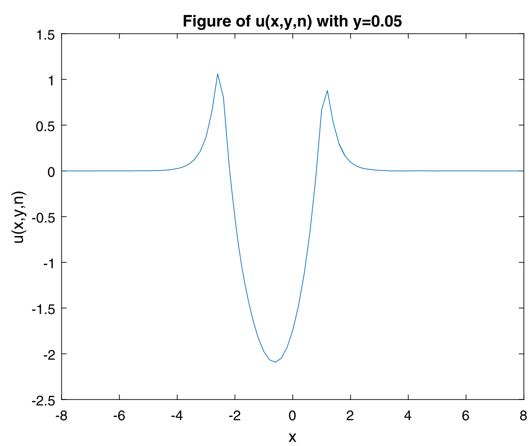

Figure 8 **The soliton solution curve of $u(x,y,n)$ for $\mu_1 = 0.05i$, $p_1 = 1$, $\mu_2 = 0.06i$, $p_2 = 2$, $l_1 = 3$, $l_2 = 6$, $n = 5$, $\eta_{01} = 2$, $\eta_{02} = 2$, $\alpha = 1.5$, $\beta = 2$, $y = 0.05$, $x \in [-10, 10]$.**

$$\begin{aligned} &+ \bigl(-16\pi^2\alpha(p_1 + p_2)^2 - 16\pi^2\beta(l_1 + l_2)^2 - 4\sinh^2\bigl(2\pi i(\mu_1 + \mu_2) + c\bigr)\bigr)\lambda_1^2\lambda_2^2\lambda_3^2 \\ &+ c + o\bigl(\lambda_1^{s_1}\lambda_2^{s_2}\bigr) = 0, \end{aligned} \tag{4.8}$$

where $s_1 + s_2 \geq 4$, as $\lambda_1 \longrightarrow 0$, $\lambda_2 \longrightarrow 0$, we obtain that $c = 0$. From

$$\tilde{G}(1,0) = \bigl(-4\pi^2\alpha p_1^2 - 4\pi^2\beta l_1^2 - 4\sinh^2\pi i\mu_1 + c\bigr)\lambda_1 + o\bigl(\lambda_1^{s_1}\lambda_2^{s_2}\bigr) = 0, \tag{4.9}$$

where $s_1 + s_2 \geq 3$, using $c = 0$, we derive the asymptotic relations

$$4\pi^2\bigl(\alpha p_1^2 + \beta l_1^2\bigr) + 4\sinh^2\pi i\mu_1 = 0, \qquad \alpha\tilde{p}_1^2 + \beta\tilde{l}_1^2 = \sinh^2\tilde{\mu}_1. \tag{4.10}$$

From

$$\tilde{G}(0,1) = \bigl(-4\pi^2\alpha p_2^2 - 4\pi^2\beta l_2^2 - 4\sinh^2\pi i\mu_2 + c\bigr)\lambda_2 + o\bigl(\lambda_1^{s_1}\lambda_2^{s_2}\bigr) = 0, \tag{4.11}$$

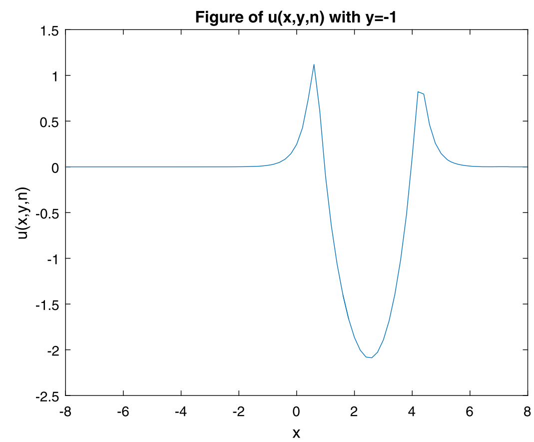

Figure 9 The soliton solution curve of $u(x,y,n)$ for $\mu_1 = 0.05i$, $p_1 = 1$, $\mu_2 = 0.06i$, $p_2 = 2$, $l_1 = 3$, $l_2 = 6$, $n = 5$, $\eta_{01} = 2$, $\eta_{02} = 2$, $\alpha = 1.5$, $\beta = 2$, $y = -1$, $x \in [-10, 10]$.

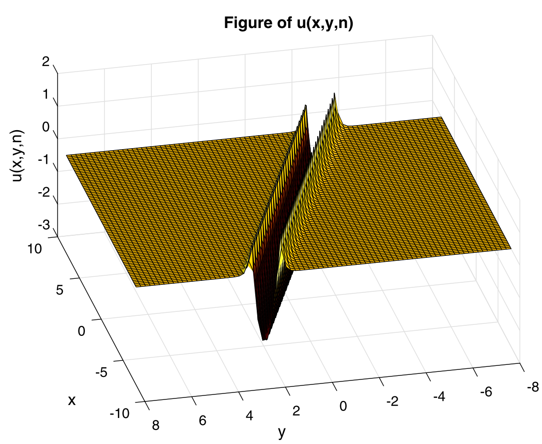

Figure 10 The soliton solution curve of $u(x,y,n)$ for $\mu_1 = 0.05i$, $p_1 = 1$, $\mu_2 = 0.06i$, $p_2 = 2$, $l_1 = 3$, $l_2 = 6$, $n = 5$, $\eta_{01} = 2$, $\eta_{02} = 2$, $\alpha = 1.5$, $\beta = 2$, $x \in [-10, 10]$.

where $s_1 + s_2 \geq 3$, using $c = 0$, we derive the asymptotic relations

$$4\pi^2\left(\alpha p_2^2 + \beta l_2^2\right) + 4\sinh^2 \pi i\mu_2 = 0, \qquad \alpha\tilde{p}_2^2 + \beta\tilde{l}_2^2 = \sinh^2 \tilde{\mu}_2. \tag{4.12}$$

From

$$\begin{aligned}\tilde{G}(1,1) = {} & 2\left(\left[-4\pi^2\left(\alpha p_1^2 + \beta l_1^2\right) - 4\pi^2\left(\alpha p_2^2 + \beta l_2^2\right) - 8\pi^2(\alpha p_1 p_2 + \beta l_1 l_2)\right.\right.\\ & \left.- 4\sinh^2\left[\pi i(\mu_1 + \mu_2)\right] + c\right]\lambda_3 + \left[-4\pi^2\left(\alpha p_1^2 + \beta l_1^2\right) - 4\pi^2\left(\alpha p_2^2 + \beta l_2^2\right)\right.\\ & \left.\left.+ 8\pi^2(\alpha p_1 p_2 + \beta l_1 l_2) - 4\sinh^2\left[\pi i(\mu_1 - \mu_2)\right] + c\right]\right)\lambda_1\lambda_2 + o\left(\lambda_1^{s_1}\lambda_2^{s_2}\right) = 0,\end{aligned} \tag{4.13}$$

where $s_1 + s_2 \geq 5$, using $c = 0$, we derive the asymptotic relations

$$e^{A_{12}} = \frac{4\pi^2(\alpha p_1^2 + \beta l_1^2) + 4\pi^2(\alpha p_2^2 + \beta l_2^2) - 8\pi^2(\alpha p_1 p_2 + \beta l_1 l_2) + 4\sinh^2[\pi i(\mu_1 - \mu_2)]}{4\pi^2(\alpha p_1^2 + \beta l_1^2) + 4\pi^2(\alpha p_2^2 + \beta l_2^2) + 8\pi^2(\alpha p_1 p_2 + \beta l_1 l_2) + 4\sinh^2[\pi i(\mu_1 + \mu_2)]}. \tag{4.14}$$

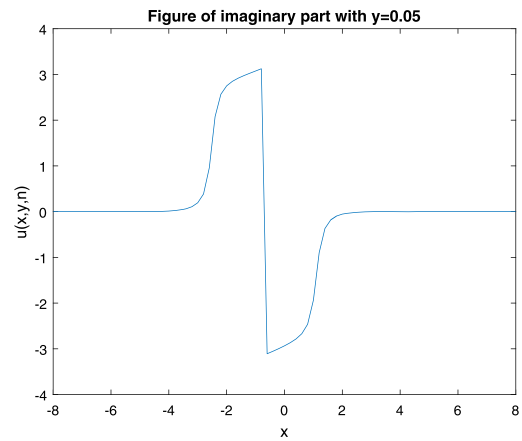

Figure 11 **The soliton solution curve of $u(x,y,n)$ for $\mu_1 = 0.05i$, $p_1 = 1$, $\mu_2 = 0.06i$, $p_2 = 2$, $l_1 = 3$, $l_2 = 6$, $n = 5$, $\eta_{01} = 2$, $\eta_{02} = 2$, $\alpha = 1.5$, $\beta = 2$, $y = 0.05$, $x \in [-10, 10]$.**

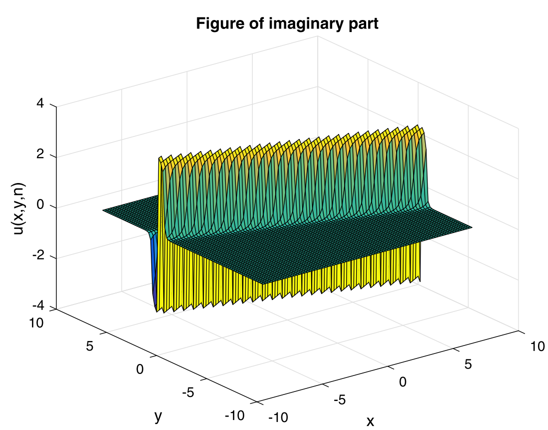

Figure 12 **The soliton solution curve of $u(x,y,n)$ for $\mu_1 = 0.05i$, $p_1 = 1$, $\mu_2 = 0.06i$, $p_2 = 2$, $l_1 = 3$, $l_2 = 6$, $n = 5$, $\eta_{01} = 2$, $\eta_{02} = 2$, $\alpha = 1.5$, $\beta = 2$, $x \in [-10, 10]$.**

We now plot the graph for $u(x,y,n)$ in order to analyze the two-periodic solutions (4.4). Figures 5 and 6 describe the curve of $u(x,y,n)$ of fixed y and real μ_1, μ_2, respectively, for $y = 0.05$ and $y = -1$. From the two curves show that the shape of the solution is not affected by y, but y has influence on translation. The corresponding solution for varying y is presented in Figure 7, from which we see that this solution is periodic in the space coordinate.

Similarly, we consider the imaginary value for $\mu_1 = 0.05i$ and $\mu_2 = 0.06i$. A solution curve of $u(x,y,n)$ for a fixed y is presented in Figures 8 and 9. The corresponding solution for varying y is presented in Figure 10, from which we see that this solution is periodic in the space coordinate.

The imaginary part of the periodic solution (4.4) is presented in Figures 11 and 12, from which we see that the solutions are periodic in the space coordinate. □

Competing interests

The authors declare that they have no competing interests.

Authors' contributions
All authors drafted the manuscript, read, and approved the final manuscript.

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References

1. Ablowitz, MJ, Clarkson, PA: Solitons, Nonlinear Evolution Equations and Inverse Scattering. Cambridge University Press, Cambridge (1991)
2. Matveev, VB, Salle, MA: Darboux Transformation and Solitons. Springer, Berlin (1991)
3. Hirota, R: Exact solution of the Korteweg-de Vries equation for multiple collisions of solitons. Phys. Rev. Lett. **27**(18), 1192-1194 (1971)
4. Hirota, R, Satsuma, J: Nonlinear evolution equations generated from the Bäcklund transformation for the Boussinesq equation. Prog. Theor. Phys. **57**(3), 797-807 (1977)
5. Hirota, R: The Direct Method in Soliton Theory. Cambridge University Press, Cambridge (2004)
6. Hu, XB, Clarkson, PA: Rational solutions of a differential-difference KdV equation, the Toda equation and the discrete KdV equation. J. Phys. A, Math. Gen. **28**(17), 5009-5023 (1995)
7. Geng, XG, Wu, YT, Cao, CW: Quasi-periodic solutions of the modified Kadometsev-Petviashvili equation. J. Phys. A, Math. Gen. **32**(20), 3733-3754 (1999)
8. Dai, HH, Geng, XG: Explicit solutions of the $2 + 1$-dimensional modified Toda lattice through straightening out of the relativistic Toda flows. J. Phys. Soc. Jpn. **72**(13), 3063-3069 (2003)
9. Geng, XG: Algebraic-geometrical solutions of some multidimensional nonlinear evolution equations. J. Phys. A, Math. Gen. **36**(9), 2289-2309 (2003)
10. Cao, CW, Wu, YT, Geng, XG: Relation between the Kadometsev-Petviashvili equation and the confocal involutive system. J. Math. Phys. **40**(8), 3948-3968 (1999)
11. Nakamura, A: A direct method of calculating periodic wave solutions to nonlinear evolution equations. I. Exact two-periodic wave solution. J. Phys. Soc. Jpn. **47**(5), 1701-1705 (1979)
12. Nakamura, A: A direct method of calculating periodic wave solutions to nonlinear evolution equations. II. Exact one- and two-periodic wave solution of the coupled bilinear equations. J. Phys. Soc. Jpn. **48**(4), 1365-1370 (1980)
13. Dai, HH, Fan, EG, Geng, XG: Periodic wave solutions of nonlinear equations by Hirota's bilinear method, 1-22 (2006). arXiv:nlin/0602015
14. Zhang, Y, Ye, LY: Rational and periodic wave solutions of two-dimensional Boussinesq equation. Commun. Theor. Phys. **49**, 815-824 (2008)
15. Fan, EG, Hon, YC: On a direct procedure for the quasi-periodic wave solutions of the supersymmetric Itô's equation. Rep. Math. Phys. **66**(3), 355-365 (2010)
16. Wu, YQ: Asymptotic behavior of the periodic wave solution for the $(3 + 1)$-dimensional Kadomtsev-Petviashvili equation. Appl. Math. Comput. **216**, 3154-3161 (2010)
17. Ma, WX, Zhou, RG, Gao, L: Exact one-periodic and two-periodic wave solutions to Hirota bilinear equations in $(2 + 1)$-dimensions. Mod. Phys. Lett. A **24**(21), 1677-1688 (2009)
18. Luo, L, Fan, EG: Quasi-periodic waves of the $N = 1$ supersymmetric modified Korteweg-de Vries equation. Nonlinear Anal. **74**(2), 666-675 (2011)
19. Nakamura, A: Exact cylindrical soliton solutions of the sine-Gordon equation, the sinh-Gordon equation and the periodic Toda equation. J. Phys. Soc. Jpn. **57**(10), 3309-3322 (1988)
20. Krichever, I, Vaninsky, KL: The periodic and open Toda lattice. In: D'Hoker, E, et al. (eds.) Mirror Symmetry IV. AMS/IP Studies in Advanced Mathematics, vol. 33, pp. 139-158 (2002)
21. Inami, T, Kanno, H: Lie superalgebraic approach to super Toda lattice and generalized super KdV equations. Commun. Math. Phys. **136**, 519-542 (1991)
22. Baleanu, D, Baskal, S: Geometrization of the Lax pairs tensors. Mod. Phys. Lett. A **15**(24), 1503-1510 (2000)
23. Baleanu, D, Karasu, A, Makhaldiani, N: About geometrization of the dynamics. Czechoslov. J. Phys. **50**(1), 17-22 (2000)
24. Ito, K, Locke, C: ODE/IM correspondence and Bethe ansatz for affine Todad field equations. Nucl. Phys. B **896**, 763-778 (2015)
25. Mahmood, I: Quasideterminant solutions of NC Painleve II equation with the Toda solution at $n = 1$ as a seed solution in its Darboux transformation. J. Geom. Phys. **95**, 127-136 (2015)
26. Klein, C, Roidot, K: Numerical study of the long wavelength limit of the Toda lattice. Mathematical Physics (2014) arXiv:1404.2593
27. Wu, GC, Baleanu, D, Zeng, SD, Deng, ZG: Discrete fractal diffusion equation. Nonlinear Dyn. **30**(1), 281-286 (2015)
28. Li, CZ: Sato theory on the q-Toda hierarchy and its extension. Chaos Solitons Fractals **76**, 10-23 (2015)
29. Bambusi, D, Kappleler, T, Paul, T: Dynamics of periodic Toda chains with a large number of particles. J. Differ. Equ. **258**(12), 4209-4274 (2015)
30. Wu, GC, Baleanu, D, Deng, ZG, Zeng, SD: Lattice fractional diffusion equation in terms of a Riesz-Caputo difference. Phys. A, Stat. Mech. Appl. **438**(15), 335-339 (2015)
31. Nakamura, A: Exact Bessel type solution of the two-dimensional Toda lattice equation. J. Phys. Soc. Jpn. **52**, 380-387 (1983)