Impulsive integral boundary value problems of the higher-order fractional differential equation with eigenvalue arguments

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Abstract

We consider a class of higher-order nonlinear Riemann-Liouville fractional differential equation with Riemann-Stieltjes integral boundary value conditions and impulses as follows:

$$\begin{cases} -D_{0^+}^{\alpha} u(t) = \lambda a(t) f(t, u(t)), & t \in (0,1) \setminus \{t_k\}_{k=1}^m, \\ \Delta u(t_k) = I_k(u(t_k)), & t = t_k, \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0), & u'(1) = \int_0^1 u(s)\, dH(s). \end{cases}$$

By converting the boundary value problem into an equivalent integral equation and applying the Schauder fixed-point theorem, fixed-point index theorem, we have established sufficient conditions for the existence and multiplicity of positive solutions. The eigenvalue intervals are also given. Some examples are presented to illustrate the validity of our main results.

Keywords: impulsive fractional differential equations; eigenvalue intervals; multiple positive solutions; Riemann-Stieltjes integral boundary value problems; fixed-point index theorem

1 Introduction

This paper is concerned with the eigenvalue intervals and positive solutions of integral boundary value problem for the following higher-order nonlinear fractional differential equation with impulses (abbreviated by BVP (1.1) throughout this paper):

$$\begin{cases} -D_{0^+}^{\alpha} u(t) = \lambda a(t) f(t, u(t)), & t \in (0,1) \setminus \{t_k\}_{k=1}^m, \\ \Delta u(t_k) = I_k(u(t_k)), & t = t_k, \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0), & u'(1) = \int_0^1 u(s)\, dH(s), \end{cases} \tag{1.1}$$

where $D_{0^+}^{\alpha}$ is the standard Riemann-Liouville fractional derivative of order $n-1 < \alpha \leq n$, $n \geq 3$. The number n is the smallest integer greater than or equal to α. The impulsive point sequence $\{t_k\}_{k=1}^m$ satisfies $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = 1$, $\Delta u(t_k) = u(t_k^+) - u(t_k^-)$. $u(t_k^-) = u(t_k)$, and $u(t_k^+) = \lim_{h \to 0} u(t_k + h)$ and $u(t_k^-) = \lim_{h \to 0} u(t_k - h)$ represent the right-

and left-hand limits of $u(t)$ at $t = t_k$, respectively; $\lambda > 0$ is a parameter, $f \in C([0,1] \times [0,+\infty) \to [0,+\infty))$, $a(t) \in C((0,1),[0,+\infty))$, $I_k \in C(\mathbb{R}^+,\mathbb{R}^+)$. The integral $\int_0^1 u(s)\,dH(s)$ is the Riemann-Stieltjes integral with $H : [0,1] \to \mathbb{R}$. By applying the Schauder fixed-point theorem, fixed-point index theorem, we obtain some sufficient conditions for the existence and multiplicity of positive solutions of BVP (1.1). Meanwhile, the eigenvalue intervals are also given.

During the past decades, the subject of fractional differential equations has aroused great attention due to both the further development of fractional-order calculus theory and important applications in the fields of science and engineering such as physics, chemistry, aerodynamics, electrodynamics of complex medium, polymer rheology, Bode's analysis of feedback amplifiers, capacitor theory, electrical circuits, electron-analytical chemistry, biology, control theory, fitting of experimental data, and so forth. Fractional derivatives provide an excellent tool for description of memory and hereditary properties of various materials and processes. This is the main advantage of fractional differential equations in comparison with classical integer-order models. As a consequence, the subject of fractional differential equations is gaining much importance and attention. In particular, there are many papers focused on the existence or multiplicity of positive solutions for the boundary value problems of fractional ordinary differential equations (see [1–20]).

However, there are a few papers [14–16] that consider the existence or multiplicity of positive solutions for fractional differential equations involving in eigenvalue parameters. For example, Bai [14] considered the following fractional ordinary differential equation boundary value problem:

$$\begin{cases} {}^cD_{0_+}^s u(t) + \lambda h(t) f(u) = 0, & 0 < t < 1, \\ u(0) = u'(1) = u''(0) = 0, \end{cases}$$

where $2 < s \le 3$ is a real number, ${}^cD_{0_+}^s$ is the standard Caputo differentiation, and $\lambda > 0$. By applying a fixed-point theorem on a cone, some sufficient conditions of multiplicity and eigenvalue intervals for the problem are established.

In order to describe the dynamics of populations subject to abrupt changes and other phenomena such as harvesting, diseases, and so on, some authors have used an impulsive differential system to describe these kinds of phenomena since the last century. Recently, some scholars have begun to study the boundary value problems of impulsive fractional differential equations (see [21–27]). This type of boundary value problems has become one of the hottest problems at present.

To the best of our knowledge, there is less research dealing with the eigenvalue intervals and positive solutions of Riemann-Stieltjes integral boundary problems for higher-order nonlinear fractional differential equation with impulses. Therefore, we will investigate the existence and multiplicity of positive solutions for BVP (1.1) under some further conditions.

The rest of this paper is organized as follows. In Section 2, we recall some useful definitions and properties and present properties of the Green functions. In Section 3, we give some sufficient conditions for the existence of single positive solutions for BVP (1.1). In Section 4, some sufficient conditions are established to guarantee the existence of multiple positive solutions for BVP (1.1). Finally, some examples are also provided to illustrate the validity of our main results in Section 5.

2 Preliminaries

For convenience, now we introduce some definitions and results of fractional calculus.

Definition 2.1 (see [28, 29]) The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $u : (0, \infty) \to \mathbb{R}$ is given by

$$I_{0+}^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} u(s) \, ds,$$

provided that the right-hand side is pointwise defined on $(0, \infty)$.

Definition 2.2 (see [28, 29]) The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $u : (0, \infty) \to \mathbb{R}$ is given by

$$D_{0+}^{\alpha} u(t) = \frac{1}{\Gamma(n - \alpha)} \frac{d^n}{dt^n} \int_0^t (t - s)^{n - \alpha - 1} u(s) \, ds,$$

where $n - 1 < \alpha \leq n$, provided that the right-hand side is pointwise defined on $(0, \infty)$.

Lemma 2.1 (see [28]) *Assume that $u \in C(0,1) \cap L(0,1)$ with a fractional derivative of order $\alpha > 0$ that belongs to $u \in C(0,1) \cap L(0,1)$. Then*

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + C_1 t^{\alpha - 1} + C_2 t^{\alpha - 2} + \cdots + C_n t^{\alpha - n}$$

for some $C_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$, where n is the smallest integer greater than or equal to α.

Lemma 2.2 (Schauder fixed-point theorem, see [30]) *If U is a close bounded convex subset of a Banach space X and $T : U \to U$ is completely continuous, then T has at least one fixed point in U.*

Lemma 2.3 (Fixed-point index theorem, see [31]) *Let E be a Banach space, and $P \subset E$ be a cone. For $r > 0$, define $\Omega_r = \{u \in P : \|u\| < r\}$. Assume that $A : \overline{\Omega}_r \to P$ is a completely continuous operator such that $Au \neq u$ for $u \in \partial \Omega_r = \{u \in P : \|u\| = r\}$.*

(1) *If $\|Au\| \geq \|u\|$ for $u \in \partial \Omega_r$, then $i(A, \Omega_r, P) = 0$.*

(2) *If $\|Au\| \leq \|u\|$ for $u \in \partial \Omega_r$, then $i(A, \Omega_r, P) = 1$.*

Now we present the Green function for the system associated with BVP (1.1).

Lemma 2.4 *If $H : [0,1] \to \mathbb{R}$ is a function of bounded variation $\delta \triangleq \int_0^1 s^{\alpha - 1} \, dH(s) \neq \alpha - 1$ and $h \in C([0,1])$, then the unique solution of*

$$\begin{cases} D_{0+}^{\alpha} u(t) + h(t) = 0, & t \in (0,1) \setminus \{t_k\}_{k=1}^m, n - 1 < \alpha \leq n, n \geq 3, \\ \Delta u(t_k) = I_k(u(t_k)), & t = t_k, \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0), & u'(1) = \int_0^1 u(s) \, dH(s), \end{cases} \tag{2.1}$$

is

$$u(t) = \int_0^1 G(t,s) h(s) \, ds + t^{\alpha - 1} \sum_{t \leq t_k < 1} t_k^{1 - \alpha} I_k(u(t_k)), \quad t \in [0,1], \tag{2.2}$$

where

$$G(t,s) = G_1(t,s) + G_2(t,s), \qquad (2.3)$$

$$G_1(t,s) = \begin{cases} \frac{t^{\alpha-1}(1-s)^{\alpha-2}-(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le s \le t \le 1, \\ \frac{t^{\alpha-1}(1-s)^{\alpha-2}}{\Gamma(\alpha)}, & 0 \le t \le s \le 1, \end{cases} \qquad (2.4)$$

$$G_2(t,s) = \frac{t^{\alpha-1}}{\alpha-1-\delta} \int_0^1 G_1(\tau,s)\,dH(\tau). \qquad (2.5)$$

Proof We denote the solution of (2.1) by $u(t) \triangleq u_k(t)$ in $[t_k, t_{k+1}]$ ($k = 0, 1, \ldots, m$).

For $t \in [0, t_1)$, applying Lemma 2.1, we have

$$u_0(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)\,ds + C_1^0 t^{\alpha-1} + C_2^0 t^{\alpha-2} + \cdots + C_n^0 t^{\alpha-n}.$$

In the light of $u(0) = u'(0) = u''(0) = \cdots = u^{(n-2)}(0) = 0$, we have $C_2^0 = C_3^0 = \cdots = C_n^0 = 0$. Thus, we get

$$u_0(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)\,ds + C_1^0 t^{\alpha-1}$$

and

$$u(t_1^-) + I_1(u(t_1)) = u(t_1) + I_1(u(t_1)) = u(t_1^+) = -\int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)\,ds + C_1^0 t_1^{\alpha-1}.$$

For $t \in [t_1, t_2)$, by applying Lemma 2.1 we have

$$u_1(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)\,ds + C_1^1 t^{\alpha-1} + C_2^1 t^{\alpha-2} + \cdots + C_n^1 t^{\alpha-n}.$$

In view of $u(0) = u'(0) = u''(0) = \cdots = u^{(n-2)}(0) = 0$, we have $C_2^1 = C_3^1 = \cdots = C_n^1 = 0$. Thus, we get

$$u_1(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)\,ds + C_1^1 t^{\alpha-1}.$$

Noting that $u(t_1) = u_1(t_1)$, we derive $C_1^1 = C_1^0 - t_1^{1-\alpha} I_1(u(t_1))$. So we obtain

$$u_1(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)\,ds + C_1^0 t^{\alpha-1} - t^{\alpha-1} t_1^{1-\alpha} I_1(u(t_1))$$

and

$$u(t_2^-) + I_2(u(t_2)) = u(t_2) + I_2(u(t_2)) = u(t_2^+) = -\int_0^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)\,ds + C_1^1 t_2^{\alpha-1}.$$

For $t \in [t_2, t_3)$, by applying Lemma 2.1 we have

$$u_2(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)\,ds + C_1^2 t^{\alpha-1} + C_2^2 t^{\alpha-2} + \cdots + C_n^2 t^{\alpha-n}.$$

In view of $u(0) = u'(0) = u''(0) = \cdots = u^{(n-2)}(0) = 0$, we have $C_2^2 = C_3^2 = \cdots = C_n^2 = 0$. Thus, we get

$$u_2(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)\, ds + C_1^2 t^{\alpha-1}.$$

Noting that $u(t_2) = u_2(t_2)$, we derive $C_1^2 = C_1^1 - t_2^{1-\alpha} I_2(u(t_2))$. So we obtain

$$u_2(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)\, ds + C_1^0 t^{\alpha-1} - t^{\alpha-1} \sum_{i=1}^2 t_i^{1-\alpha} I_i\big(u(t_i)\big)$$

and

$$u\big(t_3^-\big) + I_3\big(u(t_3)\big) = u(t_3) + I_3\big(u(t_3)\big) = u\big(t_3^+\big) = -\int_0^{t_3} \frac{(t_3-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)\, ds + C_1^2 t_3^{\alpha-1}.$$

By the recurrent method and Lemma 2.1, for $t \in [t_k, t_{k+1})$ $(k = 0, 1, 2, \ldots, m)$, we have

$$u(t) = u_k(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)\, ds + C_1^0 t^{\alpha-1} - t^{\alpha-1} \sum_{i=1}^k t_i^{1-\alpha} I_i\big(u(t_i)\big). \tag{2.6}$$

Thus, for $t \in [t_m, t_{m+1}] = [t_m, 1]$, we have

$$u(t) = u_m(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)\, ds + C_1^0 t^{\alpha-1} - t^{\alpha-1} \sum_{i=1}^m t_i^{1-\alpha} I_i\big(u(t_i)\big). \tag{2.7}$$

From $u'(1) = \int_0^1 u(s)\, dH(s)$ we obtain

$$C_1^0 = \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-2} h(s)\, ds + \frac{1}{\alpha-1} \int_0^1 u(s)\, dH(s) + \sum_{i=1}^m t_i^{1-\alpha} I_i\big(u(t_i)\big). \tag{2.8}$$

By (2.7) and (2.8), for $t \in [t_m, t_{m+1}] = [t_m, 1]$, we get

$$
\begin{aligned}
u(t) &= -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)\, ds + \frac{t^{\alpha-1}}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-2} h(s)\, ds + \frac{t^{\alpha-1}}{\alpha-1} \int_0^1 u(s)\, dH(s) \\
&= \int_0^1 G_1(t,s) h(s)\, ds + \frac{t^{\alpha-1}}{\alpha-1} \int_0^1 u(s)\, dH(s),
\end{aligned}
\tag{2.9}
$$

which implies that

$$\int_0^1 u(s)\, dH(s) = \frac{\alpha-1}{\alpha-1-\delta} \int_0^1 \left[\int_0^1 G_1(\tau,s) h(s)\, ds \right] dH(\tau). \tag{2.10}$$

According to (2.6), (2.8), and (2.10), for $t \in [t_k, t_{k+1})$ $(k = 0, 1, 2, \ldots, m)$, the unique solution of BVP (2.1) is formulated by

$$
\begin{aligned}
u(t) ={}& -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)\, ds + \frac{t^{\alpha-1}}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-2} h(s)\, ds \\
&+ \frac{t^{\alpha-1}}{\alpha-1} \int_0^1 u(s)\, dH(s) + t^{\alpha-1} \sum_{i=k+1}^m t_i^{1-\alpha} I_i\big(u(t_i)\big)
\end{aligned}
$$

$$
\begin{aligned}
&= \int_0^1 G_1(t,s)h(s)\,ds + \frac{t^{\alpha-1}}{\alpha-1}\int_0^1 u(s)\,dH(s) + t^{\alpha-1}\sum_{i=k+1}^{m} t_i^{1-\alpha}I_i\big(u(t_i)\big) \\
&= \int_0^1 G_1(t,s)h(s)\,ds + \frac{t^{\alpha-1}}{\alpha-1-\delta}\int_0^1\left[\int_0^1 G_1(\tau,s)h(s)\,ds\right]dH(\tau) \\
&\quad + t^{\alpha-1}\sum_{i=k+1}^{m} t_i^{1-\alpha}I_i\big(u(t_i)\big) \\
&= \int_0^1 G_1(t,s)h(s)\,ds + \int_0^1\left[\frac{t^{\alpha-1}}{\alpha-1-\delta}\int_0^1 G_1(\tau,s)\,dH(\tau)\right]h(s)\,ds \\
&\quad + t^{\alpha-1}\sum_{i=k+1}^{m} t_i^{1-\alpha}I_i\big(u(t_i)\big) \\
&= \int_0^1 G_1(t,s)h(s)\,ds + \int_0^1 G_2(t,s)h(s)\,ds + t^{\alpha-1}\sum_{i=k+1}^{m} t_i^{1-\alpha}I_i\big(u(t_i)\big) \\
&= \int_0^1 G(t,s)h(s)\,ds + t^{\alpha-1}\sum_{i=k+1}^{m} t_i^{1-\alpha}I_i\big(u(t_i)\big).
\end{aligned}
\tag{2.11}
$$

Therefore, for $t \in [0,1]$, the unique solution of BVP (2.1) is expressed as

$$
u(t) = \int_0^1 G(t,s)h(s)\,ds + t^{\alpha-1}\sum_{t \le t_k < 1} t_k^{1-\alpha}I_k\big(u(t_k)\big),
\tag{2.12}
$$

where $G(t,s)$, $G_1(t,s)$, and $G_2(t,s)$ are defined by (2.3), (2.4), and (2.5), respectively. The proof is complete. □

From (2.3), (2.4), and (2.5) we can prove that $G(t,s)$, $G_1(t,s)$, and $G_2(t,s)$ have the following properties.

Lemma 2.5 *The function $G_1(t,s)$ defined by (2.4) satisfies*
  (i) $G_1(t,s) \ge 0$ *is continuous for all* $t,s \in [0,1]$, *and* $G_1(t,s) > 0$ *for all* $t,s \in (0,1)$;
  (ii) *For all* $t,s \in [0,1]$, $G_1(t,s)$ *is increasing with respect to t, and*
    $G_1(t,s) \le g_1(s) \triangleq \frac{s(1-s)^{\alpha-2}}{\Gamma(\alpha)}$;
  (iii) *For* $\theta \in (0,\frac{1}{2})$, *there exists a constant $\gamma > 0$ such that* $\min_{t \in J_\theta} G_1(t,s) \ge \gamma g_1(s)$ *for* $s \in [0,1]$, *where* $J_\theta \triangleq [\theta, 1-\theta]$.

Proof (i) It is obvious that $G_1(t,s)$ is continuous on $[0,1] \times [0,1]$, and $G_1(t,s) \ge 0$ for $s \ge t$. For $0 \le s < t \le 1$, noting that $0 < t-s < 1-s \le 1$ and $t(1-s) \ge t-s$, we have

$$
t^{\alpha-1}(1-s)^{\alpha-2} - (t-s)^{\alpha-1} = t^{\alpha-1}(1-s)^{\alpha-2}\left[1 - \left(\frac{t-s}{t(1-s)}\right)^{\alpha-2}(t-s)\right] \ge 0.
\tag{2.13}
$$

So, by (2.4) we get $G_1(t,s) \ge 0$ for all $t,s \in [0,1]$. Similarly, for $s,t \in (0,1)$, we obtain $G_1(t,s) > 0$.
  (ii) In fact,

$$
\frac{\partial G_1}{\partial t} = \begin{cases} \frac{t^{\alpha-2}(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} \ge 0, & 0 \le t \le s \le 1, \\ \frac{t^{\alpha-2}(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\left[1 - (\frac{t-s}{t(1-s)})^{\alpha-2}\right] \ge 0, & 0 \le s \le t \le 1. \end{cases}
\tag{2.14}
$$

From (2.14) we see that $G_1(t,s)$ is increasing with respect to t. For $0 \le s \le t \le 1$, $G_1(t,s) \le G_1(1,s) = \frac{s(1-s)^{\alpha-2}}{\Gamma(\alpha)} = g_1(s)$; for $0 \le t \le s \le 1$, $G_1(t,s) \le G_1(s,s) = \frac{s^{\alpha-1}(1-s)^{\alpha-2}}{\Gamma(\alpha)} \le \frac{s(1-s)^{\alpha-2}}{\Gamma(\alpha)} = g_1(s)$. Therefore, for all $t,s \in [0,1]$, we have $G_1(t,s) \le g_1(s) = \frac{s(1-s)^{\alpha-2}}{\Gamma(\alpha)}$.

(iii) For $t \in J_\theta$, we divide the proof into the following three cases for $s \in [0,1]$.

Case 1. If $s \in J_\theta$, then from (i) of Lemma 2.5 we have $G_1(t,s) > 0$ and $g_1(s) > 0$ for all $t,s \in J_\theta$. It is obvious that $G_1(t,s)$ and $g_1(s)$ are bounded on J_θ. So, there exists a constant $1 > \gamma_1 > 0$ such that

$$G_1(t,s) \ge \gamma_1 g_1(s) \quad \forall t,s \in J_\theta. \tag{2.15}$$

Case 2. If $s \in [1-\theta, 1]$, then from (2.4) and (ii) of Lemma 2.5 we get

$$\min_{t \in J_\theta} G_1(t,s) = \min_{t \in J_\theta} \frac{t^{\alpha-1}(1-s)^{\alpha-2}}{\Gamma(\alpha)} = \frac{\theta^{\alpha-1}(1-s)^{\alpha-2}}{\Gamma(\alpha)}.$$

Thus, we have

$$\frac{\min_{t \in J_\theta} G_1(t,s)}{g_1(s)} = \frac{\theta^{\alpha-1}}{s} \ge \theta^{\alpha-1},$$

that is,

$$\min_{t \in J_\theta} G_1(t,s) \ge \theta^{\alpha-1} g_1(s). \tag{2.16}$$

Case 3. If $s \in [0,\theta]$, then from (2.4) and (ii) of Lemma 2.5 we obtain

$$\min_{t \in J_\theta} G_1(t,s) = \min_{t \in J_\theta} \frac{t^{\alpha-1}(1-s)^{\alpha-2} - (t-s)^{\alpha-1}}{\Gamma(\alpha)} = \frac{\theta^{\alpha-1}(1-s)^{\alpha-2} - (\theta-s)^{\alpha-1}}{\Gamma(\alpha)}.$$

It is clear that

$$0 < \min_{t \in J_\theta} G_1(t,s) \le 1, \quad 0 < g_1(s) \le 1 \ \forall s \in (0,\theta], t \in J_\theta, \tag{2.17}$$

and

$$\lim_{s \to 0} \frac{\min_{t \in J_\theta} G_1(t,s)}{g_1(s)}$$
$$= \lim_{s \to 0} \frac{\theta^{\alpha-1} - (\theta-s)^{\alpha-1}(1-s)^{2-\alpha}}{s}$$
$$= \lim_{s \to 0} \left[(\alpha-1)(\theta-s)^{\alpha-2}(1-s)^{2-\alpha} + (2-\alpha)(\theta-s)^{\alpha-1}(1-s)^{1-\alpha} \right]$$
$$= \theta^{\alpha-2} \left[(\alpha-1) - \theta(\alpha-2) \right] > 0. \tag{2.18}$$

In the light of (2.17) and (2.18), we conclude that there exists a constant $1 > \gamma_2 > 0$ such that

$$G_1(t,s) \ge \gamma_2 g_1(s) \quad \forall s \in [0,\theta], t \in J_\theta. \tag{2.19}$$

Taking $\gamma = \min\{\gamma_1, \gamma_2, \theta^{\alpha-1}\}$ and applying (2.15), (2.16), and (2.19), we get that (iii) of Lemma 2.5 holds. The proof is complete. □

Lemma 2.6 *If* $0 \leq \delta \triangleq \int_0^1 s^{\alpha-1}\, dH(s) < \alpha - 1$, *then we have*

(i) $G_2(t,s) \geq 0$ *is continuous for all* $t, s \in [0,1]$, *and* $G_2(t,s) > 0$ *for all* $t, s \in (0,1)$;

(ii) $G_2(t,s) \leq \frac{1}{\alpha-1-\delta} \int_0^1 G_1(\tau,s)\, dH(\tau)$ *for all* $t \in [0,1]$, $s \in (0,1)$.

Employing the properties of $G_1(t,s)$ and the definition of $G_2(t,s)$, it is easy to show that (i) and (ii) of Lemma 2.6 hold. So we omit it.

Lemma 2.7 *If* $0 \leq \delta \triangleq \int_0^1 s^{\alpha-1}\, dH(s) < \alpha - 1$, *then the function* $G(t,s)$ *defined by* (2.3) *satisfies*

(i) $G(t,s) \geq 0$ *is continuous for all* $t, s \in [0,1]$, *and* $G(t,s) > 0$ *for all* $t, s \in (0,1)$;

(ii) $G(t,s) \leq g(s)$ *for all* $t, s \in [0,1]$, *and*

$$\min_{t \in [\theta, 1-\theta]} G(t,s) \geq \sigma g(s) \quad \forall s \in [0,1], \tag{2.20}$$

where

$$\sigma = \min\{\gamma, \theta^{\alpha-1}\}, \qquad g(s) = g_1(s) + G_2(1,s) \tag{2.21}$$

with γ *defined in Lemma* 2.5.

Proof (i) From Lemma 2.5 and Lemma 2.6 we obtain that $G(t,s) \geq 0$ is continuous for all $t, s \in [0,1]$ and $G(t,s) > 0$ for all $t, s \in (0,1)$.

(ii) From (ii) of Lemma 2.5 and (ii) of Lemma 2.6 we have that $G(t,s) \leq g(s)$ for all $t, s \in [0,1]$. Now, we show (2.20). Indeed, from Lemma 2.6 we have

$$\min_{t \in J_\theta} G(t,s) \geq \gamma g_1(s) + \frac{\theta^{\alpha-1}}{\alpha-1-\delta} \int_0^1 G_1(\tau,s)\, dH(\tau)$$

$$\geq \sigma \left[g_1(s) + \frac{1}{\alpha-1-\delta} \int_0^1 G_1(\tau,s)\, dH(\tau) \right] = \sigma g(s) \quad \forall s \in [0,1]. \tag{2.22}$$

Then the proof of Lemma 2.7 is completed. □

Let $J_\theta \triangleq [\theta, 1-\theta]$ for $\theta \in (0, \frac{1}{2})$, and let $E = \{u(t) : u(t) \in C([0,1])\}$ be a real Banach space with the norm $\|u\| = \max_{0 \leq t \leq 1} |u(t)|$. Let

$$PC([0,1]) \triangleq \big\{ u \in E | u : [0,1] \to [0,+\infty), u(t_k^-) \text{ and } u(t_k^+)$$

$$\text{exist with } u(t_k^-) = u(t_k), 1 \leq k \leq m \big\},$$

$$K = \Big\{ u \in PC([0,1]) : u \geq 0, \min_{t \in J_\theta} u(t) \geq \sigma \|u\| \Big\}, \tag{2.23}$$

$$K_r = \big\{ u \in K : \|u\| < r \big\}, \qquad \partial K_r = \big\{ u \in K : \|u\| = r \big\}. \tag{2.24}$$

Obviously, $PC([0,1]) \subset E$ is a Banach space with the norm $\|u\| = \max_{t \in [0,1]} |u(t)|$, and $K \subset PC([0,1])$ is a positive cone.

In the following, we need the assumptions and some notation as follows:

(B$_1$) $a \in C((0,1),[0,+\infty))$, and $a(t) \not\equiv 0$ on any subinterval of $(0,1)$.

(B$_2$) $f \in C([0,1] \times [0,+\infty),[0,+\infty))$, and $f(t,0) = 0$ uniformly with respect to t on $[0,1]$.

(B$_3$) $I_k(u(t_k)) \in C([0,+\infty),[0,+\infty))$, $k = 1,2,\ldots,m$.

(B$_4$) $H : [0,1] \to [0,+\infty)$ is of bounded variation with $0 < \delta = \int_0^1 s^{\alpha-1}\, dH(s) < \alpha - 1$.

Let

$$f^\delta = \limsup_{u \to \delta} \max_{t \in [0,1]} \frac{f(t,u)}{u}, \qquad f_\delta = \liminf_{u \to \delta} \min_{t \in [0,1]} \frac{f(t,u)}{u},$$

$$A = \int_0^1 g(s)a(s)\, ds < \infty, \qquad B = \sigma^2 \int_\theta^{1-\theta} g(s)a(s)\, ds < \infty,$$

where δ denotes 0 or $+\infty$. In addition, we introduce the following conditions:

(H$_1$) $f_0 > \frac{1}{\lambda B}$, namely, $\lambda > \frac{1}{Bf_0}$ (particularly, $f_0 = +\infty$, $\lambda > 0$).

(H$_2$) $f_\infty > \frac{1}{\lambda B}$, namely, $\lambda > \frac{1}{Bf_\infty}$ (particularly, $f_\infty = +\infty$, $\lambda > 0$).

(H$_3$) $f^0 < \frac{1}{\lambda A}$, namely, $\lambda < \frac{1}{Af^0}$ (particularly, $f^0 = 0$, $\lambda < +\infty$).

(H$_4$) $f^\infty < \frac{1}{\lambda A}$, namely, $\lambda < \frac{1}{Af^\infty}$ (particularly, $f^\infty = 0$, $\lambda < +\infty$).

(H$_5$) There exists $a > 0$ such that $\min_{t \in J_\theta, u \in [\theta a, a]} f(t,u) > \frac{u}{\lambda B}$, namely,

$$\lambda > \frac{u}{B \min_{t \in J_\theta, u \in [\theta a, a]} f(t,u)}.$$

(H$_6$) There exists $b > 0$ such that $\max_{t \in [0,1], u \in [0,b]} f(t,u) < \frac{b}{\lambda A}$, namely,

$$\lambda < \frac{b}{A \max_{t \in [0,1], u \in [0,b]} f(t,u)}.$$

Remark 2.1 If there exists $a > 0$ such that $\min_{t \in J_\theta, u \in [\theta a, a]} f(t,u) > \frac{a}{\lambda B}$, then (H$_5$) holds.

Remark 2.2 If there exists $b > 0$ such that $\max_{t \in [0,1], u \in [0,b]} f(t,u) < \frac{u}{\lambda A}$, then (H$_6$) holds.

From Lemma 2.4 we obtain the following lemma.

Lemma 2.8 *If* (B$_1$)-(B$_4$) *hold, then BVP* (1.1) *has a solution* $u \in PC([0,1])$ *if and only if* $u \in PC([0,1])$ *is a solution of the integral equation*

$$u(t) = \lambda \int_0^1 G(t,s)a(s)f(s,u(s))\, ds + t^{\alpha-1} \sum_{t \leq t_k < 1} t_k^{1-\alpha} I_k(u(t_k)).$$

Let $T : K \to K$ be the operator defined as

$$(Tu)(t) = \lambda \int_0^1 G(t,s)a(s)f(s,u(s))\, ds + t^{\alpha-1} \sum_{t \leq t_k < 1} t_k^{1-\alpha} I_k(u(t_k)). \tag{2.25}$$

Then, by Lemma 2.8 the fixed point of operator T coincides with the solution of BVP (1.1).

Remark 2.3 If (B$_1$)-(B$_4$) hold, then $(Tu)'(t) \geq 0$ for all $t \in [0,1]$, that is, $(Tu)(t)$ is increasing on $[0,1]$.

Lemma 2.9 *If* (B$_1$)-(B$_4$) *hold, then* $T : K \to K$ *defined by* (2.25) *is completely continuous.*

Proof (1) For any $u \in K$, $t \in [0,1]$, it is clear that $(Tu)(t) \geq 0$. Noting that $0 < \sigma < 1$, for $t \in [0,1]$, we have $(Tu)(t) \geq \sigma(Tu)(t)$, which implies $\min_{t \in J_\theta}(Tu)(t) \geq \max_{t \in [0,1]} \sigma(Tu)(t) = \sigma \|Tu\|$. Therefore, $T(K) \subset K$, that is, $T : K \to K$ is well defined.

(2) Let $u \in K$, in view of the nonnegativeness and continuity of functions $G(t,s)$, $a(t)$, $f(t,u(t))$, $I_k(u)$ and $\lambda > 0$, we conclude that $T : K \to K$ is continuous.

Let $\Omega \subset K$ be any bounded subset in $PC([0,1])$. By the Ascoli-Arzela theorem we only need to show that $T(\Omega)$ is uniformly bounded in $PC([0,1])$ and $T : K \to K$ is equicontinuous. For any $u \in \Omega$, $t,s \in [0,1]$, there exist some constants $L_i > 0$ $(i = 1,2,3,4)$ such that

$$\max_{t,s \in [0,1]} \left|G(t,s)\right| \leq L_1, \qquad \max_{t \in [0,1]} \left|a(t)\right| \leq L_2,$$

$$\max_{u \in \Omega, t \in [0,1]} \left|f(t,u)\right| \leq L_3, \qquad \max_{1 \leq k \leq 1} \max_{u \in \Omega} \left|I_k\left(u(t_k)\right)\right| \leq L_4.$$

Then we have

$$\left|(Tu)(t)\right| \leq \int_0^1 \left|G(t,s)\right|\left[\lambda\left|a(s)\right|\left|f\left(s,u(s)\right)\right|\right]ds + \sum_{k=1}^m \left|I_k\left(u(t_k)\right)\right| \leq \lambda L_1 L_2 L_3 + m L_4.$$

Hence, $T(\Omega)$ is uniformly bounded.

Next, we will prove that $S : K \to K$ is equicontinuous. Indeed, for any $u \in \Omega$ and $t_1, t_2 \in [0,1]$, we have

$$\left|(Tu)(t_2) - (Tu)(t_1)\right|$$

$$= \lambda \int_0^1 \left|G(t_2,s) - G(t_1,s)\right| a(s) f\left(s,u(s)\right) ds$$

$$+ \left|\sum_{t_2 \leq t_k < 1}\left(\frac{t_2}{t_k}\right)^{\alpha-1} I_k\left(u(t_k)\right) - \sum_{t_1 \leq t_k < 1}\left(\frac{t_1}{t_k}\right)^{\alpha-1} I_k\left(u(t_k)\right)\right| \to 0 \quad \text{as } t_1 \to t_2.$$

Thus, it follows from the continuity of $G(t,s)$ that for any $\epsilon > 0$, there exists a positive constant $\delta = \delta(\epsilon) > 0$, independent of t_1, t_2, and u, such that $|(Tu)(t_2) - (Tu)(t_1)| < \epsilon$ whenever $|t_2 - t_1| < \delta$. Thereby, $T : K \to K$ is equicontinuous. The proof is complete. $\qquad\square$

3 Single positive solutions and eigenvalue intervals

In this section, employing the Schauder fixed-point theorem, we derive the existence of one positive solution for BVP (1.1) under weak assumptions, which improved the results of [10].

Theorem 3.1 *Assume that* (B_1)-(B_4) *hold. If* (H_3) *or* (H_4) *is satisfied, then BVP* (1.1) *has at least one positive solution.*

Proof If the condition (H_3) holds, considering $f^0 < \frac{1}{\lambda A}$, there exists $\varepsilon_1 > 0$ such that $(f^0 + \varepsilon_1)\lambda A \leq 1$. From the definition of f^0, there exists $r_1 > 0$ such that $f(t,u) \leq (f^0 + \varepsilon_1)u$ for all $0 \leq u \leq r_1$, $t \in [0,1]$. Let $u \in \Omega_1 \triangleq K_{r_1}$ be defined as in (2.24). It is easy to know that Ω_1 is a close bounded convex subset of a Banach space $PC([0,1])$. Then, for $t \in [0,1]$ and $u \in \Omega_1$, in view of the nonnegativeness and continuity of functions $G(t,s)$, $a(t)$, $f(t,u(t))$, $I_k(u(t_k))$

and $\lambda > 0$, we conclude that $Tu \in PC([0,1])$, $Tu \geq 0$, $t \in [0,1]$. According to Lemma 2.9, we have $(Tu)(t) \in K$. Next, we will prove that $\|Tu\| < r_1$. In fact, by Remark 2.3 we have

$$
\begin{aligned}
\|Tu\| &= \max_{t \in [0,1]} (Tu)(t) = (Tu)(1) = \lambda \int_0^1 G(1,s)a(s)f\big(s,u(s)\big)\,ds \\
&\leq \lambda \int_0^1 g(s)a(s)\big(f^0 + \varepsilon_1\big)u(s)\,ds \leq \|u\|\big(f^0 + \varepsilon_1\big)\lambda \int_0^1 g(s)a(s)\,ds \\
&= \|u\|\big(f^0 + \varepsilon_1\big)\lambda A \leq \|u\| < r_1.
\end{aligned}
$$

Therefore, $T : \Omega_1 \to \Omega_1$. From Lemma 2.9 we have $T : \Omega_1 \to \Omega_1$ is completely continuous. Thus, BVP (1.1) has at least one positive solution by Lemma 2.2.

If condition (H_4) holds, the proof is similar to the previous arguments. So we omit it. The proof is complete. □

Theorem 3.2 *Assume that* (B_1)-(B_4) *hold. Suppose that one of the following conditions is satisfied*:

(A_1) *there exists a constant $M > 0$ such that $f(t,u) \leq \frac{M}{\lambda A}$ for $0 \leq t \leq 1$, $0 \leq u \leq M$;*
(A_2) *there exists a constant $N > 0$ such that $f(t,u) \leq \frac{N}{\lambda A}$ for $0 \leq t \leq 1$, $u \geq N$.*

Then BVP (1.1) *has at least one positive solution.*

Proof If condition (A_1) holds, then we take $\Omega_2 \triangleq K_M = \{u \in PC([0,1]) : \|u\| < M\}$. If the condition (A_2) holds, then we take $\Omega_3 \triangleq K_d = \{u \in PC([0,1]) : \|u\| < d\}$, where $d > 0$ satisfies $d \geq 1 + N + \lambda A \max_{0 \leq t \leq 1, 0 \leq u \leq N} f(t,u)$. The rest of the proof is similar to that of Theorem 3.1. So we omit it. The proof is complete. □

4 Multiple positive solutions and eigenvalue intervals

In this section, applying the fixed-point index theorem, we will discuss the multiplicity of positive solutions for BVP (1.1).

Theorem 4.1 *Assume that* (B_1)-(B_4) *hold. If* (H_1), (H_2), *and* (H_6) *are satisfied. Then BVP* (1.1) *has at least two positive solutions u_1, u_2 with*

$$
0 < \|u_1\| < b < \|u_2\|,
\tag{4.1}
$$

and the parameter λ satisfies

$$
\max\left\{\frac{1}{Bf_0}, \frac{1}{Bf_\infty}\right\} < \lambda < \frac{b}{A \max_{t \in [0,1], u \in [0,b]} f(t,u)}.
\tag{4.2}
$$

Proof To begin with, we consider condition (H_6). Choose r, R with $0 < r < b < R$ and let $u \in \Omega_b \triangleq K_b$. For any $u \in \partial\Omega_b$, we have $\|u\| = b$ and $0 \leq u(t) \leq b$ for all $t \in [0,1]$. By condition (H_6), for $u \in \partial\Omega_b$ and $t \in [0,1]$, we get

$$
\begin{aligned}
\|Tu\| &= \max_{t \in [0,1]} (Tu)(t) = (Tu)(1) \\
&= \lambda \int_0^1 G(1,s)a(s)f\big(s,u(s)\big)\,ds
\end{aligned}
$$

$$\leq \lambda \int_0^1 g(s)a(s) \max_{s\in[0,1],u\in[0,b]} f(s,u(s))\, ds$$

$$< \lambda \int_0^1 g(s)a(s) \frac{b}{\lambda A}\, ds = b = \|u\|.$$

Therefore,

$$\|Tu\| < \|u\|, \quad u \in \partial\Omega_b. \tag{4.3}$$

By Lemma 2.3 we have

$$i(T,\Omega_b,K) = 1. \tag{4.4}$$

On the one hand, by condition (H$_1$) there exists $\epsilon_1 > 0$ that satisfies $(f_0 - \epsilon_1)\lambda B \geq 1$. By the definition of f_0 there exists $r > 0$ such that $f(t,u) \geq (f_0 - \epsilon_1)u$ for $u \in [0,r]$ and $t \in [0,1]$. Together with (2.6), let $u \in \Omega_r \triangleq K_r$. For $u \in \partial\Omega_r$ and $t \in [0,1]$, we have

$$\|Tu\| = \max_{t\in[0,1]} (Tu)(t) = (Tu)(1) = \lambda \int_0^1 G(1,s)a(s)f(s,u(s))\, ds$$

$$\geq \lambda \int_0^1 G(1,s)a(s)(f_0 - \epsilon_1)u(s)\, ds \geq (f_0 - \epsilon_1)\lambda\sigma \int_\theta^{1-\theta} G(1,s)a(s)u(s)\, ds$$

$$\geq (f_0 - \epsilon_1)\lambda\sigma^2 \int_\theta^{1-\theta} g(s)a(s)\, ds\|u\| = (f_0 - \epsilon_1)\lambda B\|u\| \geq \|u\|.$$

So

$$\|Tu\| \geq \|u\|, \quad u \in \partial\Omega_r. \tag{4.5}$$

By Lemma 2.3 we have

$$i(T,\Omega_r,K) = 0. \tag{4.6}$$

On the other hand, by condition (H$_2$) there exists $\epsilon_2 > 0$ that satisfies $(f_\infty - \epsilon_2)\lambda B \geq 1$. By the definition of f_∞ there exists $m > 0$ such that $f(t,u) \geq (f_\infty - \epsilon_2)u$ for $u \in (m,+\infty)$ and $t \in [0,1]$. Together with (2.6), let $u \in \Omega_R \triangleq K_R$. For $u \in \partial\Omega_R$ and $t \in [0,1]$, we obtain

$$\|Tu\| = \max_{t\in[0,1]} (Tu)(t) = (Tu)(1) = \lambda \int_0^1 G(1,s)a(s)f(s,u(s))\, ds$$

$$\geq \lambda \int_0^1 G(1,s)a(s)(f_\infty - \epsilon_2)u(s)\, ds \geq (f_\infty - \epsilon_2)\lambda\sigma \int_\theta^{1-\theta} G(1,s)a(s)u(s)\, ds$$

$$\geq (f_\infty - \epsilon_2)\lambda\sigma^2 \int_\theta^{1-\theta} g(s)a(s)\, ds\|u\| = (f_\infty - \epsilon_2)\lambda B\|u\| \geq \|u\|.$$

Therefore,

$$\|Tu\| \geq \|u\|, \quad u \in \partial\Omega_R. \tag{4.7}$$

By Lemma 2.3 we have

$$i(T, \Omega_R, K) = 0. \tag{4.8}$$

Combining (4.4) with (4.6) and (4.8) with (4.6), we get

$$i(T, \Omega_b \setminus \overline{\Omega}_r, K) = i(T, \Omega_b, K) - i(T, \Omega_r, K)$$
$$= 1 - 0 = 1 \tag{4.9}$$

and

$$i(T, \Omega_R \setminus \overline{\Omega}_b, K) = i(T, \Omega_R, K) - i(T, \Omega_b, K)$$
$$= 0 - 1 = -1. \tag{4.10}$$

By (4.9) and (4.10), T has a fixed point $u_1 \in \Omega_b \setminus \overline{\Omega}_r$ and a fixed point $u_2 \in \Omega_R \setminus \overline{\Omega}_b$. Thus, it follows that BVP (1.1) has at least two positive solutions u_1 and u_2. Noticing (4.3), we have $\|u_1\| \neq b$ and $\|u_2\| \neq b$. So (4.1) holds. Combing (H$_1$), (H$_2$), and (H$_6$), we derive (4.2). The proof is complete. $\square$

Similarly, we have the following results.

Theorem 4.2 *Assume that* (B$_1$)-(B$_4$) *hold. If* (H$_3$), (H$_4$), *and* (H$_5$) *are satisfied, then BVP* (1.1) *has at least two positive solutions* u_1, u_2 *with*

$$0 < \|u_1\| < a < \|u_2\|,$$

and the parameter λ satisfies

$$\frac{a}{B \min_{t \in J_\theta, u \in [\theta a, a]} f(t, u)} < \lambda < \min\left\{ \frac{1}{Af^0}, \frac{1}{Af^\infty} \right\}.$$

Theorem 4.3 *Assume that* (B$_1$)-(B$_4$) *hold. Suppose that there exist $2m$ positive numbers a_k, b_k ($k = 1, 2, \ldots, m$) with $0 < \theta a_1 < a_1 < \theta b_1 < b_1 < \theta a_2 < a_2 < \cdots < \theta a_m < a_m = a < \theta b_m < b_m = b$ such that the following two conditions are satisfied:*

(A$_3$) $\min_{t \in J_\theta, u \in [\theta a_k, a_k]} f(t, u) > \frac{a_k}{\lambda B}$;
(A$_4$) $\max_{t \in [0,1], u \in [\theta b_k, b_k]} f(t, u) < \frac{b_k}{\lambda A}$.

Then BVP (1.1) *has at least m positive solutions u_k satisfying with $a_k < \|u_k\| < \theta b_k$ ($k = 1, 2, \ldots, m$), and the parameter λ satisfies*

$$\frac{a_k}{B \min_{t \in J_\theta, u \in [\theta a_k, a_k]} f(t, u)} < \lambda < \frac{b_k}{A \max_{t \in [0,1], u \in [\theta b_k, b_k]} f(t, u)}$$

for $k = 1, 2, \ldots, m$.

Theorem 4.4 *Assume that* (B$_1$)-(B$_4$) *hold. Suppose that there exist n ($n \geq 2$) positive numbers b_i ($i = 1, 2, \ldots, n$) with $0 < \theta b_1 < b_1 < \theta b_2 < b_2 < \cdots < \theta b_n < b_n < b$ such that the following two conditions are satisfied:*

(A$_5$) $\max_{t\in[0,1],u\in\partial D_0} f(t,u) < \frac{b}{\lambda A}$;

(A$_6$) $\max_{t\in[0,1],u\in\partial D_i} f(t,u) < \frac{b_i}{\lambda A}$.

Then BVP (1.1) *has at least* $n+1$ *positive solutions* u_i $(i=0,1,\ldots,n)$ *with*

$$u_i \in D_i, \qquad u_0 \in D_0 \backslash \bigcup_{i=1}^{n} \overline{D}_i,$$

where $D_0 = \{u \in K : u \in [0,b]\}$, $D_i = \{u \in K : u \in [\theta b_i, b_i]\}$, $1 \le i \le n$.

Proof First, for each D_i $(i=1,2,\ldots,n)$, if (A$_6$) holds, then it follows from Theorem 4.1 that $i(T,D_i,K) = 1$. Therefore, BVP (1.1) has at least one positive solution $u_i \in D_i$. Thus, we show that BVP (1.1) has at least n positive solutions.

On the other hand, if (A$_5$) holds, then we get

$$i\left(T,D_0\backslash\bigcup_{i=1}^{n}\overline{D}_i,K\right) = i(T,D_0,K) - \sum_{i=1}^{n} i(T,D_i,K) = 1 - n \neq 0.$$

Thus, the operator T has a fixed point $u_0 \in D_0\backslash\bigcup_{i=1}^{n}\overline{D}_i$, that is, u_0 is the $(n+1)$th positive solution of BVP (1.1). This completes the proof. $\qquad\square$

5 Illustrative examples

Example 5.1 Consider the following boundary value problem:

$$\begin{cases} -D_{0^+}^{\frac{5}{2}}u(t) = \lambda\frac{u(t)(1+\sin(\pi t))}{1+u^2(t)}, & t \in (0,1) \backslash \{\frac{1}{2}\}, \\ \Delta u(\frac{1}{2}) = I(u(\frac{1}{2})), & t = \frac{1}{2}, \\ u(0) = u'(0) = 0, & u'(1) = \int_0^1 u(s)\,dH(s), \end{cases} \tag{5.1}$$

where $\alpha = \frac{5}{2}$, $k = 1$, $t_1 = \frac{1}{2}$, $a(t) = 1$, $f(t,u) = \frac{u(t)(1+\sin(\pi t))}{1+u^2(t)}$. Let $I(u) = \frac{u^2}{1+u^2}$, $H(s) = \frac{5}{2}s$. It is easy to verify that (B$_1$)-(B$_4$) hold. By simple calculation we have

$$f^\infty = \limsup_{u\to\infty}\max_{t\in[0,1]}\frac{u(1+\sin(\pi t))}{u(1+u^2)} = \limsup_{u\to\infty}\frac{2}{1+u^2} = 0.$$

Thus, all the assumptions of Theorem 3.1 are satisfied. Hence, BVP (5.1) has at least one positive solution for $\lambda > 0$.

Remark 5.1 Noting that $f_0 = \liminf_{u\to 0}\min_{t\in[0,1]}\frac{u(1+\sin(\pi t))}{u(1+u^2)} = \liminf_{u\to 0}\frac{1}{1+u^2} = 1 \neq \infty$, we do not conclude that BVP (5.1) has positive solutions by applying the results of [10].

Example 5.2 Consider the nonlinear fractional differential equations

$$\begin{cases} -D_{0^+}^{\frac{5}{2}}u(t) = \lambda\left|\frac{u(t)\ln u(t)}{1+2t^2}\right|, & t \in (0,1) \backslash \{\frac{1}{2}\}, \\ \Delta u(\frac{1}{2}) = I(u(\frac{1}{2})), & t = \frac{1}{2}, \\ u(0) = u'(0) = 0, & u'(1) = \int_0^1 u(s)\,dH(s), \end{cases} \tag{5.2}$$

where $\alpha = \frac{5}{2}$, $k = 1$, $t_1 = \frac{1}{2}$, $a(t) = 1$, $f(t,u) = |\frac{u(t)\ln u(t)}{1+2t^2}|$. Let $I(u) = \frac{u}{1+u}$, $H(s) = \frac{5}{2}s$, $\theta = \frac{1}{4} \in (0, \frac{1}{2})$, $b = 1$. It is easy to verify that (B$_1$)-(B$_4$) hold. By simple computation we have

$$\delta = \int_0^1 s^{\alpha-1}\,dH(s) = 1, \qquad g_1(s) = \frac{s(1-s)^{\frac{1}{2}}}{\Gamma(\frac{5}{2})},$$

$$G_1(t,s) = \begin{cases} \frac{t^{\frac{3}{2}}(1-s)^{\frac{1}{2}}-(t-s)^{\frac{3}{2}}}{\Gamma(\frac{5}{2})}, & 0 \le s \le t \le 1, \\ \frac{t^{\frac{3}{2}}(1-s)^{\frac{1}{2}}}{\Gamma(\frac{5}{2})}, & 0 \le t \le s \le 1, \end{cases}$$

$$G_2(1,s) = \frac{1}{\alpha - 1 - \delta}\int_0^1 G_1(\tau,s)\,dH(\tau) = \frac{2[(1-s)^{\frac{1}{2}} - (1-s)^{\frac{5}{2}}]}{\Gamma(\frac{5}{2})},$$

$$g(s) = g_1(s) + G_2(1,s) = \frac{s(1-s)^{\frac{1}{2}} + 2[(1-s)^{\frac{1}{2}} - (1-s)^{\frac{5}{2}}]}{\Gamma(\frac{5}{2})},$$

$$A = \int_0^1 g(s)a(s)\,ds = \int_0^1 \frac{s(1-s)^{\frac{1}{2}} + 2[(1-s)^{\frac{1}{2}} - (1-s)^{\frac{5}{2}}]}{\Gamma(\frac{5}{2})}\,ds = \frac{36}{35\Gamma(\frac{5}{2})} \approx 0.3094984,$$

$$0 < B = \sigma^2 \int_\theta^{1-\theta} g(s)a(s)\,ds \le \theta^{2(\alpha-1)}\int_\theta^{1-\theta} g(s)a(s)\,ds < A < +\infty,$$

$$f_0 = \liminf_{u\to 0}\min_{t\in[0,1]}\left|\frac{u\ln u}{(1+2t^2)u}\right| = \liminf_{u\to 0}\frac{|\ln u|}{3} = +\infty,$$

$$f_\infty = \liminf_{u\to\infty}\min_{t\in[0,1]}\left|\frac{u\ln u}{(1+2t^2)u}\right| = \liminf_{u\to\infty}\frac{|\ln u|}{3} = +\infty.$$

When $0 \le u(t) \le 1$, $0 \le t \le 1$, $f(t,u) = \frac{-u\ln u}{1+2t^2}$ arrives at maximum at $u = \frac{1}{e}$, $t = 0$. Therefore, we have

$$\max_{t\in[0,1],u\in[0,1]} f(t,u) = \max_{t\in[0,1],u\in[0,1]}\frac{-u\ln u}{1+2t^2} = \frac{1}{e} \approx 0.3678794,$$

$$\frac{b}{A\max_{t\in[0,1],u\in[0,b]} f(t,u)} = \frac{1}{0.3094984 \times 0.3678794} \approx 8.7828623.$$

Thus, by Theorem 4.1 it follows that BVP (5.2) has at least two positive solutions u_1, u_2 satisfying $0 < \|u_1\| < 1 < \|u_2\|$ for $0 < \lambda < 8.7828623$.

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