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Structure of the solution set for a partial differential inclusion

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Abstract

In this paper, we consider the biharmonic problem of a partial differential inclusion with Dirichlet boundary conditions. We prove existence theorems for related partial differential inclusions with convex and nonconvex multivalued perturbations, and obtain an existence theorem on extremal solutions, and a strong relaxation theorem. Also we prove that the solution set is compact R_δ if the perturbation term of the related partial differential inclusion is convex, and its solution set is path-connected if the perturbation term is nonconvex.

MSC: 34B15; 34B16; 37J40

Keywords: biharmonic problem; differential inclusion; set-valued mapping; path-connected; compact R_δ

1 Introduction

In this paper, we examine the following biharmonic problem of the partial differential inclusion:

$$\begin{cases} \Delta^2 u \in H(x, u, \nabla u, \Delta u) & \text{a.e. in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \frac{\partial u}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Here Ω is a bounded domain in R^N with a smooth boundary $\partial\Omega$, and $H : \Omega \times R \times R^N \times R \to 2^R \setminus \{\emptyset\}$ is a set-valued map. Biharmonic equations with Dirichlet boundary conditions were studied by Lions-Magenes [1, 2], Mozolevski-Süli [3, 4], Amrouche-Fontes [5], and Amrouche-Raudin [6, 7]. Boundary value problems involving partial differential equations with discontinuous nonlinearities which may be reduced to boundary value problems for partial differential inclusions were studied by Carl-Heikkilä [8, 9] and Chang [10, 11] (we refer the reader also to the work of Marano [12, 13]). In [14], Xue-Cheng studied periodic problems for a nonlinear evolution inclusion, defined on an evolution triple of spaces, driven by a monotone operator, and with a perturbation term which is multivalued. They established existence theorems for periodic solutions, extremal periodic solutions and a strong relaxation theorem in Banach spaces, which are similar to those in Xue-Yu [15] in infinite dimensional spaces. In [16], Cheng-Cong-Xue considered the following boundary value problem:

$$\begin{cases} \Delta u \in G(x, u, \nabla u) & \text{a.e. in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

and they established the existence of solutions for inclusions with convex- and nonconvex-valued perturbations, extremal solutions, and a strong relaxation theorem in a strong solution sense. The multivalued term in problem (1.1) that we consider contains not only the gradient but also a Laplacian item, and we obtain results in a weak solution sense. The Lipschitz condition ($H(F)_4$(iv) in [16]) with respect to the second variable u of the multifunction G is essential to get a strong relaxation theorem in Cheng-Cong-Xue [16]. However, we only need a one-sided Lipschitz condition to get this result. Furthermore, the topological structure of the solution set is discussed (this is not considered in [16] and [14]).

Inspired by Cheng-Cong-Xue [16], in this paper we prove existence theorems for both 'convex' and 'nonconvex' cases by using techniques from multivalued analysis and fixed point theory. For related works on this subject, we refer the reader to [14, 17–20] and the references therein. Based on the Baire category method, De Blasi and Pianigiani in [21] gave an existence result for the following problem:

$$\begin{cases} \nabla u \in \operatorname{ext} F(x, u) & \text{a.e. } x \in \Omega, \\ u(x) = \varphi(x), & x \in \partial\Omega. \end{cases} \tag{1.3}$$

In this paper, we will also consider the differential inclusion in which $H(x, u, \nabla u, \Delta u)$ will be replaced by its extreme point set $\operatorname{ext} H(x, u, \nabla u, \Delta u)$. We show that the resulting problem always has a solution ('extremal solutions') and the solution set is dense in the solution set of the convexified version of the problem ('strong relaxation theorem'). Also we address the structural properties of the solution sets for this type of biharmonic inclusion problem. In [22], Himmelberg and Van Vleck studied the topological structure of the solution set in R^N for the ordinary differential inclusions:

$$\dot{x}(t) \in F\bigl(t, x(t)\bigr), \qquad x(0) = 0,$$

and proved that the solution set is an R_δ-set. For Cauchy problems the topological structure of the solution set of evolution inclusions was examined by Bothe [23], Andres-Pavlackova [24], Gabor-Grudzka [25], and Chen-Wang-Zhou [26] in a Banach space, Bakowska-Gabor [27], and O'Regan [28] in Fréchet spaces.

We also refer the reader to the works of Papageorgiou-Shahzad [29] for the first-order evolution inclusion and Papageorgiou-Yannakakis [30] for the second-order evolution inclusion where the structure of solution sets was discussed. Following their lead, in this paper, we obtain the R_δ-structure of the solution set for a biharmonic differential inclusion based on the space variable $x \in \Omega$. We prove that the solution set of the biharmonic inclusion problem in the convex-valued case is compact R_δ in $C(\overline{\Omega})$, and the solution set is path-connected in the case of a nonconvex-valued orientor field.

The plan of our paper is as follows. In Section 2, we collect some preliminary results which will be used in this work. In Section 3, we present some basic assumptions and existence theorems for the both convex and nonconvex multivalued terms. Here, our results are based on the Leray-Schauder alternative. In Section 4, a relaxation theorem is established. Finally the properties of the solution set is given in Section 5.

2 Preliminaries

In this section, we introduce some basic definitions and facts which are essential tools in the later sections; see Hu-Papageorgiou [31] for details.

Let R^N ($N \geq 1$) be the N-dimensional real Euclidean space. Throughout this paper the symbol Ω denotes a nonempty, bounded, open set of R^N, with a smooth boundary $\partial\Omega$. Moreover, from now on, 'measurable' simply means Lebesgue measurable. Given two nonnegative constants $k,p \geq 1$, we denote by $W^{k,p}(\Omega)$ the space of all real-valued functions defined on Ω whose weak partial derivatives up to the order k lie in $L^p(\Omega)$, equipped with $W^{k,p}(\Omega)$ the usual norm $\|\cdot\|_{k,p}$. If $u \in W^{2,p}(\Omega)$, we set $\Delta u = \sum_{i=1}^n \frac{\partial^2 u}{\partial x_i{}^2}$, $\nabla u = \operatorname{grad} u = (\frac{\partial u}{\partial x_i})_{i=1}^N$. For any real number $p > 1$, we denote by q the dual exponent of p (and throughout the paper we assume $p > 1$).

Let V be a Hausdorff topological space and a multifunction $F : \Omega \to 2^V \setminus \{\emptyset\}$. We introduce the following notations:

$$\mathcal{P}_k(V) = \{D \subset V : D \text{ is a nonempty compact subset of } V\},$$

$$\mathcal{P}_c(V) = \{D \subset V : D \text{ is a nonempty and convex subset of } V\},$$

$$\mathcal{P}_{fc}(V) = \{D \subset V : D \text{ is a nonempty, closed. and convex subset of } V\},$$

$$\mathcal{P}_{(w)kc}(V) = \bigl\{D \subset V : D \text{ is a nonempty, (weakly) compact, and convex subset of } V\bigr\}.$$

Definition 2.1 Let X be a Banach space. A multifunction $F : \Omega \to \mathcal{P}_f(X)$ is said to be 'measurable', if for all $y \in X$, the R_+-valued function $x \to d(y, F(x)) = \inf\{\|y - v\|, v \in F(x)\}$ is measurable.

The above definition of measurability is equivalent to saying that

$$\operatorname{Gr} F = \bigl\{(x, v) \in \Omega \times X : v \in F(x)\bigr\} \in \Sigma \times B(X),$$

with $B(X)$ being the Borel σ-field of X, Σ is Lebesgue σ-field of Ω, that is, $x \to F(x)$ is graph measurable. In general, however, we can only say that measurability implies graph measurability.

Definition 2.2 A generalized metric known in the literature as the 'Hausdorff metric', is obtained by setting

$$h(A,B) = \max\Bigl\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\Bigr\}$$

for all $A, B \in \mathcal{P}_{kc}(V)$.

Definition 2.3 Let Y, Z be Hausdorff topological spaces and $\beta : Y \to 2^Z \setminus \{\emptyset\}$. $\beta(\cdot)$ is called 'upper semicontinuous (USC)' (resp., 'lower semicontinuous (LSC)'), if for any nonempty closed set $C \subseteq Z$, $\beta^-(C) = \{y \in Y : \beta(y) \cap C \neq \emptyset\}$ (resp., $\beta^+(C) = \{y \in Y : \beta(y) \subseteq C\}$) is closed in Y.

A multifunction which is both USC and LSC is said to be continuous. From Theorem 2.1 and Remark 2.6 of [32], we note the following result.

Theorem 2.1 *Let X be a Banach Space with the weak topology, and $D \subseteq X$ a weakly compact, convex subset of X, Then any weakly sequentially upper semicontinuous map $F : D \to \mathcal{P}_{wkc}(D)$ has a fixed point, i.e., there exists $x \in D$, such that $x \in F(x)$.*

Remark 2.1 Recall $F : D \to \mathcal{P}_{wkc}(D)$ is weakly sequentially upper semicontinuous if for any weakly closed set A of D, $F^{-1}(A)$ is sequentially closed for the weak topology on D.

We now use Theorem 2.1 to obtain the following result.

Lemma 2.1 *Let Ω be a nonempty, closed, convex subset of a Banach space X. Suppose $F : \Omega \to \mathcal{P}_c(\Omega)$ has a weakly sequentially closed graph and $F(\Omega)$ is weakly relatively compact. Then F has a fixed point.*

Proof Let $D = \overline{\mathrm{co}}(F(\Omega))$. It follows from the Krein-Šmulian theorem that D is a weakly compact convex set. Note $F(\Omega) \subseteq D \subseteq \Omega$. Notice also that $T = F|_D : D \to \mathcal{P}_c(D)$. First we prove that $\mathrm{Gr}\, T$ is weakly compact. Since $(X \times X)_w = X_w \times X_w$ (X_w is the space X endowed its weak topology), it follows that $D \times D$ is a weakly compact subset of $X \times X$. Also, $\mathrm{Gr}\, T = \{(x, y) \in D \times X : y \in T(x)\} \subset D \times D$, so, $\mathrm{Gr}\, T$ is weakly relatively compact. Let $(x, y) \in D \times D$ be weakly adherent to $\mathrm{Gr}\, T$. Then from the Eberlein-Šmulian theorem we can find $\{(\{x_n\}, \{y_n\})\}_n \subseteq \mathrm{Gr}\, T$ such that $y_n \in T(x_n)$, $x_n \to x$ weakly and $y_n \to y$ weakly in X. Because F has weakly sequentially closed graph, $y \in T(x)$ and so $(x, y) \in \mathrm{Gr}\, T$. Therefore, $\mathrm{Gr}\, T$ is a weakly closed subset of $D \times D$ and so weakly compact. Consequently $T(x)$ is weakly closed and so a weakly compact subset of D for every $x \in D$. In view of Theorem 2.1 it suffices to show that T is weakly sequentially upper semicontinuous. First we note that $\mathrm{Gr}\, T$ is weakly closed and therefore is sequentially weakly closed. Let $A \subset D$ be a weakly closed set and let $x_n \in T^{-1}(A)$ with $x_n \to x$ weakly. Now since $T(x_n) \cap A \neq \emptyset$ and $T(x_n) \subset D$, then for $y_n \in T(x_n) \cap A$ we may assume $y_n \to y$ weakly for some $y \in A$. Since $(x_n, y_n) \in \mathrm{Gr}\, T$ and $\mathrm{Gr}\, T$ is sequentially weakly closed, we have $y \in T(x) \cap A$ and so $x \in T^{-1}(A)$. Thus $T^{-1}(A)$ is sequentially weakly closed. Applying Theorem 2.1, we see that T has a fixed point $x \in D \subset \Omega$. Therefore F has a fixed point. □

Let X be a Banach space and $L^p(\Omega, X)$ be the Banach space of all functions $u : \Omega \to X$ which are Bochner integrable. Let $D(L^p(\Omega, X))$ be the collection of nonempty decomposable subsets of $L^p(\Omega, X)$. Next is the Bressan-Colombo continuous selection theorem.

Lemma 2.2 (see, *e.g.*, [33]) *Let X be a separable metric space and let $F : X \to D(L^p(\Omega, X))$ be a LSC multifunction with closed decomposable values. Then F has a continuous selection.*

Let X be a separable Banach Space and $C(\Omega, X)$ be the Banach space of all continuous functions. A multifunction $F : \Omega \times X \to \mathcal{P}_{wkc}(X)$ is said to be of Carathéodory type, if for every $u \in X$, $F(\cdot, u)$ is measurable, and for almost all $x \in \Omega$, $F(x, \cdot)$ is h-continuous. A nonempty subset $\eta_0 \subset C(\Omega, X)$ is called σ-compact if there is a sequence $\{\eta_k\}_{k\geq 1}$ of compact subsets η_k such that $\eta_0 = \bigcup_{k\geq 1}\eta_k$. Let $\eta_0 \subset \eta$, such that η_0 is dense in η and σ-compact. The following continuous selection theorem in the extreme point case is due to Tolstonogov [34].

Lemma 2.3 (see, *e.g.*, [34]) *Let the multifunction $F : \Omega \times X \to \mathcal{P}_{wkc}(X)$ be of Carathéodory type and be integrably bounded. Then there exists a continuous function $g : \eta \to L^p(\Omega, X)$ such that for almost $x \in \Omega$, if $u(\cdot) \in \eta_0$, then $g(u)(x) \in \operatorname{ext} F(x, u(x))$, and if $u(\cdot) \in \eta \setminus \eta_0$, then $g(u)(x) \in \overline{\operatorname{ext}} F(x, u(x))$.*

Definition 2.4 A nonempty subset D of Y is said to be contractible if there exist a point $y_0 \in D$ and a continuous function $h : [0,1] \times D \to D$ such that $h(0, y) = y_0$ and $h(1, y) = y$ for every $y \in D$.

Definition 2.5 A subset D of a metric space is called an R_δ-set if there exists a decreasing sequence $\{D_n\}$ of compact and contractible sets such that

$$D = \bigcap_{n=1}^{\infty} D_n.$$

Note that a compact R_δ set D is nonempty, compact, and connected. However, in contrast to contractible sets, a compact R_δ set D need not be path-connected. We also need the following approximation result that can be proved from Proposition 4.1 of [29] with minor modifications to accommodate the presence of $x \in \Omega$.

Lemma 2.4 *Let $G : \Omega \times R \times R^N \to \mathcal{P}_{fc}(R)$ be a multifunction such that*

(i) *$\forall (u, s) \in R \times R^N$, $x \to G(x, u, s)$ is measurable*;
(ii) *$\forall x \in \Omega$, $(u, s) \to G(x, u, s)$ is USC*;
(iii) *$\forall (x, u, s) \in \Omega \times R \times R^N$, $|G(x, u, s)| \leq \varphi(x)$ a.e. with $\varphi(x) \in L^q_+(\Omega)$.*

Then there exists a sequence of multifunctions $G_n : \Omega \times R \times R^N \to \mathcal{P}_{fc}(R)$, $n \geq 1$ with the following properties:

(a) *For every $x \in \Omega$, and $(u, s) \in R \times R^N$ there exist $\mu_n(u, s) > 0$ and $\varepsilon_n > 0$ such that if $u_1, u_2 \in B_{\varepsilon_n}(u) = \{y \in R : |u - y| \leq \varepsilon_n\}$, $s_1, s_2 \in B_{\varepsilon_n}(s)$, then $h(G_n(x, u_1, s_1), G_n(x, u_2, s_2)) \leq \mu_n(x, u, s)\varphi(x)(|u_1 - u_2| + \|s_1 - s_2\|)$ a.e. (i.e., $G_n(x, u, s)$ is locally h-Lipschitz with respect to (u, s)).*
(b) *$G(x, u, s) \subseteq \cdots \subseteq G_n(x, u, s) \subseteq G_{n-1}(x, u, s) \subseteq \cdots$, $|G_1(x, u, s)| \leq \varphi(x)$ a.e. $n \geq 1$, $G_n(x, u, s) \to G(x, u, s)$ as $n \geq 1$ for every $(x, u, s) \in \Omega \times R \times R^N$, and finally there exists $g_n : \Omega \times R \times R^N \to R$, measurable in x, locally Lipschitz in (u, s) and $g_n(x, u, s) \in G_n(x, u, s)$ for every $(x, u, s) \in \Omega \times R \times R^N$. Moreover, if $G(x, \cdot, \cdot)$ is h-continuous, then $x \to G_n(x, u, s)$ is measurable (hence $(x, u, s) \to G_n(x, u, s)$ is measurable too*; *see* [35]).

Theorem 2.2 (see, *e.g.*, [36]) *Let X and Y be two normed spaces. If $T : X \to Y$ is a compact linear operator and $\{x_n\}_n$ is a sequence in X such that $x_n \to x$ weakly then $T(x_n) \to T(x)$ strongly.*

3 Existence theorems of solutions

Definition 3.1 A function $u \in W_0^{2,p}(\Omega)$ is called a solution of problem (1.1) if

$$\int_\Omega \Delta u \cdot \Delta v \, dx = \langle f, v\rangle, \quad \forall v \in W_0^{2,q}(\Omega),$$

where $f(x) \in H(x, u, \nabla u, \Delta u)$ and $\langle \cdot, \cdot \rangle$ denotes (here and in the sequel) the duality pairing between $W_0^{-2,p}(\Omega)$ and $W_0^{2,q}(\Omega)$.

Consider the kernel $D_0^p(\Omega)$ of a biharmonic operator with Dirichlet boundary conditions:

$$D_0^p(\Omega) := \left\{ z \in W_0^{2,p}(\Omega) : \Delta^2 z = 0, z = \frac{\partial z}{\partial \mathbf{n}} = 0 \text{ on } \partial\Omega \right\}, \tag{3.1}$$

where $\mathbf{n}$ is the unit vector normal to $\partial\Omega$ pointing outside Ω. Then $W_p(\Omega) := W^{2,p}(\Omega)/D_0^p(\Omega)$ is a reflexive Banach space, with norm $\|\cdot\|_W$, which is equivalent to the quotient norm $\|\cdot\|_{W^{2,p}}$. From Theorem 2.7 of [5], $D_0^p(\Omega) = 0$ when $p < N$. We prove an existence theorem for nonconvex problems under the following assumptions:

$H(F)_1$: $H : \Omega \times R \times R^N \times R \to \mathcal{P}_k(R)$ is a multifunction satisfying the following properties:

(a) $(x, u, s, t) \to H(x, u, s, t)$ is graph measurable.

(b) For almost all $x \in \Omega$, $(u, s, t) \to H(x, u, s, t)$ is LSC.

(c) For every $(u, s, t) \in R \times R^N \times R$, there exist $\omega_0(x) \in L^p(\Omega)$, $\omega_1(x) \in L^{\frac{p}{1-\alpha}}(\Omega)$, $\omega_2(x) \in L^{\frac{p}{1-\beta}}(\Omega)$, $\omega_3(x) \in L^{\frac{p}{1-\gamma}}(\Omega)$ such that

$$\begin{aligned} \bigl|H(x, u, s, t)\bigr| &= \bigl\{ |v| : v \in H(x, u, s, t) \bigr\} \\ &\leq \omega_0(x) + \omega_1(x)|u|^\alpha + \omega_2(x)\|s\|^\beta + \omega_3(x)|t|^\gamma \quad \text{a.e. } x \in \Omega, \end{aligned}$$

where $0 \leq \alpha, \beta, \gamma < 1$.

Theorem 3.1 *If assumption $H(F)_1$ holds, then the partial differential inclusion* (1.1) *has a solution $u \in W_p(\Omega)$.*

Proof Following [5], we define the biharmonic operator $L := \Delta^2 : W_p(\Omega) \to W_0^{-2,p}(\Omega)$, and then $L : W_p(\Omega) \to W^{-2,p}(\Omega)$ is a linear mapping. According to Theorem 2.12 of [5], for each $f \in W_0^{-2,p}(\Omega)$ the following problem:

$$\begin{cases} \Delta^2 u = f(x) & \text{a.e. in } \Omega, \\ u = \frac{\partial u}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.2}$$

has only one solution $u \in W_p(\Omega)$, and

$$\|u\|_W \leq C\|f\|_{-2,p}, \tag{3.3}$$

where the constant C depends only on N, p, and Ω. Thus $L : W_p(\Omega) \to W_0^{-2,p}(\Omega)$ is one to one and surjective, which implies $L^{-1} : W_0^{-2,p}(\Omega) \to W_p(\Omega)$ is well defined. Thus, $u = L^{-1}(f)$, and from (3.3), we have $\|L^{-1}(f)\|_W \leq C\|f\|_{-2,p}$. Then we see that $L^{-1} : W_0^{-2,p}(\Omega) \to W_p(\Omega)$ is a bounded linear operator, which implies L^{-1} is continuous from $W_0^{-2,p}(\Omega)$ to $W_p(\Omega)$. Let $K \subset W_0^{-2,p}(\Omega)$ be a bounded subset, and then we find that $L^{-1}(K)$ is also bounded in $W_p(\Omega)$. From the Sobolev embedding theorem, we see that the embedding $W^{2,p}(\Omega) \subset W_0^{-2,p}(\Omega)$ is compact. Now since $W^{2,p}(\Omega)$ is compactly embedded in $W_0^{-2,p}(\Omega)$, it follows that $L^{-1}(K)$ is relatively compact in $W_0^{-2,p}(\Omega)$. Hence, $L^{-1} : W_0^{-2,p}(\Omega) \to W_0^{-2,p}(\Omega)$ $(W_p(\Omega) \subset W_0^{-2,p}(\Omega))$ is completely continuous.

Now let $S_H^p : W_p(\Omega) \to 2^{W_0^{-2,p}(\Omega)}$ ($L^p(\Omega) \subset W_0^{-2,p}(\Omega)$) be the multivalued Nemytskii operator corresponding to H defined by

$$S_H^p(u) = \{v \in L^p(\Omega) : v \in H(x, u, \nabla u, \Delta u) \text{ a.e. on } \Omega\}.$$

We show that $S_H^p(\cdot)$ has nonempty, closed, decomposable values in $L^p(\Omega)$ and is LSC. The closedness and decomposability of the values of $S_H^p(\cdot)$ are easy to check. For the nonemptiness, note that if $u \in W_p(\Omega)$, by hypothesis $H(F)_1$(i), $(x, u, s, t) \to H(x, u, s, t)$ is graph measurable, so we can apply Aumann's selection theorem and get a measurable map $v : \Omega \to R$ such that $v(x) \in H(x, u, s, t)$ a.e. on Ω. By hypothesis $H(F)_1$(iii), $v \in L^p(\Omega)$. Thus for every $u \in W_p(\Omega)$, $S_H^p(u) \neq \emptyset$. To prove the lower semicontinuity of $S_H^p(\cdot)$, it is sufficient to show that for every $w \in L^p(\Omega)$, $u \to d(w, S_H^p(u))$ is an USC R_+-valued function. Note that

$$\begin{aligned} d\big(w, S_H^p(u)\big) &= \inf\big\{\|w - v\|_p : v \in S_H^p(u)\big\} \\ &= \inf\left\{\left(\int_\Omega \big|w(x) - v(x)\big|^p\, dx\right)^{\frac{1}{p}} : v \in S_H^p(u)\right\} \\ &= \left(\int_\Omega \inf\big\{\big|w(x) - v(x)\big|^p : v \in H(x, u, \nabla u, \Delta u)\big\}\, dx\right)^{\frac{1}{p}} \\ &= \left(\int_\Omega d\big(w(x), H(x, u, \nabla u, \Delta u)\big)^p\, dx\right)^{\frac{1}{p}}. \end{aligned}$$

We claim that for every $\lambda \geq 0$, the superlevel set $R_\lambda = \{u \in L^p(\Omega) : d(w, S_H^p(u)) \geq \lambda\}$ is closed in $L^p(\Omega)$. Let $\{u_n\}_{n\geq 1} \subseteq R_\lambda$ and assume that $u_n \to u$ in $L^p(\Omega)$. By passing to a subsequence if necessary, we may assume that $u_n(x) \to u(x)$ a.e. on Ω as $n \to \infty$. By hypothesis $H(F)_1$(ii), $(u, s, t) \to d(w, H(x, u, s, t))$ is an upper semicontinuous R_+-valued function. Thus via Fatou's lemma, we have

$$\begin{aligned} \lambda^p &\leq \overline{\lim}\big[d\big(w, S_H^p(u_n)\big)\big]^p \\ &= \overline{\lim} \int_\Omega \big[d\big(w(x), H(x, u_n, \nabla u_n, \Delta u_n)\big)\big]^p\, dx \\ &\leq \int_\Omega \overline{\lim}\big[d\big(w(x), H(x, u_n, \nabla u_n, \Delta u_n)\big)\big]^p\, dx \\ &\leq \int_\Omega \big[d\big(w, H(x, u, \nabla u, \Delta u)\big)\big]^p\, dt = \big[d\big(w, S_H^p(u)\big)\big]^p. \end{aligned}$$

Therefore $u \in R_\lambda$ and this proves the LSC of $S_H^p(\cdot)$.

Hence, by Lemma 2.2, there exists a continuous map $g : W_p(\Omega) \to L^p(\Omega) \subset W_0^{-2,p}(\Omega)$ such that $g(u) \in S_H^p(u)$. To complete the proof, we need to consider the fixed point problem: $u = L^{-1} \circ g(u)$.

To this aim, we show that the set $\Gamma = \{u \in W_0^{-2,p}(\Omega) : u = \sigma L^{-1} \circ g(u), \sigma \in (0,1)\}$ is bounded. Let $u \in \Gamma$, then $Lu = \sigma g(u)$ and so $\|Lu\|_p = \sigma \|g(u)\|_p$. By hypothesis $H(F)_1$(iii), we derive

$$\big|g(u)\big| \leq \omega_0(x) + \omega_1(x)|u|^\alpha + \omega_2(x)\|\nabla u\|^\beta + \omega_3(x)|\Delta u|^\gamma,$$

and then

$$\begin{aligned}
\|g(u)\|_p &\le \|\omega_0(x)\|_p + \|\omega_1(x)|u|^\alpha\|_p + \|\omega_2(x)|\nabla u|^\beta\|_p + \|\omega_3(x)|\Delta u|^\gamma\|_p \\
&\le \|\omega_0\|_p + \left(\|\omega_1^p\|_{\frac{1}{1-\alpha}}\||u|^{p\alpha}\|_{\frac{1}{\alpha}}\right)^{1/p} + \left(\|\omega_2^p\|_{\frac{1}{1-\beta}}\||\nabla u|^{p\beta}\|_{\frac{1}{\beta}}\right)^{1/p} \\
&\quad + \left(\|\omega_3^p\|_{\frac{1}{1-\gamma}}\||\Delta u|^{p\gamma}\|_{\frac{1}{\gamma}}\right)^{1/p} \\
&= \|\omega_0\|_p + \|\omega_1\|_{\frac{p}{1-\alpha}}\|u\|_p^\alpha + \|\omega_2\|_{\frac{p}{1-\beta}}\|\nabla u\|_p^\beta + \|\omega_3\|_{\frac{p}{1-\gamma}}\|\Delta u\|_p^\gamma.
\end{aligned}$$

Thus by (3.3), it follows that

$$\begin{aligned}
\|u\|_W &\le C\|Lu\|_{-2,p} \\
&\le C_1\|g(u)\|_p \\
&\le C_1\|\omega_0\|_p + C_1\|\omega_1\|_{\frac{p}{1-\alpha}}\|u\|_p^\alpha + C_1\|\omega_2\|_{\frac{p}{1-\beta}}\|\nabla u\|_p^\beta + C_1\|\omega_3\|_{\frac{p}{1-\gamma}}\|\Delta u\|_p^\gamma \\
&\le C_1\|\omega_0\|_p + \widehat{C}\|u\|_W^\theta
\end{aligned} \tag{3.4}$$

for some $\theta \in (0,1)$. Now since $0 < \theta < 1$, we can find a constant $M_2 > 0$ such that $\|u\|_W \le M_2$. Thus in view of the continuity of the embedding $W_p(\Omega) \to W_0^{-2,p}(\Omega)$, it follows that Γ is bounded in $W_0^{-2,p}(\Omega)$. Apply the Leray-Schauder alternative, and we find that there exists $u \in W_p(\Omega)$ such that $u = L^{-1} \circ g(u)$, *i.e.*, u is a solution of problem (1.1). This completes the proof. □

The assumptions we need for convex problems are as follows:

$H(F)_2$: $H : \Omega \times R \times R^N \times R \to \mathcal{P}_{kc}(R)$ is a multifunction satisfying the following properties:
- (a) $(x,u,s,t) \to H(x,u,s,t)$ is graph measurable.
- (b) For almost all $x \in \Omega$, $(u,s,t) \to H(x,u,s,t)$ is USC; and $H(F)_1$(iii) holds.

Theorem 3.2 *If assumption $H(F)_2$ holds, then the solution set of the partial differential inclusion* (1.1) *is nonempty in $W_p(\Omega)$. Moreover, the solution set is weakly compact in $W_p(\Omega)$.*

Proof In view of the proof of Theorem 3.1, we only need to emphasis those steps where the proofs differ.

In this case the multivalued Nemytskii operator $S_H^p : W_p(\Omega) \to 2^{L^p(\Omega)}$ has nonempty convex values in $L^p(\Omega)$. The convexity of the values of $S_H^p(\cdot)$ are clear. To prove the nonemptiness, let $u \in W_p(\Omega)$, and let $\{s_n\}_{n\ge1}$, $\{r_n\}_{n\ge1}$, $\{t_n\}_{n\ge1}$ be three sequences of step functions such that

$$s_n \to u, \qquad r_n \to \nabla u, \qquad t_n \to \Delta u$$

and

$$|s_n| \le |u|, \qquad \|r_n\| \le \|\nabla u\|, \qquad \|t_n\| \le |\Delta u| \quad \text{a.e. on } \Omega.$$

Then by virtue of hypothesis $H(F)_2$(i), for every $n \ge 1$, $x \to H(x,s_n,r_n,t_n)$ admits a measurable selector $g_n(x)$. From hypothesis $H(F)_1$(iii) and (3.4), we have

$$\sup_{g_n \in H(x,s_n,r_n,t_n)} \|g_n\|_p \le C_1\|\omega_0\|_p + \widehat{C}\|u\|_W^\theta.$$

Now $p > 1$, and the set $\{g_n, n \geq 1\}$ is bounded in the reflexive space $L^p(\Omega)$, so it is relatively weakly compact in $L^p(\Omega)$. Consequently by Eberlein-Šmulian's theorem and passing to a subsequence if necessary, we may assume that $g_n \to g$ weakly in $L^p(\Omega)$. Then from Theorem 3.1 in [37], we find

$$\begin{aligned} g(x) &\in \overline{\text{conv}}\ \overline{\lim}\{g_n(x)\}_{n\geq 1} \\ &\subseteq \overline{\text{conv}}\ \overline{\lim} H(x, s_n, r_n, t_n) \\ &\subseteq H(x, u, \nabla u, \Delta u) \quad \text{a.e. on } \Omega, \end{aligned} \tag{3.5}$$

here, the last inclusion being a consequence of hypothesis $H(F)_2$(ii). Thus $g \in S_H^p(u)$, which shows that $S_H^p(\cdot)$ is nonempty.

As in Theorem 3.1, we obtain the *a priori* bound for the solution set of problem (1.1). Let

$$\mathcal{D} = \{x \in W_p(\Omega) : \|x\|_W \leq M\},$$

where $M > 0$ is a constant (constructed as in Theorem 3.1). Note that the set $\mathcal{D}$ is closed, convex, and bounded in $W_p(\Omega)$ (which is reflexive), so it is compact in a weak topology. Due to the convexity of the values of $S_H^p(\cdot)$ and L^{-1} being a linear map, it follows that $L^{-1} \circ S_H^p(u)$ is convex for every $u \in \mathcal{D}$. Using an argument similar to that in (3.4), we obtain $L^{-1} \circ S_H^p(\mathcal{D}) \subseteq \mathcal{D}$. Next we show that $L^{-1} \circ S_H^p : \mathcal{D} \to \mathcal{P}_c(\mathcal{D})$ has weakly sequentially closed graph. Let $(u_n, v_n) \in \mathcal{D} \times \mathcal{D}$, $n \geq 1$, be in the graph of $L^{-1} \circ S_H^p$ with (u_n, v_n) converging weakly to (u, v). Note

$$v_n \in L^{-1} \circ S_H^p(u_n), \quad n \geq 1, \tag{3.6}$$

and passing to a subsequence, we note that $u_n(x) \to u(x)$ a.e. on Ω. Now for every $n \in N$ there exists $f_n \in S_H^p(u_n)$ such that $v_n = L^{-1}(f_n)$. Again from hypothesis $H(F)_1$(iii) and (3.4), we have

$$\sup_n \|f_n\|_p \leq C_1\|\omega_0\|_p + \widehat{C}\|u\|_W^\theta.$$

Now $p > 1$, and the set $\{f_n, n \geq 1\}$ is bounded in the reflexive space $L^p(\Omega)$, so it is relatively weakly compact in $L^p(\Omega)$. Consequently by the Eberlein-Šmulian theorem and passing to a subsequence if necessary, we may assume that $f_n \to f$ weakly in $L^p(\Omega)$. As above we have

$$\begin{aligned} f(x) &\in \overline{\text{conv}}\,\overline{\lim}\{f_n(x)\}_{n\geq 1} \\ &\subseteq \overline{\text{conv}}\,\overline{\lim} H(x, u_n, \nabla u_n, \Delta u_n) \\ &\subseteq H(x, u, \nabla u, \Delta u) \quad \text{a.e. on } \Omega, \end{aligned} \tag{3.7}$$

and thus $f \in S_H^p(u)$. Let $w = L^{-1}(f)$, and note $w \in L^{-1} \circ S_H^p(u)$. From Theorem 3.1, we know that $L^{-1} : W^{-2,p}(\Omega) \to W_p(\Omega)$ is continuous. Since the embedding $L^p(\Omega) \subseteq W^{-2,p}(\Omega)$ is compact, we get (see Theorem 2.2) that $f_n \to f$ in $W^{-2,p}(\Omega)$. Thus, $v_n = L^{-1}(f_n) \to w = L^{-1}(f)$. Therefore, $v \in L^{-1} \circ S_H^p(u)$, and so $L^{-1} \circ S_H^p : \mathcal{D} \to \mathcal{P}_c(\mathcal{D})$ has weakly sequentially closed graph.

Invoking Lemma 2.1, there exists $u \in \mathcal{D}$ such that $u \in L^{-1} \circ S_H^p(u)$. Evidently this is a solution of problem (1.1). Let S denote the solution set of problem (1.1). Again, as in Theorem 3.1, we have

$$|S| = \sup\{\|u\|_W : u \in S\} \leq M,$$

where $M > 0$. Since S is bounded and $W_p(\Omega)$ is reflexive, it follows that $\overline{S}^w$ is weakly compact. Let $u \in W_p(\Omega)$ be weakly adherent to S. Since $\overline{S}^w$ is weakly compact, by Eberlein-Šmulian's theorem there exists a sequence $\{u_n\}_n \subseteq S$ such that $u_n \to u$ weakly in $u \in \mathcal{D} \subset W_p(\Omega)$. Since $L^{-1} \circ S_H^p$ has weakly sequentially closed graph, it follows that $u \in L^{-1} \circ S_H^p(u)$ which implies that $u \in S$. Hence $\overline{S}^w = S$ and S is weakly closed. Therefore S is weakly compact in $W_p(\Omega)$. □

4 Relation theorem of solutions

In this section, we are concerned with the following extremal problem:

$$\begin{cases} \Delta^2 u \in \operatorname{ext} H(x, u, \nabla u, \Delta u) & \text{a.e. in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \frac{\partial u}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where $\operatorname{ext} H(x, u, \nabla u, \Delta u)$ denotes the extremal point set of $H(x, u, \nabla u, \Delta u)$.

The precise assumptions on the data of problem (4.1) are the following:

$H(F)_3$: $H : \Omega \times R \times R^N \times R \to \mathcal{P}_{kc}(R)$ is multifunction such that for almost all $x \in \Omega$, $(u, s, t) \to H(x, u, s, t)$ is h-continuous, and $H(F)_1$(i), (iii) holds, where $p > \frac{N}{2} \geq 2$.

In the following let S denote the solution set of (1.1), and S_e denote the solution set of (4.1).

Theorem 4.1 *If assumption $H(F)_3$ holds, then the partial differential inclusion* (4.1) *has a solution $u \in W_p(\Omega) \cap C(\overline{\Omega})$.*

Proof As H is replaced by $\operatorname{ext} H$, we also obtain an *a priori* bound for S_e. Let

$$|S_e| = \sup\{\|u\|_W : u \in S_e\} \leq M$$

for some constant $M > 0$. By virtue of hypothesis $H(F)_1$(iii) and (3.3), there exists $a(x) \in L_+^p(\Omega)$ such that for every $u \in S_e$, $|H(x, u, \nabla u, \Delta u)| \leq a(x)$. Let

$$V = \{v \in L^p(\Omega) : |v(x)| \leq a(x) \text{ a.e. on } \Omega\},$$

and we now show that $\widehat{Q} = L^{-1}(V)$ is a compact convex subset in $C(\overline{\Omega})$. The closedness and convexity of $\widehat{Q}$ are clear. We only need to show its compactness. To this aim, since $\widehat{Q} \subseteq W_0^{2,p}(\Omega)$ and $p > \frac{N}{2}$, the embedding $W_0^{2,p}(\Omega) \hookrightarrow C(\overline{\Omega})$ is compact, and hence $\widehat{Q} \subseteq C(\overline{\Omega})$ is compact. From Lemma 2.3, we can find a continuous map: $g : \widehat{Q} \to L^p(\Omega)$ such that $g(u) \in \operatorname{ext} H(x, u, \nabla u, \Delta u)$ a.e. Ω for all $u \in \widehat{Q}$. Thus $L^{-1} \circ g$ is a compact operator. Hence, Schauder's fixed point theorem is applicable, and there exists a $u \in \widehat{Q}$ such that

$u = L^{-1} \circ g(u)$, which is a solution of (4.1). Therefore $S_e \neq \emptyset$ in $W_p(\Omega) \cap C(\overline{\Omega})$. This complete the proof. □

To prove our next result, we need the following definition.

Definition 4.1 (see [38]) The multifunction $H : \Omega \times R \times R^N \times R \to \mathcal{P}_k(R)$ is called 'one-sided Lipschitz (OSL)' continuous if there is an integrable function $\mathcal{L} : \Omega \to R$ such that for every $u_1, u_2 \in R$, $x \in \Omega$, $s \in R^N$, $t \in R$, and $v_1 \in H(x, u_1, s, t)$ there exists $v_2 \in H(x, u_2, s, t)$ such that $(v_2 - v_1) \cdot (u_2 - u_1) \leq \mathcal{L}(x)|u_2 - u_1|^2$.

We also recall Poincaré's inequality: there exists a constant $\lambda > 0$ depending on N, p, Ω such that

$$\int_\Omega |u|^2 \, dx \leq \lambda \int_\Omega |\Delta u|^2 \, dx \tag{4.2}$$

for every $u \in W_p(\Omega)$.

Theorem 4.2 *If the hypothesis $H(F)_3$ holds, and*

(i) *H is one-sided Lipschitz* (*OSL*) *continuous*;

(ii) *$\mathcal{L}(x) \leq \alpha < \frac{1}{\lambda}$, where $\mathcal{L}$ is from Definition* 4.1 *and λ from* (4.2);

then $\overline{S}_e^{C(\overline{\Omega})} = S$, the closure is taken in $C(\overline{\Omega})$.

Proof Let $u_g \in S$, then there exist $g \in L^p(\Omega)$ and $g(x) \in H(x, u, \nabla u, \Delta u)$ a.e. on Ω, such that

$$\begin{cases} \Delta^2 u_g = g(x) & \text{a.e. in } \Omega, \\ u_g = 0 & \text{on } \partial\Omega, \\ \frac{\partial u_g}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.3}$$

As earlier, we let $V = \{v \in L^p(\Omega) : |v(x)| \leq a(x) \text{ a.e. on } \Omega\}$ and $\widehat{Q} = L^{-1}(V)$, and then $\widehat{Q}$ is a compact convex subset of $C(\overline{\Omega})$. For any $u \in \widehat{Q}$ and $\varepsilon > 0$, we define the multifunction

$$Q_\varepsilon(x) = \left\{ v \in H(x, u, \nabla u, \Delta u) : (g - v) \cdot (u_g - u) \leq \mathcal{L}(x)|u_g - u|^2 + \varepsilon \right\}.$$

Clearly, for every $x \in \Omega$, $Q_\varepsilon(x) \neq \emptyset$, and it is graph measurable. On applying Aumann's selection theorem, we get a measurable function $v : x \to R$ such that $v(x) \in Q_\varepsilon(x)$ almost everywhere on Ω. We define the multifunction

$$R_\varepsilon(u) = \left\{ v \in S^p_{H(\cdot, u(\cdot), \nabla u(\cdot), \Delta u(\cdot))} : (g - v) \cdot (u_g - u) \leq \mathcal{L}(x)|u_g - u|^2 + \varepsilon \right\}.$$

It is clear that $R_\varepsilon : \widehat{Q} \to 2^{L^p(\Omega)}$ has nonempty and decomposable values. Moreover, from Theorem 3 of [39] $R_\varepsilon(\cdot)$ is LSC. Therefore $u \to \overline{R_\varepsilon(u)}$ is LSC where $\overline{R_\varepsilon(u)}$ is the closed hull of $R_\varepsilon(u)$, and has closed and decomposable values. Thus in view of Lemma 2.2 there exists a continuous map $f_\varepsilon : \widehat{Q} \to L^p(\Omega)$ such that $f_\varepsilon(u) \in \overline{R_\varepsilon(u)}$ for all $u \in \widehat{Q}$. Now invoking the Relaxation Theorem in [40], we can find a continuous map $g_\varepsilon : \widehat{Q} \to L^p(\Omega)$ such that $g_\varepsilon(u)(x) \in \operatorname{ext} H(x, u, \nabla u, \Delta u)$ almost everywhere on Ω, and $\|f_\varepsilon(u) - g_\varepsilon(u)\|_p \leq \varepsilon$ for all $u \in \widehat{Q}$.

Next let $\varepsilon = \frac{1}{n}$, $g_\varepsilon = g_n$, $f_\varepsilon = f_n$, and consider the following biharmonic boundary value problem:

$$\begin{cases} \Delta^2 u_n = g_n(u_n) \in \operatorname{ext} H(x, u_n, \nabla u_n, \Delta u_n) & \text{a.e. in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \\ \frac{\partial u_n}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.4}$$

It is easy to see that $L^{-1} \circ g_n : \widehat{Q} \to \widehat{Q}$ is a compact operator, and hence by the Schauder fixed point theorem, there exists a solution $u_n \in W_p(\Omega)$ of (4.4). It is clear that $\{u_n\}_{n\geq 1} \subseteq \widehat{Q}$ and $\{u_n\}_{n\geq 1}$ is uniformly bounded. Thus on passing to a subsequence if necessary, we may assume that $u_n \to \hat{u}$ weakly in $W_p(\Omega)$. From an earlier argument, we know that $u_n \to \hat{u}$ in $C(\overline{\Omega})$ as $n \to +\infty$, and hence

$$\Delta^2 u_n - \Delta^2 u_g = g_n(u_n)(x) - g(x)$$

and

$$\int_\Omega \left(\Delta^2 u_n - \Delta^2 u_g\right)(u_n - u_g)\,dx = \int_\Omega \left(g_n(u_n) - g(x)\right)(u_n - u_g)\,dx.$$

Since $u_n - u_g \in W_p(\Omega)$, the embedding $W_0^{2,p}(\Omega) \subseteq W_0^{2,2}(\Omega)$ is continuous when $p \geq 2$. Thus

$$\begin{aligned} \int_\Omega \Delta^2(u_n - u_g)\cdot(u_n - u_g)\,dx &= \int_\Omega \left|\Delta(u_n - u_g)\right|^2 dx \\ &\geq \frac{1}{\lambda}\int_\Omega |u_n - u_g|^2\,dx \\ &\to \frac{1}{\lambda}\int_\Omega |\hat{u} - u_g|^2\,dx \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} \int_\Omega \left(g_n(u_n) - g(x)\right)(u_n - u_g)\,dx \leq{}& \int_\Omega \left(g_n(u_n) - f_n(u_n)\right)(u_n - u_g)\,dx \\ &+ \int_\Omega \left(f_n(u_n) - g(x)\right)(u_n - u_g)\,dx. \end{aligned} \tag{4.6}$$

Now since

$$\left\|f_n(u_n) - g_n(u_n)\right\|_p \leq \frac{1}{n},$$

it follows that

$$\int_\Omega \left(g_n(u_n) - f_n(u_n)\right)(u_n - u_g)\,dx \to 0$$

as $n \to \infty$. Also, we have

$$\begin{aligned} &\int_\Omega \left(f_n(u_n) - g(x)\right)(u_n - u_g)\,dx \\ &\quad\leq \int_\Omega \left[\mathcal{L}(x)|u_n - u_g|^2 + \frac{1}{n}\right] dx \end{aligned}$$

$$\begin{aligned} &\leq \int_\Omega \left(\alpha|u_n - u_g|^2 + \frac{1}{n}\right) dx \\ &\to \alpha \int_\Omega |\hat{u} - u_g|^2\,dx \end{aligned} \tag{4.7}$$

as $n \to \infty$. Hence, from (4.5), (4.6), and (4.7), we find

$$\frac{1}{\lambda}\int_\Omega |\hat{u} - u_g|^2\,dx \leq \alpha \int_\Omega |\hat{u} - u_g|^2\,dx. \tag{4.8}$$

Finally since $\frac{1}{\lambda} > \alpha$, it follows that $u_g = \hat{u}$, *i.e.*, $u_n \to u_g$ and $u_n \in S_e$ for $n \geq 1$, and hence $S \subseteq \overline{S}_e$. Also, S is closed in $C(\overline{\Omega})$ (see the proof of Theorem 3.2), and thus $S = \overline{S}_e^{C(\overline{\Omega})}$. □

5 Properties of the solutions set

In this section, keeping our above hypotheses on the orientor field $H(x,u,s)$, we will establish the topological regularity of the solution set S for the following biharmonic problem:

$$\begin{cases} \Delta^2 u \in H(x,u,\nabla u) \quad \text{a.e. in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \\ \frac{\partial u}{\partial \mathbf{n}} = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{5.1}$$

Remark 5.1 The results obtained above for the problem (1.1) hold for the partial differential inclusion (5.1).

From Theorem 3.1, it is easy to show that for every $f \in H(x,u,\nabla u) \subseteq L^p(\Omega)$, problem (5.1) has at least one weak solution $u = L^{-1}(f) \in W_p(\Omega)$ and $\|u\|_W \leq C\|f\|_p$, where C is a constant independent of u and f. Let $L^p(\Omega)_w$ denote the Lebesgue-Bochner space furnished with the weak topology. From the proof of Theorem 3.1, it follows that the map $P = L^{-1} : L^p(\Omega)_w \to W_p(\Omega)$ is sequentially continuous.

Remark 5.2 Since the embedding of $W_p(\Omega) \hookrightarrow C^1(\overline{\Omega})$ is compact when $p \geq N$, $P : L^p(\Omega)_w \to C^1(\overline{\Omega})$ is completely continuous.

Next we introduce the following hypothesis:

$H(F)_4$: $H : \Omega \times R \times R^N \to \mathcal{P}_{kc}(R)$ is a multifunction satisfying the following properties:

(i) $(x,s,t) \to H(x,s,t)$ is graph measurable.

(ii) For almost all $x \in \Omega$, $(s,t) \to H(x,s,t)$ is USC.

(iii) For every $(s,t) \in R \times R^N$, there exist $\xi_0(x) \in L^p(\Omega)$, $\xi_1(x) \in L^{\frac{p}{1-\gamma}}(\Omega)$, $\xi_2(x) \in L^{\frac{p}{1-\eta}}(\Omega)$, such that

$$\left|H(x,s,t)\right| = \sup\bigl\{|v| : v \in H(x,s,t)\bigr\} \leq \xi_0(x) + \xi_1(x)|u|^\gamma + \xi_2(x)|s|^\eta$$

for a.e. $x \in \Omega$, where $0 \leq \gamma, \eta < 1$.

Theorem 5.1 *If hypothesis* $H(F)_4$ *holds and* $p \geq N$, *then the solution set* S *of problem* (5.1) *is an* R_δ *set in* $C(\overline{\Omega})$.

Proof Similar reasoning as in the proof of Theorem 3.1, for almost all $x \in \Omega$, all $u \in S$, and all $v \in H(x, u, \nabla u)$, we may assume that $|v| \leq a(x)$ where $a(x) \in L^p_+(\Omega)$. From Lemma 2.4, we obtain a sequence of multifunctions $H_n : \Omega \times R \times R^N \to \mathcal{P}_{fc}(R)$. For every $n \geq 1$, consider the following biharmonic problem of the partial differential inclusion:

$$\begin{cases} \Delta^2 u \in H_n(x, u, \nabla u) \quad \text{a.e. in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \\ \frac{\partial u}{\partial \mathbf{n}} = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{5.2}$$

Now from the proof of Theorem 3.2 and Remark 5.2, we find that problem (5.2) has a nonempty solution set $S_n \subseteq W_p(\Omega)$ which is compact in $C(\overline{\Omega})$.

First, we prove that the set S_n is contractible. Let $g_n(x, u, \nabla u)$ be locally Lipschitz with respect to u, measurable selector of $H_n(x, u, \nabla u)$ postulated from Lemma 2.4. Let $d = \operatorname{dist}(\Omega', \partial\Omega)$ where $\operatorname{dist}(\Omega', \partial\Omega)$ denotes the distance from Ω' to $\partial\Omega$ and $\Omega' \subseteq \Omega$, then $d \in I := [0, \tau]$ where $\tau = \max_{x\in\Omega}\{\operatorname{dist}(x, \partial\Omega)\}$. Let

$$\Omega_\delta = \{x \in \Omega | \operatorname{dist}(x, \partial\Omega) \geq \delta\}.$$

For $\delta \in [0, \tau]$, given $z \in S_n$, there exists a $g_n \in H_n(x, u, \nabla u)$ such that $z = P(g_n)$. For each $u \in S_n$, let $z_u(\delta)(x) \in W_p(\Omega)$ be the unique solution of

$$\begin{cases} \Delta^2 z(x) = g_n(x, z(x), \nabla z) \quad \text{a.e. in } \Omega_\delta, \\ z(x) = u(x) \quad \text{on } \partial\Omega_\delta, \\ \frac{\partial z}{\partial n} = \frac{\partial u}{\partial n} \quad \text{on } \partial\Omega_\delta. \end{cases} \tag{5.3}$$

Define $\mu : I \times S_n \to S_n$ by

$$\mu(\delta, u)(x) = \begin{cases} u(x) & \text{for } x \in \Omega \setminus \Omega_\delta, \\ z_u(\delta)(x) & \text{for } x \in \Omega_\delta. \end{cases} \tag{5.4}$$

Evidently $\mu(0, u) = z$, and $\mu(\tau, u) = u$ for every $u \in S_n$. In the following we show that $\mu(\delta, u) \in S_n(u)$ for each $(\delta, u) \in I \times S_n$. Note that for each $u \in S_n(u)$ there exists $f_n \in H_n(x, u, \nabla u)$, such that $u = P(f_n)$. Set $\overline{g}_n(x) = f_n\chi_{[0,\delta]}(x) + g_n\chi_{(\delta,\tau]}(x)$ for each $x \in \Omega$ where χ is the characteristic function. It is easy to see that $\overline{g}_n(x) \in H_n(x, u, \nabla u)$. Thus $P(\overline{g}_n) = u(x)$ for all $\Omega \setminus \Omega_\delta$, $P(\overline{g}_n) = z_u(x)$ for all $x \in \Omega_\delta$, which implies that $P(\overline{g}_n) = \mu(\delta, u)(x)$, and thus $\mu(\delta, u)(x) \in S_n$. In order to prove that S_n is contractible in $C(\overline{\Omega})$, we only need to show that $\mu(\delta, u)$ is continuous in $I \times C(\overline{\Omega})$. To this aim, we let $(\delta_m, u_m) \to (\delta, u)$ in $I \times S_n$, and consider the following two distinct cases.

Case 1: $\delta_m \geq \delta$ *for every* $m \geq 1$, *and then* $\Omega_{\delta_m} \subseteq \Omega_\delta$. Let $v_m(x) = \mu(\delta_m, u_m)(x)$, $x \in \Omega$. Evidently, $v_m(x) \in S_n$, $m \geq 1$, and hence passing to a subsequence if necessary, we may assume that $v_m \to v$ in $C(\overline{\Omega})$ as $m \to \infty$. Clearly, $v(x) = u(x)$ for a.e. $\Omega \setminus \Omega_\delta$. Also let $w \in W_p(\Omega)$ be the unique solution of

$$\begin{cases} \Delta^2 w = g_n(t, v, \nabla v) \quad \text{a.e. on } \Omega_\delta, \\ w(x) = u(x), \qquad \frac{\partial w}{\partial n} = \frac{\partial u}{\partial n} \quad \text{on } \partial\Omega_\delta. \end{cases} \tag{5.5}$$

Let $N \geq 1$, $m \geq N$, then for m large enough $v_m(\cdot)$ satisfies $\Delta^2 v_m = g_n(x, v_m, \nabla v_m)$ a.e. on Ω_{δ_m}, and $v_m = u_m$, $\frac{\partial v_m}{\partial n} = \frac{\partial u_m}{\partial n}$ on $\partial\Omega_{\delta_m}$. From (5.5) and $\Omega_{\delta_m} \subseteq \Omega_\delta$, we have

$$\Delta^2(v_m - w) = g_n(x, v_m, \nabla v_m) - g_n(x, v, \nabla v) \quad \text{a.e. on } \Omega_{\delta_m}. \tag{5.6}$$

From $(\delta_m, u_m) \to (\delta, u)$, we have

$$\begin{aligned} &v_m|_{\partial\Omega_{\delta_m}} = u_m|_{\partial\Omega_{\delta_m}} \to v|_{\partial\Omega_\delta} = u|_{\partial\Omega_\delta}, \\ &\left.\frac{\partial v_m}{\partial n}\right|_{\partial\Omega_{\delta_m}} = \left.\frac{\partial u_m}{\partial n}\right|_{\partial\Omega_{\delta_m}} \to \left.\frac{\partial v}{\partial n}\right|_{\partial\Omega_\delta} = \left.\frac{\partial u}{\partial n}\right|_{\partial\Omega_\delta} \end{aligned}$$

as $m \to \infty$. Passing to the limit in (5.6) as $m \to \infty$, and recalling that $\delta_m \to \delta$, $v_m \to v$ in S_n, we get

$$\begin{cases} \Delta^2(v - w) = 0 \quad \text{a.e. on } \Omega_\delta, \\ v - w = \frac{\partial(v-w)}{\partial n} = 0 \quad \text{on } \partial\Omega_\delta. \end{cases} \tag{5.7}$$

Clearly, $v - w = 0$ a.e. on Ω_δ. Thus $v(x) = w(x)$ for every $x \in \Omega_\delta$. Hence, from (5.6) we obtain

$$\begin{cases} \Delta^2 v = g_n(t, v, \nabla v) \quad \text{a.e. on } \Omega_\delta, \\ v(x) = u(x), \qquad \frac{\partial v}{\partial n} = \frac{\partial u}{\partial n} \quad \text{on } \partial\Omega_\delta. \end{cases} \tag{5.8}$$

Therefore $v = \mu(\delta, u)$, and we can conclude that $\mu(\delta_m, u_m) \to \mu(\delta, u)$ in $I \times C(\overline{\Omega})$.

Case 2: $\delta_m \leq \delta$ for every $m \geq 1$, *and then* $\Omega_\delta \subseteq \Omega_{\delta_m}$. Keeping the notation used in Case 1, from $(\delta_m, u_m) \to (\delta, u)$, we find $u_m|_{\partial\Omega_{\delta_m}} \to u|_{\partial\Omega_\delta}$, $\frac{\partial u_m}{\partial n}|_{\partial\Omega_{\delta_m}} \to \frac{\partial u}{\partial n}|_{\partial\Omega_\delta}$ as $m \to \infty$. Hence, $v_m \to v$, in $C^1(\overline{\Omega})$ and $v(x) = u(x)$ for a.e. $\Omega \setminus \Omega_\delta$. Also using the same argument as in Case 1, we have

$$\Delta^2(v_m - w) = g_n(x, v_m, \nabla v_m) - g_n(x, v, \nabla v) \quad \text{a.e. on } \Omega_\delta.$$

From $v_m \to v$ in $C^1(\overline{\Omega})$, $\nabla v_m \to \nabla v$ in $C(\overline{\Omega})$, and $g_n(x, u, \nabla u)$ is locally Lipschitz continuous in u, it follows that $g_n(x, v_m, \nabla v_m) \to g_n(x, v, \nabla v)$ as $m \to \infty$. Thus

$$\Delta^2(v - w) = 0 \quad \text{a.e. on } \Omega_\delta$$

for $m \to \infty$. Now since $v = u$ and $\frac{\partial v}{\partial n} = \frac{\partial u}{\partial n}$ on $\partial\Omega_\delta$, we have $w = v$, *i.e.*, $v_m \to v$ in $C(\overline{\Omega})$. Hence, $v = \mu(\delta, u)$, and thus $\mu(\delta_m, u_m) \to \mu(\delta, u)$ as $m \to \infty$.

In general we can always get a subsequence of $\{\delta_m\}_{m\geq 1}$ satisfying either Case 1 or Case 2. Thus in conclusion, $\mu(\delta, u)$ is continuous, and hence for every $n \geq 1$, $S_n \subseteq C(\overline{\Omega})$ is compact and contractible.

Next we claim that $S = \bigcap_{n\geq 1} S_n$. Obviously, $S \subseteq \bigcap_{n\geq 1} S_n$. Let $u \in \bigcap_{n\geq 1} S_n$. Then from definition $u = P(v_n)$, $v_n \in S^p_{H_n(\cdot,u_n,\nabla u_n)}$ for some $n \geq 1$. On passing to a subsequence if necessary we may assume that $v_n \to v$ weakly in $L^p(\Omega)$. Then $v \in S^p_{H(\cdot,u,\nabla u)}$ (see Theorem 3.2). Thus $u = P(v)$ with $v \in S^p_{H_n(\cdot,u,\nabla u)}$, from which we can conclude that $u \in S$ *i.e.*, $S = \bigcap_{n\geq 1} S_n$. Finally from Hyman's result [41] we see that S is an R_δ set in $C(\overline{\Omega})$. □

The following remark is given as an immediate consequence of Theorem 5.1 for the multivalued problem (5.1).

Remark 5.3 If hypothesis $H(F)_4$ holds, then for every $x \in \Omega$, $S(x) = \{u(x) | u \in S\}$ (the reachable set at $x \in \Omega$) is compact and connected in R.

When $H(x, u, \nabla u)$ has nonconvex values, an analogous result for the topological structure can be obtained if we modify our hypothesis on the continuity of $H(x, u, \nabla u)$. Therefore, in this case we can prove that the solution set is path-connected. In our next result we prove that the solution set of (5.1) is path-connected under the following assumption:

$H(F)_5$: $H : \Omega \times R \times R^N \to \mathcal{P}_k(R)$ is a multifunction such that for almost all $x \in \Omega$, $(u, s) \to H(x, u, s)$ is LSC, and $H(F)_4$(i), (iii) holds.

Theorem 5.2 *If hypothesis $H(F)_5$ holds and there exists a function $b(x) \in L_+^\infty(\Omega)$, such that*

$$h\big(H(x, u, \nabla u), H(x, v, \nabla v)\big) \leq b(x)\big|u(x) - v(x)\big| \quad \textit{a.e. on } \Omega,$$

where $C\|b\|_\infty < 1$ with C in (3.3), *then $S \subseteq C(\overline{\Omega})$ is nonempty and path-connected.*

Proof As in Theorem 3.1, *a priori* estimates for the problem (5.1) can be obtained easily. Thus we may assume that $|H(x, u, \nabla u)| \leq a(x)$ a.e. Ω with $a(x) \in L_+^p(\Omega)$ for all $u \in S$. Let

$$O_\alpha = \big\{g \in L^p(\Omega) : \big|g(x)\big| \leq a(x) \text{ a.e. on } \Omega\big\}$$

and consider the multifunction $N : O_\alpha \to \mathcal{P}_f(L^p(\Omega))$ defined by $N(g) = S^p_{H(\cdot,P(g),\nabla P(g))}$ (here $P(g)$ is the solution map as before). Let $f, g \in O_\alpha$ and let $v \in N(g)$. Let $\varepsilon > 0$ and define

$$D_\varepsilon(x) = \big\{u \in H\big(x, P(f), \nabla P(f)\big) : \big|u(x) - v(x)\big| \leq d\big(v(x), H\big(x, P(f), \nabla P(f)\big)\big) + \varepsilon\big\}.$$

Set $\chi(x, u) = d(v(x), H(x, P(f), \nabla P(f))) - |v(x) - u(x)| + \varepsilon$. Clearly, for every $x \in \Omega$, $D_\varepsilon(x) \neq \emptyset$ and $\operatorname{Gr} D_\varepsilon = \{(x, u) \in \operatorname{Gr} H(\cdot, P(f), \nabla P(f)) : 0 \leq \chi(x, u)\}$. By $H(F)_5$, $(x, u, s) \to H(x, u, s)$ is measurable, thus $x \to H(x, P(f), \nabla P(f))$ is measurable. Since $(x, u) \to \chi(x, u)$ is a Carathéodory function, it is jointly measurable. Hence, $\operatorname{Gr} D_\varepsilon \in \Sigma \times B(R)$. Applying Aumann's selection theorem we find $u : \Omega \to R$ measurable and $u(x) \in D_\varepsilon(x)$ a.e. on Ω. Thus it follows that

$$\begin{aligned} d\big(v, N(f)\big) &\leq \|v - u\|_p \\ &= \left(\int_\Omega |v - u|^p \, dx\right)^{\frac{1}{p}} \\ &\leq \left(\int_\Omega \big[d\big(v(x), H\big(x, P(f), \nabla P(f)\big)\big) + \varepsilon\big]^p \, dx\right)^{\frac{1}{p}} \\ &\leq \left(\int_\Omega \big[h\big(H\big(x, P(f), \nabla P(f)\big), H\big(x, P(g), \nabla P(g)\big)\big)\big]^p \, dx\right)^{\frac{1}{p}} + |\Omega|^{\frac{1}{p}} \varepsilon \\ &\leq \left(\int_\Omega b(x)\big|P(f)(x) - P(g)(x)\big|^p \, dx\right)^{\frac{1}{p}} + |\Omega|^{\frac{1}{p}} \varepsilon \\ &\leq \|b\|_\infty C\big\|f(x) - g(x)\big\|_p + \varepsilon|\Omega|^{\frac{1}{p}}. \end{aligned} \tag{5.9}$$

As $\varepsilon \to 0$, we find $d(v, N(f)) \leq \|b\|_\infty C\|f(x) - g(x)\|_p$. Exchanging the roles of f and g we also have $d(u, N(g)) \leq \|b\|_\infty C\|f(x) - g(x)\|_p$. Thus $h(N(f), N(g)) \leq \|b\|_\infty C\|f(x) - g(x)\|_p$ with $\|b\|_\infty C < 1$. Let $\Theta := \{g \in O_\alpha : g \in N(g)\}$. By Nadler's fixed point theorem [42] we find that $\Theta \neq \emptyset$ and from [43] we know that 0 is an absolute retract in $L^p(\Omega)$. Since an absolute retract is path-connected, Θ is path-connected. Thus $P(\Theta)$ is path-connected in $C(\overline{\Omega})$. Since $P(\Theta) = S$, we see that S is nonempty and path-connected in $C(\overline{\Omega})$. This completes the proof. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript. Each of the authors YC, RPA, ABA, and DO contributed to each part of this work equally and read and approved the final version of the manuscript.

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