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Construction of measures of noncompactness of $DC^n[J,E]$ and $C_0^n[J,E]$ with application to the solvability of nth-order integro-differential equations in Banach spaces

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Abstract

In the present paper, we first investigate the construction of compact sets of $DC^n[J,E]$ and $C_0^n[J,E]$, and then we introduce new measures of noncompactness on these spaces. In addition, as an application, we discuss the existence of solutions of initial value problems for nth-order nonlinear integro-differential equations of mixed type on an infinite interval in Banach spaces. We will also state an interesting example which shows that our results can apply for solving infinite systems of integro-differential equations.

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Keywords: measure of noncompactness; Darbo fixed point theorem; Arzelà-Ascoli theorem; nth-order integro-differential equations

1 Introduction

The integro-differential equation (IDE) can be considered in different branches of sciences and engineering. It is connected naturally with a variety of models obtained from biological science, applied mathematics, physics, and other disciplines, such as theory of elasticity, biomechanics, electromagnetic, electrodynamics, fluid dynamics, heat and mass transfer, oscillating magnetic fields, *etc.* [1–4]. In recent years, several authors have studied different techniques such as the mixed monotone iterative method [5–7], the numerical methods [8–10] and the variational iteration method [11, 12] for solving initial value problems (IVP) of nonlinear integro-differential equations.

On the other hand, measures of noncompactness are very useful tools in functional analysis, for instance in metric fixed point theory and in the theory of operator equations in Banach spaces. They are also used in the studies of functional equations, ordinary and partial differential equations, integral and integro-differential equations, fractional partial differential equations, and optimal control theory (see [3, 13–23]).

Now, in this paper, we shall investigate the existence of solutions of an IVP for nth-order nonlinear integro-differential equations of mixed type on an infinite interval in E by a new

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measure of noncompactness. Please notice that our new measure of noncompactness is not necessarily equal to zero on the family of all relatively compact sets and is also very fruitful in applications. In this situation, new spaces $DC^n[J,E]$ and $C_0^n[J,E]$ are introduced and it is verified that the corresponding operators are completely continuous (*i.e.*, continuous and compact). Consider the IVP for the nth-order nonlinear integro-differential equation of mixed type,

$$\begin{aligned} u^{(n)}(t) = f\biggl(&t, u\bigl(\xi(t)\bigr), u'\bigl(\xi(t)\bigr), \ldots, u^{(n-1)}\bigl(\xi(t)\bigr), \\ &\int_0^\infty k_1(t,s)h\bigl(s, u(s), u'(s), \ldots, u^{(n-1)}(s)\bigr)\,ds, \\ &\int_0^{\beta(t)} k_2(t,s)g\bigl(s, u(s), u'(s), \ldots, u^{(n-1)}(s)\bigr)\,ds\biggr), \\ u(0) = u_0, \qquad &u'(0) = u_1, \qquad \ldots, \qquad u^{(n-1)}(0) = u_{n-1}. \end{aligned} \tag{1.1}$$

We are going to show that the above functional integro-differential equation has at least one solution in the space $DC^n[J,E]$ where $J = [0,\infty)$ and E is a Banach space. Also notice that the results of this paper extend those obtained in [5, 7, 24, 25].

Throughout this paper, we assume some basic facts concerning measures of noncompactness in [17]. Denote by $\mathbb{R}$ the set of real numbers and put $\mathbb{R}_+ = [0,+\infty)$. Let $(E, \|\cdot\|)$ be a real Banach space with zero element 0. Let $\overline{B}(x,r)$ denote the closed ball centered at x with radius r. The symbol $\overline{B}_r$ stands for the ball $\overline{B}(0,r)$. For X, a nonempty subset of E, we denote by $\overline{X}$ and $\operatorname{Conv} X$ the closure and the closed convex hull of X, respectively. Moreover, let us denote by $\mathfrak{M}_E$ the family of nonempty bounded subsets of E and by $\mathfrak{N}_E$ its subfamily consisting of all relatively compact sets.

Definition 1.1 [17] A mapping $\mu : \mathfrak{M}_E \to \mathbb{R}_+$ is said to be a measure of noncompactness in E if it satisfies the following conditions:

1° The family $\ker\mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\}$ is nonempty and $\ker\mu \subset \mathfrak{N}_E$.
2° $X \subset Y \Longrightarrow \mu(X) \le \mu(Y)$.
3° $\mu(\overline{X}) = \mu(X)$.
4° $\mu(\operatorname{Conv} X) = \mu(X)$.
5° $\mu(\lambda X + (1-\lambda)Y) \le \lambda\mu(X) + (1-\lambda)\mu(Y)$ for $\lambda \in [0,1]$.
6° If $\{X_n\}$ is a sequence of closed sets from $\mathfrak{M}_E$ such that $X_{n+1} \subset X_n$ for $n = 1,2,\ldots$ and if $\lim_{n\to\infty} \mu(X_n) = 0$ then $X_\infty = \bigcap_{n=1}^\infty X_n \neq \emptyset$.

We recall that a measure of noncompactness is regular [17] if it additionally satisfies the following conditions:

7° $\mu(X \cup Y) = \max\{\mu(X), \mu(Y)\}$.
8° $\mu(X + Y) \le \mu(X) + \mu(Y)$.
9° $\mu(\lambda X) = |\lambda|\mu(X)$ for $\lambda \in \mathbb{R}$.
10° $\ker\mu = \mathfrak{N}_E$.

It is worth mentioning that the Kuratowski and Hausdorff measures of noncompactness, two important ones, have all the properties above.

The following theorem, Darbo's fixed point theorem, will be needed in Section 3.

Theorem 1.1 [17] *Let Ω be a nonempty, bounded, closed, and convex subset of a Banach space E and also let $F:\Omega\to\Omega$ be a continuous mapping such that there exists a constant $k\in[0,1)$ with the property*

$$\mu(FX)\le k\mu(X) \tag{1.2}$$

for any nonempty subset X of Ω. Then F has a fixed point in the set Ω.

2 Main results

In this section, we first introduce the Banach spaces $DC^n[J,E]$ and $C_0^n[J,E]$. Then we characterize the compact subsets of $DC^n[J,E]$ and $C_0^n[J,E]$. Also, we introduce a new measure of noncompactness on $DC^n[J,E]$ and $C_0^n[J,E]$.

Let $BC[J,E]=\{u\in C[J,E]:e^{-t}\|u(t)\|_E$ bounded for all $t\ge 0\}$ and

$$DC^n[J,E]=\left\{u\in C^n[J,E]:e^{-t}\left\|u^{(i)}(t)\right\| \text{ bounded for all } t\ge 0, i=1,2,\dots,n\right\},$$

where $u^{(0)}(t)=u(t)$. It is easy to see that $BC[J,E]$ is a Banach space with norm

$$\|u\|_B=\sup_{t\in J}\left\|u(t)\right\|_E,$$

and $DC^n[J,E]$ is a Banach space with norm

$$\|u\|_D=\max\left\{\|u\|_B,\left\|u'\right\|_B,\left\|u''\right\|_B,\dots,\left\|u^{(n)}\right\|_B\right\}.$$

If we define $C_0[J,E]=\{u\in C[J,E]:e^{-t}\|u(t)\|_E\to 0$ as $t\to\infty\}$ and

$$C_0^n[J,E]=\left\{u\in C^n[J,E]:e^{-t}\left\|u^{(i)}(t)\right\|_E\to 0 \text{ as } t\to\infty, i=1,2,\dots,n\right\},$$

then $C_0[J,E]$ with norm $\|\cdot\|_B$ is a Banach subspace of $BC[J,E]$ and $C_0^n[J,E]$ with $\|\cdot\|_D$ is a Banach subspace of $DC^n[J,E]$. Now, we need to characterize the compact subsets of $DC^n[J,E]$.

Theorem 2.1 *Let E be a Banach space, $n\in\mathbb{N}$, and $\mathcal{B}$ be a bounded set in $DC^n[J,E]$. Then $\mathcal{B}$ is relatively compact if the following conditions are satisfied*:

(i) *$\mathcal{B}^{(k)}{}_{|[0,T]}$ are equicontinuous on $[0,T]$ for any $T>0$ where $\mathcal{B}^{(k)}{}_{|[0,T]}$ denotes the restrictions to $[0,T]$ of the functions $\mathcal{B}^{(k)}=\{u^{(k)}:u\in\mathcal{B}\}$.*

(ii) *$\mathcal{B}^{(k)}(t)=\{u^{(k)}(t):u\in\mathcal{B}\}$ is a relatively compact subset of E for all $t\in J$ and $k\in\{0,1,\dots,n\}$.*

(iii) *For each $\varepsilon>0$, there exists $T>0$ such that*

$$e^{-t}\left\|u^{(k)}(t)\right\|_E\le\varepsilon$$

for all $t>T$, $k\in\{0,1,\dots,n\}$, and $u\in\mathcal{B}$.

The proof relies on the following useful observation.

Lemma 2.2 [26] *Let X be a metric space. Assume that, for every $\varepsilon>0$, there exist some $\delta>0$, a metric space W, and a mapping $\Phi:X\to W$ so that $\Phi[X]$ is totally bounded,* *and whenever $x, y \in X$ are such that $d(\Phi(x), \Phi(y)) < \delta$, then $d(x,y) < \varepsilon$. Then X is totally bounded.*

Proof of Theorem 2.1 Let $\varepsilon > 0$. From condition (iii) of Theorem 2.1, for $\varepsilon > 0$ there exists $T > 0$ such that

$$e^{-t}\left\|u^{(k)}(t)\right\|_E \leq \varepsilon$$

for all $t > T$, $k \in \{0,1,\dots,n\}$ and $u \in \mathcal{B}$. Using the equicontinuity of $\mathcal{B}^{(k)}{}_{|[0,T]}$, we can find a finite set of points $s_1,\dots,s_m \in [0,T]$ with neighborhoods $I_1,\dots,I_m$ covering all of $[0,T]$ so that

$$\left\|u^{(k)}(t) - u^{(k)}(s_j)\right\|_E < \varepsilon,$$

whenever $u \in \mathcal{B}$, $t \in I_j$ and $0 \leq k \leq n$. Define $\Phi : \mathcal{B} \to E^{m(n+1)}$ by

$$\Phi(u) = \bigl(u(s_1),\dots,u(s_m),u'(s_1),\dots,u'(s_m),\dots,u^{(n)}(s_1),\dots,u^{(n)}(s_m)\bigr).$$

By the boundedness of $\mathcal{B}^{(k)}$ for all $0 \leq k \leq n$, the image $\Phi[\mathcal{B}]$ is bounded and $\Phi[\mathcal{B}] \subseteq \prod_{k=0}^{n}\prod_{i=1}^{m}\mathcal{B}^{(k)}(s_i)$. Since $\mathcal{B}^{(k)}(s_i)$ is relatively compact for all $0 \leq k \leq n$ and $1 \leq i \leq m$, $\Phi[\mathcal{B}]$ is totally bounded in $E^{m(n+1)}$. Furthermore, if $u,f \in \mathcal{B}$ with $\|\Phi(u) - \Phi(f)\|_{E^{m(n+1)}} < \varepsilon$, then since any $t \in [0,T]$ belongs to some I_j, we get

$$\begin{aligned}\left\|e^{-t}u^{(k)}(t) - e^{-t}f^{(k)}(t)\right\|_E &\leq \left\|u^{(k)}(t) - u^{(k)}(s_j)\right\|_E + \left\|u^{(k)}(s_j) - f^{(k)}(s_j)\right\|_E \\ &\quad + \left\|f^{(k)}(s_j) - f^{(k)}(t)\right\|_E \\ &< 3\varepsilon. \end{aligned} \tag{2.1}$$

On the other hand, for any $t \in [T,\infty)$, we have

$$\left\|e^{-t}u^{(k)}(t) - e^{-t}f^{(k)}(t)\right\|_E \leq \left\|e^{-t}u^{(k)}(t)\right\|_E + \left\|e^{-t}f^{(k)}(t)\right\|_E \leq 2\varepsilon. \tag{2.2}$$

So from (2.1) and (2.2), we get $\|u - f\|_D \leq 3\varepsilon$. Now by Lemma 2.2, $\mathcal{B}$ is totally bounded. □

The next result is also a consequence of Theorem 2.1.

Corollary 2.3 *Let $n \in \mathbb{N}$ and $\mathcal{B}$ be a bounded set in $C_0^n[J,E]$. Then $\mathcal{B}$ is totally bounded in $C_0^n[J,E]$ if and only if the following conditions are satisfied:*

(i) *$\mathcal{B}^{(k)}{}_{|[0,T]}$ are equicontinuous on $[0,T]$ for any $T > 0$, where $\mathcal{B}^{(k)}{}_{|[0,T]}$ denotes the restrictions to $[0,T]$ of the functions $\mathcal{B}^{(k)} = \{u^{(k)} : u \in \mathcal{B}\}$.*

(ii) *$\mathcal{B}^{(k)}(t) = \{u^{(k)}(t) : u \in \mathcal{B}\}$ is a relatively compact subset of E for all $t \in J$ and $k \in \{0,1,\dots,n\}$.*

(iii) *For each $\varepsilon > 0$, there exists $T > 0$ such that*

$$e^{-t}\left\|u^{(k)}(t)\right\|_E \leq \varepsilon$$

for all $t > T$, $k \in \{0,1,\dots,n\}$ and $u \in \mathcal{B}$.

Proof Assume that $\mathcal{B}$ satisfies conditions (i)-(iii). Since $C_0^n[J,E]$ is a subspace of $DC^n[J,E]$, $\mathcal{B}$ is totally bounded.

Conversely, assume that $\mathcal{B}$ is totally bounded. Let us fix arbitrarily $k \in \{0,1,\dots,n\}$ and $T > 0$. To prove the equicontinuity of $\mathcal{B}^{(k)}{}_{|[0,T]}$, let $t \in [0,T]$ and $\varepsilon > 0$ be given. Pick an ε-cover $\{U_1,\dots,U_m\}$ of $\mathcal{B}$, and choose $g_j \in U_j$ for $j = 1,\dots,m$. Pick a neighborhood I_j of t, so that

$$\|g_j{}^{(k)}(t) - g_j{}^{(k)}(s)\|_E < \varepsilon,$$

whenever $s \in I_j$, for $j = 1,\dots,m$. Let $I = I_1 \cap \cdots \cap I_m$. If $u \in U_j$ then $\|u - g_j\|_D < \varepsilon$, and so when $s \in I$,

$$\|u^{(k)}(s) - u^{(k)}(t)\|_E \leq \|u^{(k)}(s) - g_j{}^{(k)}(s)\|_E + \|g_j{}^{(k)}(s) - g_j{}^{(k)}(t)\|_E + \|g_j{}^{(k)}(t) - u^{(k)}(t)\|_E < 3\varepsilon.$$

Now, since $[0,T]$ is compact, we have the equicontinuity of $\mathcal{B}^{(k)}{}_{|[0,T]}$.

Next, we show that (ii) holds. Let $t \in J$ and $k \in \{0,1,\dots,n\}$ be given. We define the function $P_k : DC^{(n-1)}[J,E] \to E$ by $P_k(u) = u^{(k)}(t)$. Since P_k are continuous for all $0 \leq k \leq n-1$ and $\mathcal{B}$ is compact subset of $DC^n[J,E]$, so $\mathcal{B}^k(t)$ is compact.

Finally, take an arbitrary $\varepsilon > 0$. Thus, there exist $u_1,\dots,u_m \in \mathcal{B}$ such that $\mathcal{B} \subseteq \bigcup_{i=1}^{i=m} \overline{B}(u_i, \varepsilon)$. Since $u_i \in C_0^k[J,E]$, there exists $T > 0$ such that

$$\|u_i^{(k)}(t)\|_E < \varepsilon$$

for all $1 \leq i \leq m$, $0 \leq k \leq n$ and $t > T$. Hence for each $u \in \mathcal{B}$, there exists an $1 \leq i \leq m$ such that u belongs to $\overline{B}(u_i,\varepsilon)$, and therefore we get

$$\begin{aligned}\|u^{(k)}(t)\|_E &\leq \|u^{(k)}(t) - u_i^{(k)}(t)\|_E + \|u_i^{(k)}(t)\|_E \\ &\leq 2\varepsilon\end{aligned}$$

for all $t > T$ and $0 \leq k \leq n$, and consequently conditions (i)-(iii) are satisfied. □

Now, we are ready to define a new measure of noncompactness on $DC^n[J,E]$.

Let E be a Banach space, μ be a measure of noncompactness on E, $n \in \mathbb{N}$ and $\mathcal{B}$ be a bounded set in $DC^n[J,E]$. For $u \in \mathcal{B}$ and $\varepsilon > 0$. Let us denote

$$\begin{aligned}&\omega^T(u,\varepsilon) = \sup\{\|u^{(k)}(t) - u^{(k)}(s)\|_E : t,s \in [0,T], |t-s| < \varepsilon, 0 \leq k \leq n\},\\ &\omega^T(\mathcal{B},\varepsilon) = \sup\{\omega^T(u,\varepsilon) : u \in \mathcal{B}\},\\ &\omega^T(\mathcal{B}) = \lim_{\varepsilon\to 0} \omega^T(\mathcal{B},\varepsilon),\\ &\omega(\mathcal{B}) = \lim_{T\to\infty} \omega^T(\mathcal{B}),\end{aligned}$$

and

$$\begin{aligned}&W(\mathcal{B}) = \sup\{\mu(\mathcal{B}^{(k)}(t)) : t \in J, 0 \leq k \leq n\},\\ &d(\mathcal{B}) = \lim_{t\to\infty} \sup\{e^{-t}\|u^{(k)}(t)\|_E : u \in \mathcal{B}, 0 \leq k \leq n\}.\end{aligned}$$

Moreover, let us put

$$\omega_0(\mathcal{B}) = \omega(\mathcal{B}) + d(\mathcal{B}) + W(\mathcal{B}).$$

Theorem 2.4 *The function ω_0, where $\omega_0 : \mathfrak{M}_{DC^n[J,E]} \to \mathbb{R}$, is a measure of noncompactness on $DC^n[J,E]$.*

Proof First we show 1° holds. To do this, take $\mathcal{B} \in \mathfrak{M}_{DC^n[J,E]}$ such that $\omega_0(\mathcal{B}) = 0$. Let $\eta > 0$ be arbitrary, since $\omega_0(\mathcal{B}) = 0$,

$$\lim_{T\to\infty}\lim_{\varepsilon\to 0}\omega^T(\mathcal{B},\varepsilon) = 0.$$

Thus, there exist $\delta > 0$ and $T' > 0$ such that $\omega^T(\mathcal{B},\delta) < \eta$ for all $T > T'$. This implies that

$$\left\| u^{(k)}(t) - u^{(k)}(s) \right\|_E < \eta$$

for all $u \in \mathcal{B}$, $0 \leq k \leq n$ and $t,s \in [0,T]$ such that $|t-s| < \delta$. Then ${\mathcal{B}^{(k)}}_{|[0,T]}$ is bounded and equicontinuous for all $0 \leq k \leq n$ and $T > T'$. On the other hand, since ${\mathcal{B}^{(k)}}_{|[0,T']}$ is bounded and equicontinuous for all $0 \leq k \leq n$, then ${\mathcal{B}^{(k)}}_{|[0,T]}$ is bounded and equicontinuous for all $0 \leq k \leq n$ and $T < T'$. Using again the fact that $\omega_0(\mathcal{B}) = 0$ we have $d(\mathcal{B}) = 0$ and $W(\mathcal{B}) = 0$. Hence the condition (ii) and (iii) of Theorem 2.1 is valid and we conclude that 1° holds.

2° follows directly from the definition ω_0.

Next, we show that 3° holds. Suppose that $\mathcal{B} \in \mathfrak{M}_{DC^n[J,E]}$ and $\{u_m\} \subset \mathcal{B}$ such that $u_m \to u \in \overline{\mathcal{B}}$ in $DC^n[J,E]$. By the definition of $\omega^T(\mathcal{B},\varepsilon)$ we have

$$\left\| u_m^{(k)}(t) - u_m^{(k)}(s) \right\|_E \leq \omega^T(\mathcal{B},\varepsilon)$$

for all $m \in \mathbb{N}$, $0 \leq k \leq n$ and $t,s \in [0,T]$ with $|t-s| < \varepsilon$. Letting $m \to \infty$, we get

$$\left\| u^{(k)}(t) - u^{(k)}(s) \right\|_E \leq \omega^T(\mathcal{B},\varepsilon)$$

for any $0 \leq k \leq n$ and $t,s \in [0,T]$ with $|t-s| < \varepsilon$, and hence

$$\lim_{T\to\infty}\lim_{\varepsilon\to 0}\omega^T(\overline{\mathcal{B}},\varepsilon) \leq \lim_{T\to\infty}\lim_{\varepsilon\to 0}\omega^T(\mathcal{B},\varepsilon).$$

This implies that $\omega(\overline{\mathcal{B}}) \leq \omega(\mathcal{B})$ and by means of 2°, we obtain

$$\omega(\overline{\mathcal{B}}) = \omega(\mathcal{B}). \tag{2.3}$$

Also, since

$$\sup\left\{e^{-t}\left\| u^k(t) \right\|_E : u \in \mathcal{B}, 0 \leq k \leq n\right\} = \sup\left\{e^{-t}\left\| u^k(t) \right\|_E : u \in \overline{\mathcal{B}}, 0 \leq k \leq n\right\}$$

and

$$\sup\left\{\mu\left(\mathcal{B}^{(k)}(t)\right) : t \in [0,T], 0 \leq k \leq n\right\} = \sup\left\{\mu\left(\overline{\mathcal{B}}^{(k)}(t)\right) : t \in [0,T], 0 \leq k \leq n\right\},$$

we have $d(\overline{\mathcal{B}}) = d(\mathcal{B})$, $W(\overline{\mathcal{B}}) = W(\mathcal{B})$, and from (2.3) we get $\omega_0(\overline{\mathcal{B}}) = \omega_0(\mathcal{B})$. Hence ω_0 satisfies condition 3° of Definition 1.1.

Condition 4° follows directly from $[\mathrm{Conv}(\mathcal{B})]^{(k)} = \mathrm{Conv}(\mathcal{B}^{(k)})$ and is left to the reader.

The proof of condition 5° can be carried out by using the equality

$$\bigl(\lambda f + (1-\lambda)g\bigr)^{(k)} = \lambda f^{(k)} + (1-\lambda)g^{(k)}$$

for all $\lambda \in [0,1]$.

It remains to prove 6°, suppose that $\{\mathcal{B}_m\}$ is a sequence of closed and nonempty sets of $\mathfrak{M}_{DC^n[J,E]}$ such that $\mathcal{B}_{m+1} \subset \mathcal{B}_m$ for $m = 1,2,\ldots$, and $\lim_{m\to\infty} \omega_0(\mathcal{B}_m) = 0$. Now for any $m \in \mathbb{N}$, take $u_m \in \mathcal{B}_m$. Suppose that A is a countable dense subset of J.

Claim 1 *$\{u_m\}$ has a subsequence $\{u_{m_k}\}$ such that $u_{m_k}^{(k)}(t)$ converges for every $t \in A$ and $0 \le k \le n$.*

Let $\{t_i\}$ be the points of A, arranged in the sequence. Since $\omega_0(\mathcal{B}_m) \to 0$, $\mu(\mathcal{B}_m^{(k)}(t)) \to 0$ and there exists a subsequence, which we denote by $\{u_{1,i}\}$ such that $\{u_{1,i}^{(k)}\}$ $(0 \le k \le n)$ converges, as $i \to \infty$. Let us now consider the sequence $S_1, S_2, \ldots$, which we represent by the array

$$\begin{array}{lllll} S_1: & u_{1,1}, & u_{1,2}, & u_{1,3}, & \ldots \\ S_2: & u_{2,1}, & u_{2,2}, & u_{2,3}, & \ldots \\ S_3: & u_{3,1}, & u_{3,2}, & u_{3,3}, & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

such that S_j is a subsequence of S_{j-1} for $j = 1,2,\ldots$ and $u_{j,i}^{(k)}(t_j)$ $(0 \le k \le n)$ converges, as $i \to \infty$. We now go down the diagonal of the array, *i.e.*, we consider the sequence

$$S: \quad u_{1,1}, u_{2,2}, u_{3,3}, \ldots.$$

Hence $u_{m,m}^{(k)}(t)$ converges for every $t \in A$ and $0 \le k \le n$.

Without loss of generality, we can suppose that A be a countable dense subset of J and $\{u_n(t)\}$ converges for every $t \in A$.

Claim 2 *$\{u_m\}$ has a subsequence $\{u_{m_k}\}$ and there exists a $u_0 \in DC^n[J,E]$ such that $u_{m_k} \to u_0$ in $DC^n[J,E]$.*

Let $\varepsilon > 0$ be fixed, since $\lim_{m\to\infty} \omega_0(\mathcal{B}_m) = 0$, there exists $N_1 \in \mathbb{N}$ such that $\omega_0(\mathcal{B}_{N_1}) < \varepsilon$. Hence, we can find $T > 0$ and $\delta > 0$ such that

$$\omega^T(\mathcal{B}_{N_1}, \delta) < \varepsilon \tag{2.4}$$

and

$$e^{-t}\bigl\|u^{(k)}(t)\bigr\|_E < \varepsilon \tag{2.5}$$

for all $u \in \mathcal{B}_{N_1}$, $0 \le k \le n$, and $t \in J$ with $t > T$. Also, we can find a finite set of points $y_1, \dots, y_l \in A \cap [0,T]$ with neighborhoods $B_\delta(y_1), \dots, B_\delta(y_l)$ that cover $[0,T]$, so that for all $t \in [0,T]$ we have $i \in \{1,2,\dots,l\}$ where

$$\left\| u_n^{(k)}(y_i) - u_n^{(k)}(t) \right\|_E \le \omega^T(u_n,\delta) \le \omega^T(\mathcal{B}_N,\delta) < \varepsilon \tag{2.6}$$

for all $0 \le k \le n$. Since $\{u_n^{(k)}(y)\}$ converges for every $y \in A$ and $0 \le k \le n$, there exists $N_2 > N_1$ such that for all $p,q > N_2$ and $1 \le i \le l$, we have

$$\left\| u_p^{(k)}(y_i) - u_q^{(k)}(y_i) \right\|_E < \varepsilon.$$

There are now two cases.

Case 1: If $t \le T$, then there exists i_0, such that for all $p,q > N_2$, using (2.6) we get

$$\begin{aligned} e^{-t}\left\| u_p^{(k)}(t) - u_q^{(k)}(t) \right\|_E &\le \left\| u_p^{(k)}(t) - u_q^{(k)}(t) \right\|_E \\ &\le \left\| u_p^{(k)}(t) - u_p^{(k)}(y_{i_0}) \right\|_E + \left\| u_p^{(k)}(y_{i_0}) - u_q^{(k)}(y_{i_0}) \right\|_E \\ &\quad + \left\| u_p^{(k)}(y_{i_0}) - u_q^{(k)}(t) \right\|_E \\ &< 3\varepsilon. \end{aligned}$$

Case 2: If $\|x\| > T$, by (2.5), we have

$$e^{-t}\left\| u_p^{(k)}(t) - u_q^{(k)}(t) \right\|_E \le e^{-t}\left(\left\| u_p^{(k)}(t) \right\|_E + \left\| u_q^{(k)}(t) \right\|_E \right) \le \varepsilon$$

for all $p,q > N_2$ and $0 \le k \le n$. Thus, $\{u_n\}$ is a Cauchy sequence in $DC^n[J,E]$ and there exists $u_0 \in DC^n[J,E]$ such that $u_n \to u_0$ and this completes the proof of the claim.

Now, since $u_n \in \mathcal{B}_n$, $\mathcal{B}_{n+1} \subset \mathcal{B}_n$, and $\mathcal{B}_n$ is closed for all $n \in \mathbb{N}$ we obtain

$$u_0 \in \bigcap_{n=1}^{\infty} \mathcal{B}_n = \mathcal{B}_\infty,$$

which completes the proof of 6°. □

Corollary 2.5 *The function $\omega_0 : \mathfrak{M}_{C_0^n[J,E]} \to \mathbb{R}$ is a regular measure of noncompactness on $C_0^n[J,E]$.*

Proof Since $C_0^n[J,E]$ is a subspace of $DC^n[J,E]$, ω_0 defines a measure of noncompactness on $C_0^n[J,E]$. Also, it is easy to see that ω_0 satisfies conditions 7°-9°. Suppose that $\mathcal{B} \in \mathfrak{N}_{C_0^k[J,E]}$. Thus, the closure of $\mathcal{B}$ in $C_0^k[J,E]$ is compact. Let us fix arbitrarily $\varepsilon > 0$. Since $\mathcal{B}^{(k)}{}_{|[0,T]}$ are bounded and equicontinuous on $[0,T]$ for all $0 \le k \le n$ and $T > 0$, there exists $\delta > 0$ such that

$$\left\| u^{(k)}(t) - u^{(k)}(s) \right\| < \varepsilon$$

for all $0 \le k \le n$, $u \in \mathcal{B}$, and $t,s \in [0,T]$ such that $|t-s| \le \delta$. Then for all $u \in \mathcal{B}$ we have

$$\omega^T(u,\delta) = \sup\left\{ \left\| u^{(k)}(t) - u^{(k)}(s) \right\| : t,s \in [0,T], |t-s| < \delta, 0 \le k \le n \right\} \le \varepsilon$$

and

$$\omega^T(\mathcal{B},\varepsilon) = \sup\{\omega^T(u,\varepsilon) : u \in \mathcal{B}\} \leq \varepsilon.$$

This implies that

$$\omega(\mathcal{B}) = \lim_{T\to\infty}\lim_{\delta\to 0}\omega^T(\mathcal{B},\delta) = 0. \tag{2.7}$$

Also, using conditions (ii) and (iii) of Corollary 2.3 implies that $d(\mathcal{B}) = 0$, $W(\mathcal{B}) = 0$, and the condition $\ker(\omega_0) = \mathfrak{N}_{C_0^k[J,E]}$ holds. □

3 Application

In this section, we will investigate the solvability of the functional integral equation (1.1) in the space $DC^n[J,E]$. We will assume that the following conditions are satisfied:

(i) $k_i : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ $(i = 1,2)$ and $\xi, \beta : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous functions such there exist two positive constants Q_1 and Q_2 such that

$$Q_1 := \sup\left\{e^{-t}\left|\int_0^\infty k_1(t,s)\,ds\right| : t \in \mathbb{R}_+\right\} < \infty,$$
$$Q_2 := \sup\left\{e^{-t}\left|\int_0^{\beta(t)} k_2(t,s)\,ds\right| : t \in \mathbb{R}_+\right\} < \infty,$$
$$\lim_{t\to\infty} e^{-t}\left|\int_0^\infty k_1(t,s)\,ds\right| = 0,$$

and

$$\lim_{t\to\infty} e^{-t}\left|\int_0^{\beta(t)} k_2(t,s)\,ds\right| = 0.$$

(ii) $u_0, u_1, \ldots, u_{n-1} \in E$.

(iii) $f : \mathbb{R}_+ \times E^{n+2} \to E$ is continuous and there exist two continuous functions $a_1, a_2 : \mathbb{R}_+ \to \mathbb{R}_+$ and a nondecreasing and continuous function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\left\|f(t,x_1,x_2,\ldots,x_{n+1},x_{n+2})\right\|_E \leq a_1(t) + a_2(t)\psi\left(e^{-t}\max_{1\leq i\leq n+2}\|x_i\|_E\right). \tag{3.1}$$

Moreover, there exist two positive constants M_1 and M_2 such that

$$M_i = \sup\left\{\frac{e^{-t}}{(n-1-k)!}\left|\int_0^t (t-s)^{n-1-k}a_i(s)ds\right| : t \in \mathbb{R}_+, 0 \leq k \leq n-1\right\} < \infty$$

for $i = 1,2$ and

$$\lim_{t\to\infty}\sup\left\{e^{-t}\left|\int_0^t (t-s)^{n-1-k}a_i(s)ds\right| : i = 1,2, 0 \leq k \leq n-1\right\} = 0.$$

(iv) For any $r > 0$, h and g are uniformly continuous on $[0,r] \times \overline{B}_r^n$, and $f([0,r] \times \overline{B}_r^{n+2})$ is uniformly continuous and relatively compact in E.

(v) $h, g : \mathbb{R}_+ \times E^n \to E$ are continuous such that for any $r > 0$, h and g are uniformly continuous on $[0, r] \times \overline{B}_r^n$,

$$\|h(t, x_0, x_1, \ldots, x_{n-1})\|_E \le e^{-t} \max_{0 \le i \le n-1} \|x_i\|_E$$

and

$$\|g(t, x_0, x_1, \ldots, x_{n-1})\|_E \le e^{-t} \max_{0 \le i \le n-1} \|x_i\|_E.$$

(vi) There exists a positive solution r_0 of the inequality

$$M_2 \psi\bigl(\max\{r, Q_1 r, Q_2 r\}\bigr) + M_1 + M_3 \le r,$$

where $M_3 = \max\{\|u_0\|_E, \|u_1\|_E, \ldots, \|u_{n-1}\|_E\}$.

Now, we will need the following lemmas later.

Lemma 3.1 *Assume that h satisfies the hypothesis* (v), *then* $H : DC^{(n-1)}[J, E] \to BC[J, E]$, *defined by*

$$Hx(t) = \int_0^\infty k_1(t, s) h\bigl(s, x(s), x'(s), \ldots, x^{(n-1)}(s)\bigr)\, ds, \tag{3.2}$$

is a continuous operator and $\|Hx\|_B \le Q_1 \|x\|_D$.

Proof Obviously, $Hx(t)$ for any $x \in DC^{(n-1)}[J, E]$ is continuous on J, and we have

$$\begin{aligned} e^{-t}\|Hx(t)\|_E &= e^{-t}\left\| \int_0^\infty k_1(t, s) h\bigl(s, x(s), x'(s), \ldots, x^{(n-1)}(s)\bigr)\, ds \right\|_E \\ &\le e^{-t}\left\| \int_0^\infty k_1(t, s) e^{-s} \max_{0 \le i \le n-1} x^{(i)}\, ds \right\|_E \\ &\le Q_1 \|x\|_D. \end{aligned}$$

Thus, we obtain

$$\|Hx\|_B \le Q_1 \|x\|_D. \tag{3.3}$$

Hence $H(x) \in BC[J, E]$ and H is well defined. Now we show that H is continuous. To verify this, take $x \in DC^{n-1}[J, E]$ and $\varepsilon > 0$ arbitrarily. Moreover, take $y \in DC^{n-1}[J, E]$ with $\|x - y\|_D < \varepsilon$. Then, considering condition (i), there exists $T > 0$ such that for $t > T$, we obtain

$$\begin{aligned} &e^{-t}\|Hx(t) - Hy(t)\|_E \\ &\quad\le e^{-t}\left\| \int_0^\infty k_1(t, s)\bigl[h\bigl(s, x(s), x'(s), \ldots, x^{(n-1)}(s)\bigr) - h\bigl(s, y(s), y'(s), \ldots, y^{(n-1)}(s)\bigr)\bigr]\, ds \right\|_E \\ &\quad\le e^{-t} \int_0^\infty k(t, s)\bigl[\|h\bigl(s, x(s), x'(s), \ldots, x^{(n-1)}(s)\bigr)\|_E \\ &\qquad + \|h\bigl(s, y(s), y'(s), \ldots, y^{(n-1)}(s)\bigr)\|_E\bigr]\, ds \end{aligned}$$

$$\begin{aligned} &\le 2\bigl(\|x\|_D + \varepsilon\bigr)e^{-t}\int_0^\infty k(t,s)\,ds \\ &\le 2\bigl(\|x\|_D + \varepsilon\bigr)\varepsilon. \end{aligned} \tag{3.4}$$

Also if $t \in [0,T]$, then from the first inequality in (3.4) follows that

$$e^{-t}\bigl\|Hx(t) - Hy(t)\bigr\|_E \le Q_1\omega_b^T(h,\varepsilon), \tag{3.5}$$

where

$$\begin{aligned} &b = \|x\|_D + \varepsilon, \\ &\omega_b^T(h,\varepsilon) = \sup\bigl\{\bigl|h(s,x_0,x_1,\dots,x_{n-1}) - h(s,y_0,y_1,\dots,y_{n-1})\bigr| : \\ &\qquad s \in [0,T], x_i, y_i \in \overline{B}_b, \|x_i - y_i\|_E \le \varepsilon\bigr\}. \end{aligned}$$

Since h is uniformly continuous on $[0,T] \times \overline{B}_b \times \cdots \times \overline{B}_b$, we have $\omega_b^T(h,\varepsilon) \to 0$ as $\varepsilon \to 0$. Thus, H is a continuous function. □

Lemma 3.2 *Assume that g satisfies the hypothesis* (v), *then $G : DC^{(n-1)}[J,E] \to BC[J,E]$ defined by*

$$Gx(t) = \int_0^{\beta(t)} k_2(t,s)g\bigl(s,x(s),x'(s),\dots,x^{(n-1)}(s)\bigr)\,ds, \tag{3.6}$$

is a continuous operator and $\|Gx\|_B \le Q_2\|x\|_D$.

Proof The proof is similar to Lemma 3.1. □

Theorem 3.3 *Under assumptions* (i)-(vi), *equation* (1.1) *has at least one solution in the space $DC^n[J,E]$.*

Proof By Taylor's formula, we have

$$u(t) = u(0) + u'(0)t + \cdots + \frac{u^{(n-1)}(0)}{(n-1)!}t^{n-1} + \frac{1}{(n-1)!}\int_0^t (t-s)^{n-1}u^{(n)}(s)\,ds$$

for all $C^n[J,E]$. Then the nth-order nonlinear integro-differential equation (1.1) has at least one solution in the space $DC^n[J,E]$ if and only if nonlinear integral equation

$$u(t) = p(t) + \frac{1}{(n-1)!}\int_0^t (t-s)^{n-1}f\bigl(s,u(s),u'(s),\dots,u^{(n-1)}(s),Hu(s),Gu(s)\bigr)\,ds \tag{3.7}$$

has at least one solution in the space $DC^{(n-1)}[J,E]$ where

$$p(t) = u_0 + u_1t + \cdots + \frac{u_{n-1}}{(n-1)!}t^{n-1}.$$

Now, we define the operator $F : DC^{(n-1)}[J,E] \to DC^{(n-1)}[J,E]$ by

$$Fu(t) = p(t) + \frac{1}{(n-1)!}\int_0^t (t-s)^{n-1}f\bigl(s,u(s),u'(s),\dots,u^{(n-1)}(s),Hu(s),Gu(s)\bigr)\,ds. \tag{3.8}$$

First, notice that the continuity of $\frac{d^k(Fu)}{dt^k}(t)$ for any $u \in DC^{(n-1)}[J,E]$ and $0 \leq k \leq n-1$ is obvious. Also, for any $t \in \mathbb{R}_+$, $1 \leq k \leq n-1$, and by (3.8) we have

$$\begin{aligned}\frac{d^k(Fu)}{dt^k}(t) &= p^{(k)}(t) + \frac{1}{(n-1-k)!}\int_0^t (t-s)^{n-1-k} \\ &\quad \times f\bigl(s, u(s), u'(s), \ldots, u^{(n-1)}(s), Hu(s), Gu(s)\bigr)\, ds.\end{aligned}$$

Using conditions (i)-(iv), for arbitrarily fixed $t \in J$, we have

$$\begin{aligned}&e^{-t}\left\|\frac{d^k(Fu)}{dt^k}(t)\right\|_E \\ &\quad \leq e^{-t}\bigl\|p^{(k)}(t)\bigr\|_E \\ &\qquad + \frac{e^{-t}}{(n-1-k)!}\left\|\int_0^t (t-s)^{n-1-k} f\bigl(s, u(s), u'(s), \ldots, u^{(n-1)}(s), Hu(s), Gu(s)\bigr)\, ds\right\|_E \\ &\quad \leq \|p\|_D + \frac{e^{-t}}{(n-1-k)!}\int_0^t (t-s)^{n-1-k}\bigl[a_1(s) \\ &\qquad + a_2(s)\psi\bigl(e^{-s}\max\bigl\{\|u(s)\|_E, \|u'(s)\|_E \ldots, \|u^{(n-1)}(s)\|_E, \|Hu(s)\|_E, \|Gu(s)\|_E\bigr\}\bigr)\bigr]\, ds \\ &\quad \leq \|p\|_D + M_1 + M_2\psi\bigl(\max\bigl\{\|u\|_D, Q_1\|u\|_D, Q_2\|u\|_D\bigr\}\bigr).\end{aligned}$$

Since $M_3 = \|p\|_D$, we have

$$\|Fu\|_D \leq M_3 + M_1 + M_2\psi\bigl(\max\bigl\{\|u\|_D, Q_1\|u\|_D, Q_2\|u\|_D\bigr\}\bigr) \tag{3.9}$$

and $F(u) \in DC^{(n-1)}[J,E]$ for any $u \in DC^{(n-1)}[J,E]$. Due to inequality (3.9) and using (vi), the function F maps $\overline{B}_{r_0}$ into $\overline{B}_{r_0}$. Now we show that F is a continuous function on $\overline{B}_{r_0}$. To do this, let us fix $\varepsilon > 0$ and take arbitrary $u, v \in \overline{B}_{r_0}$ such that $\|u - v\|_D < \varepsilon$. Then for $k \in \{0, 1, \ldots, n-1\}$, we get

$$\begin{aligned}&e^{-t}\left\|\frac{d^k(Fu)}{dt^k}(t) - \frac{d^k(Fv)}{dt^k}(t)\right\|_E \\ &\quad \leq \frac{e^{-t}}{(n-1-k)!}\left\|\int_0^t (t-s)^{n-1-k}\bigl[f\bigl(s, u(s), u'(s), \ldots, u^{(n-1)}(s), Hu(s), Gu(s)\bigr) \right. \\ &\qquad \left. - f\bigl(s, v(s), v'(s), \ldots, v^{(n-1)}(s), Hv(s), Gv(s)\bigr)\bigr]\, ds\right\|_E.\end{aligned} \tag{3.10}$$

Furthermore, considering condition (iii), there exists $T > 0$ such that for $t > T$, we have

$$\begin{aligned}&e^{-t}\left\|\frac{d^k(Fu)}{dt^k}(t) - \frac{d^k(Fv)}{dt^k}(t)\right\|_E \\ &\quad \leq \frac{e^{-t}}{(n-1-k)!}\left\|\int_0^t (t-s)^{n-1-k}\bigl[f\bigl(s, u(s), u'(s), \ldots, u^{(n-1)}(s), Hu(s), Gu(s)\bigr) \right. \\ &\qquad \left. - f\bigl(s, v(s), v'(s), \ldots, v^{(n-1)}(s), Hv(s), Gv(s)\bigr)\bigr]\, ds\right\|_E \\ &\quad \leq \frac{e^{-t}}{(n-1-k)!}\int_0^t (t-s)^{n-1-k}\bigl[\bigl\|f\bigl(s, u(s), u'(s), \ldots, u^{(n-1)}(s), Hu(s), Gu(s)\bigr)\bigr\|_E\end{aligned}$$

$$\begin{aligned}
&\quad + \bigl\|f\bigl(s,v(s),v'(s),\ldots,v^{(n-1)}(s),Hv(s),Gv(s)\bigr)\bigr\|_E\Bigr]\,ds \\
&\le \frac{2e^{-t}}{(n-1-k)!}\int_0^t (t-s)^{n-1-k}\bigl[a_1(s)+a_2(s)\psi(r_0)\bigr]\,ds \\
&\le 2\bigl(\varepsilon+\psi(r_0)\varepsilon\bigr).
\end{aligned} \tag{3.11}$$

Now we assume that $t \in [0,T]$. By applying the assumptions, we have

$$\begin{aligned}
&e^{-t}\left\|\frac{d^k(Fu)}{dt^k}(t)-\frac{d^k(Fv)}{dt^k}(t)\right\|_E \\
&\le \frac{e^{-t}t^{n-1-k}}{(n-k-1)!}\left\|\int_0^t \bigl[f\bigl(s,u(s),u'(s),\ldots,u^{(n-1)}(s),Hu(s),Gu(s)\bigr)\right. \\
&\quad \left. -f\bigl(s,v(s),v'(s),\ldots,v^{(n-1)}(s),Hv(s),Gv(s)\bigr)\bigr]\,ds\right\|_E \\
&\le \frac{T^{n-1-k}}{(n-k-1)!}\vartheta(\varepsilon),
\end{aligned} \tag{3.12}$$

where

$$\begin{aligned}
\vartheta(\varepsilon) = \sup\bigl\{&\bigl\|f(t,u_0,u_1,\ldots,u_{n-1})-f(t,v_0,v_1,\ldots,v_{n-1})\bigr\| : \\
&t\in[0,T], u_i,v_i\in \overline{B}_{r_0}, \|u_i-v_i\|_E\le\varepsilon\bigr\}.
\end{aligned}$$

Since f is uniformly continuous on $[0,T]\times\overline{B}^n_{r_0}$, we have $\vartheta(\varepsilon)\to 0$ as $\varepsilon\to 0$. Thus F is a continuous operator on $DC^{(n-1)}[J,E]$ into $DC^{(n-1)}[J,E]$. Now, let X be a nonempty and bounded subset of $\overline{B}_{r_0}$, and assume that $T>0$ and $\varepsilon>0$ are arbitrary constants. Let $t_1,t_2\in[0,T]$, with $|t_2-t_1|\le\varepsilon$, $u\in X$, and $k\in\{0,1,\ldots,n-1\}$. We obtain

$$\begin{aligned}
&\left\|\frac{d^k(Fu)}{dt^k}(t_1)-\frac{d^k(Fu)}{dt^k}(t_2)\right\|_E \\
&\le \left\|p^{(k)}(t_1)+\frac{1}{(n-1-k)!}\int_0^{t_1}(t_1-s)^{n-1-k}f\bigl(s,u(s),u'(s),\ldots,u^{(n-1)}(s),Hu(s),Gu(s)\bigr)\right. \\
&\quad -p^{(k)}(t_2)-\frac{1}{(n-1-k)!}\int_0^{t_2}(t_2-s)^{n-1-k} \\
&\quad \left.\times f\bigl(s,u(s),u'(s),\ldots,u^{(n-1)}(s),Hu(s),Gu(s)\bigr)\,ds\right\|_E \\
&\le \bigl\|p^{(k)}(t_1)-p^{(k)}(t_2)\bigr\|_E \\
&\quad +\left\|\frac{1}{(n-1-k)!}\int_0^{t_1}(t_1-s)^{n-1-k}f\bigl(s,u(s),u'(s),\ldots,u^{(n-1)}(s),Hu(s),Gu(s)\bigr)\right. \\
&\quad \left.-\frac{1}{(n-1-k)!}\int_0^{t_2}(t_1-s)^{n-1-k}f\bigl(s,u(s),u'(s),\ldots,u^{(n-1)}(s),Hu(s),Gu(s)\bigr)\,ds\right\|_E \\
&\quad +\left\|\frac{1}{(n-1-k)!}\int_0^{t_2}\bigl[(t_1-s)^{n-1-k}-(t_2-s)^{n-1-k}\bigr]\right. \\
&\quad \left.\times f\bigl(s,u(s),u'(s),\ldots,u^{(n-1)}(s),Hu(s),Gu(s)\bigr)\,ds\right\|_E
\end{aligned}$$

$$
\begin{aligned}
&\leq \omega^T(p,\varepsilon) + \frac{1}{(n-1-k)!}\biggl[\biggl|\int_{t_2}^{t_1}(t_1-s)^{n-1-k}\bigl[a_1(s)+a_2(s)\psi(r_0)\bigr]\,ds\biggr| \\
&\quad + \int_0^{t_2}\bigl|(t_1-s)^{n-1-k}-(t_1-s)^{n-1-k}\bigr|\bigl[a_1(s)+a_2(s)\psi(r_0)\bigr]\,ds\biggr] \\
&\leq \omega^T(p,\varepsilon) + \frac{1}{(n-1-k)!}\bigl[\varepsilon T^n U_{r_0}^T + \phi^T(\varepsilon)TU_{r_0}^T\bigr],
\end{aligned}
\tag{3.13}
$$

where

$$
\begin{aligned}
&\phi^T(\varepsilon) = \sup\bigl\{\bigl|(t_1-s)^{n-1-k}-(t_2-s)^{n-1-k}\bigr| : t_1,t_2,s\in[0,T], 0\leq k\leq n-1, |t_1-t_2|\leq\varepsilon\bigr\}, \\
&U_{r_0}^T = \sup\bigl\{\bigl|a_1(s)+a_2(s)\psi(r_0)\bigr| : s\in[0,T]\bigr\}.
\end{aligned}
$$

Since u was arbitrary element of X in (3.13), we obtain

$$
\omega^T\bigl(F(X),\varepsilon\bigr) \leq \omega^T(p,\varepsilon) + \frac{1}{(n-1-k)!}\bigl[\varepsilon T^n U_{r_0}^T + \phi^T(\varepsilon)TU_{r_0}^T\bigr]. \tag{3.14}
$$

Thus, by the uniform continuity of $p^{(k)}$ and $(t-s)^{n-1-k}$ on the compact set $[0,T]$ and $[0,T]^2$ for all $0\leq k\leq n-1$, we have $\omega^T(p,\varepsilon)\to 0$ and $\phi^T(\varepsilon)\to 0$ as $\varepsilon\to 0$. Therefore we obtain ${\omega_0}^T(F(X)) = 0$ and, finally,

$$
\omega_0\bigl(F(X)\bigr) = 0. \tag{3.15}
$$

In addition, for arbitrary $u,v\in X$, $k\in\{0,1,\ldots,n-1\}$, and $t\in\mathbb{R}_+$, we have

$$
\begin{aligned}
&e^{-t}\biggl\|\frac{d^k(Fu)}{dt^k}(t)-\frac{d^k(Fv)}{dt^k}(t)\biggr\|_E \\
&\leq \frac{e^{-t}}{(n-1-k)!}\biggl\|\int_0^t (t-s)^{n-1-k}\bigl[f\bigl(s,u(s),u'(s),\ldots,u^{(n-1)}(s),Hu(s),Gu(s)\bigr) \\
&\quad - f\bigl(s,v(s),v'(s),\ldots,v^{(n-1)}(s),Hv(s),Gv(s)\bigr)\bigr]\,ds\biggr\|_E \\
&\leq \frac{e^{-t}}{(n-1-k)!}\int_0^t (t-s)^{n-1-k}\bigl[\bigl\|f\bigl(s,u(s),u'(s),\ldots,u^{(n-1)}(s),Hu(s),Gu(s)\bigr)\bigr\|_E \\
&\quad + \bigl\|f\bigl(s,v(s),v'(s),\ldots,v^{(n-1)}(s),Hv(s),Gv(s)\bigr)\bigr\|_E\bigr]\,ds \\
&\leq \frac{2e^{-t}}{(n-1-k)!}\int_0^t (t-s)^{n-1-k}\bigl[a_1(s)+a_2(s)\psi(r_0)\bigr]\,ds.
\end{aligned}
$$

Thus, we have

$$
\operatorname{diam} F(X) \leq \frac{2e^{-t}}{(n-1-k)!}\int_0^t (t-s)^{n-1-k}\bigl[a_1(s)+a_2(s)\psi(r_0)\bigr]\,ds. \tag{3.16}
$$

Taking $t\to\infty$ in the inequality (3.16), then using (iii) we arrive at

$$
d\bigl(F(X)\bigr) = \limsup_{t\to\infty}\operatorname{diam} F(X) = 0. \tag{3.17}
$$

On the other hand, by using (iv) we obtain

$$W\big(F(X)\big) = 0. \tag{3.18}$$

Further, combining (3.15)-(3.18) we get

$$\omega_0\big(F(X)\big) = 0$$

or, equivalently,

$$\omega_0\big(F(X)\big) \le \lambda\omega_0(X)$$

with $\lambda = 0$. From Theorem 1.1 we see that the operator F has a fixed point u in $\overline{B}_{r_0}$ and thus the functional integral-differential equation (1.1) has at least one solution in $DC^n[J,E]$. □

Example 3.1 Consider the infinite system of scalar fourth-order integro-differential equations

$$\begin{aligned} u_n^{(4)}(t) &= \frac{\cos^4(t)}{n^3} + \frac{e^{\frac{-t}{2}}}{n}\ln\Big(1 + \sqrt{u_n\big(t^2\big) + u'_{n+1}\big(t^2\big) + u''_{n+3}\big(t^2\big) + u^{(3)}_{n+5}\big(t^2\big)}\Big) \\ &\quad + \frac{e^{\frac{-t}{2}}}{n^2}\bigg(\int_0^{t^2}\frac{e^{-3s}u^{(3)}_{2n}(s)}{1+ts^2}\,ds\bigg)^{\frac{1}{2}} + \frac{e^{\frac{-t}{2}}}{2\sqrt{n}}\bigg(\int_0^\infty e^{-2s}\sin(t-s)u'_n(s)\,ds\bigg)^{\frac{1}{2}}. \end{aligned} \tag{3.19}$$

Let $J = [0,\infty)$, $J_r = [0,r]$, $E = C_0 = \{u = (u_1,\dots,u_n,\dots) : u_n \to 0\}$ with norm $\|u\|_E = \sup_n |u_n|$. Then the infinite system equation (3.19) can be regarded as an IVP of the form of equation (1.1) in E. In this situation, $k_1(t,s) = e^{-s}\sin(t-s)$, $k_2(t,s) = (1+ts^2)^{-1}$, $\xi(t) = \beta(t) = t^2$, $u = (u_1,\dots,u_n,\dots)$, $v = (v_1,\dots,v_n,\dots)$,$w = (w_1,\dots,w_n,\dots)$, $x = (x_1,\dots,x_n,\dots)$, $y = (y_1,\dots,y_n,\dots)$, $z = (z_1,\dots,z_n,\dots)$,$f = (f_1,\dots,f_n,\dots)$, $h = (h_1,\dots,h_n,\dots)$, $g = (g_1,\dots,g_n,\dots)$, in which

$$\begin{aligned} f_n(t,u,v,w,x,y,z) &= \frac{\cos^4(t)}{n^3} + \frac{e^{\frac{-t}{2}}}{n}\ln(1 + \sqrt{u_n + v_{n+1} + w_{n+3} + x_{n+5}}) \\ &\quad + \frac{e^{\frac{-t}{2}}t}{n^2}\sqrt{y_{2n}} + \frac{e^{\frac{-t}{2}}}{2\sqrt{n}}\sqrt{z_n} \end{aligned}$$

and

$$h_n(s,x_0,x_1,x_2,x_3) = e^{-3s}x_{3_{2n}}, \qquad g_n(s,x_0,x_1,x_2,x_3) = e^{-s}x_{1_n}.$$

It is clear that $f \in C(\mathbb{R}_+ \times E^6, E)$, and $u_0 = (1,\dots,\frac{1}{n},\dots) \in E$, $u_1 = (0,\dots,0,\dots) \in E$, $u_2 = (1,\dots,\frac{1}{n^2},\dots) \in E$, $u_3 = (1,\dots,\frac{1}{n^3},\dots) \in E$. So, condition (ii) is valid. On the other hand,

$$\begin{aligned} Q_1 &= \sup\bigg\{e^{-t}\bigg|\int_0^\infty e^{-s}\sin(t-s)\,ds\bigg| : t \in \mathbb{R}_+\bigg\} \le \sup\big\{e^{-t} : t \in \mathbb{R}_+\big\} = 1 < \infty, \\ Q_2 &= \sup\bigg\{e^{-t}\bigg|\int_0^{\beta(t)}\big(1+ts^2\big)^{-1}\,ds\bigg| : t \in \mathbb{R}_+\bigg\} \le \sup\big\{t^2e^{-t} : t \in \mathbb{R}_+\big\} = \frac{4}{e^2} < \infty, \end{aligned}$$

$$\lim_{t\to\infty} e^{-t}\left|\int_0^\infty e^{-s}\sin(t-s)\,ds\right| \le \lim_{t\to\infty} e^{-t} = 0,$$
$$\lim_{t\to\infty} e^{-t}\left|\int_0^{\beta(t)} \left(1+ts^2\right)^{-1}\,ds\right| \le \lim_{t\to\infty} t^2e^{-t} = 0.$$

Hence, condition (i) is satisfied. Also, we see that

$$\begin{aligned}\left|f_n(t,u,v,w,x,y,z)\right| &\le \frac{\cos^4(t)}{n^3} + \frac{e^{\frac{-t}{2}}}{n}\ln(1+\sqrt{\|u\|_E+\|v\|_E+\|w\|_E+\|x\|_E}) \\ &\quad + \frac{e^{\frac{-t}{2}}}{n^2}\sqrt{\|y\|_E} + \frac{e^{\frac{-t}{2}}}{2\sqrt{n}}\sqrt{\|z\|_E} \end{aligned} \tag{3.20}$$

and

$$\left|h_n(s,x_0,x_1,x_2,x_3)\right| \le e^{-3s}\|x_3\|_E,$$
$$\left|g_n(s,x_0,x_1,x_2,x_3)\right| \le e^{-s}\|x_1\|_E,$$

and therefore, by taking $\psi(t)=4\sqrt{t}$ we obtain

$$\begin{aligned}\left\|f(t,u,v,w,x,y,z)\right\|_E &\le 1+\sqrt{16e^{-t}\Big[\max_{1\le i\le 6}\|x_i\|_E\Big]} \\ &\le a_1(t)+a_2(t)\psi\Big(e^{-t}\max_{1\le i\le 6}\|x_i\|_E\Big),\end{aligned}$$

where $a_1(t)=a_2(t)=1$, and

$$\left\|h(s,x_0,x_1,x_2,x_3)\right\|_E \le e^{-s}\max_{0\le i\le 3}\|x_i\|_E,$$
$$\left\|g(s,x_0,x_1,x_2,x_3)\right\|_E \le e^{-s}\max_{0\le i\le 3}\|x_i\|_E,$$

which imply conditions (iii) and (v) are true. Now, we get

$$M_i = \sup\left\{\frac{e^{-t}}{(3-k)!}\left|\int_0^t (t-s)^{3-k}\,ds\right| : t\in\mathbb{R}_+, 0\le k\le 3\right\} = \frac{1}{e} < \infty$$

for $i=1,2$,

$$M_3 = \max\left\{\|u_0\|_E,\|u_1\|_E,\|u_2\|_E,\|u_3\|_E\right\} = 1$$

and

$$\lim_{t\to\infty}\sup\left\{e^{-t}\left|\int_0^t (t-s)^{3-k}\,ds\right| : 0\le k\le 3\right\} = 0.$$

It can readily be seen that each number $r\ge 5$ satisfies the inequality in condition (vi), *i.e.*,

$$M_2\psi\left(\max\{r,Q_1r,Q_2r\}\right)+M_1+M_3 = \frac{4}{e}\sqrt{r}+\frac{1}{e}+1\le r.$$

Thus, as the number r_0 we can take $r_0 = 5$. Finally, we check condition (iv). Let $r > 0$ be arbitrarily given. It is clear that f is uniformly continuous on $J_r \times \overline{B}_r \times \overline{B}_r \times \overline{B}_r \times \overline{B}_r \times \overline{B}_r \times \overline{B}_r$. Let $\{t^{(m)}\} \subset J_r$, $\{u^{(m)}\} \subset \overline{B}_r$, $\{v^{(m)}\} \subset \overline{B}_r$, $\{w^{(m)}\} \subset \overline{B}_r$, $\{x^{(m)}\} \subset \overline{B}_r$, $\{y^{(m)}\} \subset \overline{B}_r$, $\{z^{(m)}\} \subset \overline{B}_r$. By virtue of (3.20), we have

$$\begin{aligned} &\bigl|f_n\bigl(t^{(m)}, u^{(m)}, v^{(m)}, w^{(m)}, x^{(m)}, y^{(m)}, z^{(m)}\bigr)\bigr| \\ &\quad \leq \frac{1}{n^3} + \frac{1}{n}\ln(1+\sqrt{4r}) + \frac{1}{n^2}\sqrt{r} + \frac{1}{2\sqrt{n}}\sqrt{r} \quad (n, m = 1, 2, 3, \ldots), \end{aligned} \tag{3.21}$$

therefore, $|f_n(t^{(m)}, u^{(m)}, v^{(m)}, w^{(m)}, x^{(m)}, y^{(m)}, z^{(m)})|$ is bounded, and so, by a diagonal method, we can choose a subsequence $\{m_i\} \subset \{m\}$ such that

$$\bigl|f_n\bigl(t^{(m_i)}, u^{(m_i)}, v^{(m_i)}, w^{(m_i)}, x^{(m_i)}, y^{(m_i)}, z^{(m_i)}\bigr)\bigr| \to d_n \quad \text{as } i \to \infty\ (n = 1, 2, \ldots). \tag{3.22}$$

Now, (3.21) and (3.22) imply

$$|d_n| \leq \frac{1}{n^3} + \frac{1}{n}\ln(1+\sqrt{4r}) + \frac{1}{n^2}\sqrt{r} + \frac{1}{2\sqrt{n}}\sqrt{r} \quad (n = 1, 2, \ldots). \tag{3.23}$$

so $d = (d_1, \ldots, d_n, \ldots) \in C_0 = E$, and it is easy to see from (3.21)-(3.23) that

$$\begin{aligned} &\bigl\|f\bigl(t^{(m_i)}, u^{(m_i)}, v^{(m_i)}, w^{(m_i)}, x^{(m_i)}, y^{(m_i)}, z^{(m_i)}\bigr) - d\bigr\|_E \\ &\quad = \sup_n \bigl\{\bigl|f_n\bigl(t^{(m_i)}, u^{(m_i)}, v^{(m_i)}, w^{(m_i)}, x^{(m_i)}, y^{(m_i)}, z^{(m_i)}\bigr) - d_n\bigr|\bigr\} \to 0 \quad \text{as } i \to \infty. \end{aligned}$$

Thus, we have proved that $f(J_r \times \overline{B}_r \times \overline{B}_r \times \overline{B}_r \times \overline{B}_r \times \overline{B}_r \times \overline{B}_r)$ is relatively compact in E. Also, it is clear that h and g are uniformly continuous on $J_r \times \overline{B}_r^n$, and condition (iv) is satisfied. Consequently, all the conditions of Theorem 3.3 are satisfied. Hence equation (3.19) has at least one solution belonging to the ball $\overline{B}_5$ in the space $DC^n[J, E]$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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