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Approximate controllability of fractional partial differential equation

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Abstract

In the present paper, by using the theory of semigroup operators and the Schauder fixed point theorem, we define the mild solution of a fractional partial differential equation and obtain the existence and uniqueness of the mild solution. Then we study the approximate controllability of fractional partial differential equation and give an example to illustrate the theory.

MSC: 34H05

Keywords: fractional partial differential equation; compact semigroup; Schauder fixed point theorem; approximate controllability

1 Introduction

Fractional partial differential systems have recently been proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering, such as diffusion process, electrical science, electrochemistry, viscoelasticity, control science, electro magnetic theory, *etc.* (see [1–4]). For example, fractional diffusion equations are abstract partial differential equations that involve fractional derivatives in space and time. They are more accurate to model anomalous diffusion, where a plume of particles spreads in a different manner from what the classical diffusion equation predicts. Therefore, it is very necessary for us to consider the anomalous diffusion equation which is obtained from the standard diffusion equation by replacing the integer derivative with a fractional derivative of order $q \in (0,1)$.

As is well known there has been a great deal of interest in the solution of fractional differential equations in the analytic and numerical sense [4–9]. In order to study the fractional systems in the infinite dimensional space, the first important step is to introduce the concept of mild solutions. Some pioneering work has been reported by El-Borai [10] and Zhou and Jiao [11]. On the basis of the well posedness of mild solutions, controllability problems for various types of nonlinear fractional dynamical systems have also been considered in many publications (see [12–30]). The main tool used in these papers is to convert the controllability problem into a fixed point problem with the assumption that the controllability operator has an induced inverse on a quotient space. But Hernández *et al.* [31], Sukavanam and Tomar [32] pointed out that some papers on the controllability of abstract control systems contain a similar technical error when the compactness of semigroup and other hypotheses are satisfied, more precisely, in this case the applications of controllability results are restricted to a finite dimensional space. Thus, the concept of exact controllability is too strong in infinite dimensional spaces and the approximate controllability is more appropriate.

The approximate controllability of the systems with integer order has been proved in [12, 13] among others. However, there are only few papers concerned with the approximate controllability of a fractional partial differential equation system. Russell and Zhang [14] discussed the controllability and stabilizability of the following third-order linear dispersion equation of (1.1) on a periodic domain. In [15], George *et al.* proved the exact controllability of

$$\begin{cases} \frac{\partial w}{\partial t}(x,t) + \frac{\partial^3 w}{\partial x^3}(x,t) = (Gu)(x,t) + f(t, w(x,t)), \quad t \in [0,b], x \in [0,2\pi], \\ \frac{\partial^k w}{\partial x^k}(0,t) = \frac{\partial^k w}{\partial x^k}(2\pi,t), \quad k = 0,1,2, \\ w(x,0) = 0, \end{cases} \tag{1.1}$$

on a periodic domain by using two standard types of nonlinearities and the approach of integral contractors. In [16], Sakthivel *et al.* proved the approximate controllability of (1.1) with the initial and periodic boundary condition by the assumption that the C_0 semigroup $T(t)$ is a compact and nonlinear function and is uniformly bounded. All these conclusions have provided the better theory analysis for the mild solution of the (1.1). In order to describe the physical process more precisely, we use the theory of semigroups of operators to prove the approximate controllability of the fractional diffusion equation. The fractional diffusion differential equation in the present paper generalizes the third-order diffusion equation (1.1) appearing in [16]. Compared to [15], we use the different method and discuss the unique solution of fractional diffusion equations with the weaker conditions. So the conclusion in the present paper is a continuation of the conclusions in [15] and [16].

2 Preliminaries

We consider the following fractional partial differential equation:

$$\begin{cases} {}^{c}D_t^q w(x,t) + Aw(x,t) = Gu(x,t) + f(t, w(x,t)), \quad t \in [0,b], x \in [0,2\pi], \\ w(x,0) = 0, \end{cases} \tag{2.1}$$

where ${}^{c}D_t^q$ is the Caputo fractional derivative of order $0 < q < 1$. The state function $w(\cdot,t)$ takes its value in the space $X = L^2(0,2\pi)$ with the norm $\|\cdot\|$ and the control function $u(\cdot,t)$ takes its value in the space $L^2(0,2\pi)$. Define an operator $-A$ to be the infinitesimal generator of the analytic semigroup $T(t)$ of operators on X, G is a bounded linear operator from X to X.

It is suitable to rewrite (2.1) in the equivalent integral equation

$$\begin{cases} w(x,t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} (-Aw(x,s) + Gu(x,s) + f(s, w(x,s)))\, ds, \\ \quad t \in [0,b], x \in [0,2\pi], \\ w(x,0) = 0, \end{cases} \tag{2.2}$$

provided that the integral in (2.2) exists.

Before giving the definition of mild solution of (2.1), we first prove the following lemma.

Lemma 2.1 *If* (2.2) *holds, then we have*

$$\begin{cases} w(x,t) = q\int_0^t \int_0^\infty (t-s)^{q-1}\theta\Psi_q(\theta)T(t^q\theta)(Gu(x,s)+f(s,w(x,s)))\,d\theta\,ds, \\ \quad t\in[0,b], x\in[0,2\pi], \\ w(x,0)=0. \end{cases} \tag{2.3}$$

Proof Let $\lambda > 0$. Applying the Laplace transforms

$$\begin{aligned} W(\lambda,x) &= \int_0^\infty e^{-\lambda s}w(x,s)\,ds, \\ U(\lambda,x) &= \int_0^\infty e^{-\lambda s}u(x,s)\,ds, \\ F(\lambda,x) &= \int_0^\infty e^{-\lambda s}f\bigl(x,w(x,s)\bigr)\,ds \end{aligned}$$

to (2.2), we have

$$\begin{aligned} &W(\lambda,x) = -\frac{A}{\lambda^q}W(\lambda,x) + \frac{1}{\lambda^q}GU(\lambda,x) + \frac{1}{\lambda^q}F(\lambda,x), \\ &\lambda^q W(\lambda,x) = -AW(\lambda,x) + GU(\lambda,x) + F(\lambda,x), \\ &\bigl(\lambda^q I + A\bigr)W(\lambda,x) = GU(\lambda,x) + F(\lambda,x), \\ &W(\lambda,x) = \bigl(\lambda^q I + A\bigr)^{-1}\bigl[GU(\lambda,x) + F(\lambda,x)\bigr] \\ &\qquad = \int_0^\infty e^{-\lambda^q s}T(s)\bigl(GU(\lambda,x)+F(\lambda,x)\bigr)\,ds. \end{aligned}$$

Since $\int_0^\infty e^{-\lambda^q s}T(s)\,ds = \frac{1}{\lambda^q I + A}$, we may consider the one-side stable probability density [22],

$$\Phi_q(\theta) = \frac{1}{\pi}\sum_{n=1}^\infty (-1)^{n-1}\vartheta^{-qn-1}\frac{\Gamma(nq+1)}{n!}\sin(n\pi q), \quad \theta\in(0,\infty),$$

whose Laplace transform is given by

$$\int_0^\infty e^{-\lambda\theta}\Phi_q(\theta)\,d\theta = e^{-\lambda^q}, \quad q\in(0,1). \tag{2.4}$$

Using (2.4), we get

$$\begin{aligned} &\int_0^\infty e^{-\lambda^q s}T(s)GU(\lambda,x)\,ds \\ &\quad = \int_0^\infty \int_0^\infty e^{-\lambda^q t}T(t)e^{-\lambda s}Gu(x,s)\,ds\,dt \\ &\quad = \int_0^\infty \int_0^\infty qt^{q-1}e^{(-\lambda t)^q}T\bigl(t^q\bigr)e^{-\lambda s}Gu(x,s)\,ds\,dt \\ &\quad = \int_0^\infty \int_0^\infty qt^{q-1}\biggl(\int_0^\infty e^{-\lambda t\theta}\Phi_q(\theta)\,d\theta\biggr)T\bigl(t^q\bigr)e^{-\lambda s}Gu(x,s)\,ds\,dt \\ &\quad = \int_0^\infty \int_0^\infty \int_0^\infty \bigl[qt^{q-1}\bigl(e^{-\lambda t\theta}\Phi_q(\theta)\bigr)T\bigl(t^q\bigr)e^{-\lambda s}Gu(x,s)\bigr]\,d\theta\,ds\,dt \end{aligned}$$

$$\begin{aligned}
&= \int_0^\infty \int_0^\infty \int_0^\infty \left[q e^{-\lambda(t+s)} \Phi_q\left(\frac{t}{\theta}\right) \frac{t^{q-1}}{\theta^q} T\left(\frac{t^q}{\theta^q}\right) Gu(x,s) \right] d\theta\, ds\, dt \\
&= \int_0^\infty \int_0^\infty \int_s^\infty \left[q \Phi_q\left(\frac{t-s}{\theta}\right) e^{-\lambda t} T\left(\frac{(t-s)^q}{\theta^q}\right) \frac{(t-s)^q}{\theta^q} Gu(x,s) \right] dt\, d\theta\, ds \\
&= \int_0^\infty \int_0^\infty \int_0^t \left[q \Phi_q(\theta) e^{-\lambda t} T\left(\frac{(t-s)^q}{\theta^q}\right) \frac{(t-s)^q}{\theta^q} Gu(x,s) \right] ds\, dt\, d\theta \\
&= \int_0^\infty e^{-\lambda t} \left[q \int_0^t \int_0^\infty \left(\Phi_q(\theta) T\left(\frac{(t-s)^q}{\theta^q}\right) \frac{(t-s)^q}{\theta^q} Gu(x,s) \right) d\theta\, ds \right] dt,
\end{aligned}$$

$$\begin{aligned}
&\int_0^\infty e^{-\lambda^q s} T(s) F(\lambda, x)\, ds \\
&\quad = \int_0^\infty e^{-\lambda t} \left[q \int_0^t \int_0^\infty \left(\Phi_q(\theta) T\left(\frac{(t-s)^q}{\theta^q}\right) \frac{(t-s)^q}{\theta^q} f\bigl(s, w(x,s)\bigr) \right) d\theta\, ds \right] dt.
\end{aligned}$$

Then we have

$$\begin{aligned}
&U(x,\lambda) \\
&\quad = \int_0^\infty e^{-\lambda t} \left[q \int_0^t \int_0^\infty \Phi_q(\theta) T\left(\frac{(t-s)^q}{\theta^q}\right) \frac{(t-s)^q}{\theta^q} \bigl(Gu(x,s) + f\bigl(s, w(x,s)\bigr)\bigr)\, d\theta\, ds \right] dt.
\end{aligned}$$

Now we can invert the last Laplace transform to get

$$w(x,t) = q \int_0^t \int_0^\infty \theta (t-s)^{q-1} \Psi_q(\theta) T\bigl((t-s)^q \theta\bigr) \bigl(Gu(x,s) + f\bigl(s, w(x,s)\bigr)\bigr)\, d\theta\, ds,$$

where $\Psi_q(\theta) = \frac{1}{q}\theta^{-1-\frac{1}{q}} \Phi_q(\theta^{-\frac{1}{q}})$ is the probability density function defined on $(0,\infty)$. This completes the proof. □

For any $x \in X$, define operators $\{S_q(t)\}_{t\geq 0}$ and $\{T_q(t)\}_{t\geq 0}$ by

$$\begin{aligned}
S_q(t)x &= \int_0^\infty \Psi_q(\theta) T\bigl(t^q \theta\bigr) x\, d\theta, \\
T_q(t)x &= q \int_0^\infty \theta \Psi_q(\theta) T\bigl(t^q \theta\bigr) x\, d\theta.
\end{aligned}$$

Due to Lemma 2.1, we give the following definition of the mild solution of (2.1).

Definition 2.2 By the mild solution of fractional partial differential equation (2.1), we mean that the function $w(x,t) \in L^2([0,b], L^2[0,2\pi])$ satisfies

$$w(x,t) = \int_0^t (t-s)^{q-1} T_q(t-s) \bigl(Gu(x,s) + f\bigl(s, w(x,s)\bigr)\bigr)\, ds.$$

Lemma 2.3 ([17]) *For any fixed $t \geq 0$, $S_q(t)$ and $T_q(t)$ are bounded linear operators. Hence*

$$\bigl\| S_q(t)x \bigr\| \leq M\|x\|$$

and

$$\bigl\| T_q(t)x \bigr\| \leq \frac{Mq}{\Gamma(1+q)} \|x\|$$

for all $x \in X$, where M is a constant such that $\|T(t)\| \leq M$ for all $t \in [0,b]$.

3 Existence and uniqueness of mild solution

In this section we prove the existence and uniqueness of the mild solution of (2.1). To prove the result let us assume the following conditions:

(H_1) For each $t \in [0,b]$, $f(t,\cdot)$ is continuous and $f(t,x)$ satisfy the generalized Lipschitz condition, *i.e.* there exists a function $H(t) \in L^{\frac{1}{l}}([0,b];X)$, $0 < l < q$, such that

$$\|f(t,x) - f(t,y)\| \le H(t)\|x - y\|.$$

(H_2) We have

$$\frac{Mq}{\Gamma(1+q)}\|H\|_{\frac{1}{l}}\left(\frac{1-l}{q-l}\right)^{1-l} b^{q-l} < 1,$$

where $\|H\|_{\frac{1}{l}} = (\int_0^b (H(s))^{\frac{1}{l}}\,ds)^l$.

Theorem 3.1 *If the conditions* (H_1)-(H_2) *hold, the system* (2.1) *admits a unique mild solution in X for each control function* $u(x,t) \in L^2([0,b],L^2[0,2\pi])$.

Proof Define the operator $N : X \to X$ by

$$Nw(x,t) = \int_0^t (t-s)^{q-1} T_q(t-s)\big(Gu(x,s) + f\big(s,w(x,s)\big)\big)\,ds.$$

Let

$$B_r = \big\{w(x,t) \in L^2\big([0,b],L^2(0,2\pi)\big); w(x,0) = 0, \big\|w(x,t)\big\| \le r\big\}$$

which is a bounded and closed subset of $L^2([0,b],L^2[0,2\pi])$, and

$$H_g = \max_{0\le t\le b}\big\|f(t,0)\big\|.$$

For any $w(x,t) \in B_r$, we have

$$\begin{aligned}
&\big\|Nw(x,t)\big\| \\
&\quad = \left\|\int_0^t (t-s)^{q-1} T_q(t-s)\big(Gu(x,s) + f(s,w)\big)\,ds\right\| \\
&\quad \le \frac{Mq}{\Gamma(1+q)}\left\|\int_0^t (t-s)^{q-1}\big(Gu(x,s) + f(s,w)\big)\,ds\right\| \\
&\quad \le \frac{Mq}{\Gamma(1+q)}\int_0^t (t-s)^{q-1}\big\|\big(Gu(x,s) + f(s,w)\big)\big\|\,ds \\
&\quad \le \frac{Mq}{\Gamma(1+q)}\|G\|\int_0^t (t-s)^{q-1}\|u\|\,ds \\
&\qquad + \frac{Mq}{\Gamma(1+q)}\int_0^t (t-s)^{q-1}\big\|f(s,w) - f(s,0) + f(s,0)\big\|\,ds \\
&\quad \le \frac{Mq}{\Gamma(1+q)}\|G\|\sqrt{\frac{b^{2q-1}}{2q-1}}\|u\|_{L^2}
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{Mq}{\Gamma(1+q)} \|w\|_{L^2} \|H\|_{\frac{1}{l}} \left(\frac{1-l}{q-l}\right)^{1-l} b^{q-l} + \frac{b^q Mq}{\Gamma(1+q)} H_g \\
&\le \frac{Mq}{\Gamma(1+q)} \|G\| \sqrt{\frac{b^{2q-1}}{2q-1}} \|u\|_{L^2} \\
&+ \frac{Mq}{\Gamma(1+q)} \|H\|_{\frac{1}{l}} \left(\frac{1-l}{q-l}\right)^{1-l} b^{q-l} r + \frac{b^q Mq}{\Gamma(1+q)} H_g.
\end{aligned}
$$

Now let $\|Nw(x,t)\| < r$, then

$$
\frac{Mq}{\Gamma(1+q)} \|G\| \sqrt{\frac{b^{2q-1}}{2q-1}} \|u\|_{L^2} + \frac{Mq}{\Gamma(1+q)} \|H\|_{\frac{1}{l}} \left(\frac{1-l}{q-l}\right)^{1-l} b^{q-l} r + \frac{b^q Mq}{\Gamma(1+q)} H_g < r.
$$

Since we have the condition (H_3), we can obtain

$$
\frac{Mq}{\Gamma(1+q)} \|H\|_{\frac{1}{l}} \left(\frac{1-l}{q-l}\right)^{1-l} b^{q-l} < 1.
$$

So we obtain this result: N maps the ball B_r of radius r into itself.

Next we show that N is a contraction on B_r. To this aim, let us take $w_1, w_2 \in B_r$, we get

$$
\begin{aligned}
&\|Nw_1(t) - Nw_2(t)\| \\
&\le \left\| \int_0^t (t-s)^{q-1} T_q(t-s) \big(f(s,w_1) - f(s,w_2)\big)\, ds \right\| \\
&\le \int_0^t (t-s)^{q-1} T_q(t-s) H(s)^{\frac{1}{l}} \|w_1 - w_2\|\, ds \\
&\le \frac{Mq}{\Gamma(1+q)} \|H\|_{\frac{1}{l}} \left(\frac{1-l}{q-l}\right)^{1-l} b^{q-l} \|w_1 - w_2\| \\
&< \|w_1 - w_2\|,
\end{aligned}
$$

then N has a unique fixed point in B_r. According to the extension theorem of a solution, Theorem 3.1 is proved. □

4 Approximate controllability of system (2.1)

Let $w(x,t)$ be the state value of system (2.1) at time t and space X corresponding to the control function $u(x,t)$. The system (2.1) is said be approximately controllable on the interval $[0,b]$, if for any given $w_1 \in L^2(0,2\pi)$ the solution $w(\cdot,t)$ of (2.1) satisfies $\|w(\cdot,b) - w_1\| < \epsilon$.

We introduce two relevant operators and the basic assumption on these operators

$$
\Gamma_0^b = \int_0^b (b-s)^{q-1} T_q(b-s) G G^* T_q^*(b-s)\, ds
$$

and

$$
R\big(\lambda, \Gamma_0^b\big) = \big(\lambda I + \Gamma_0^b\big)^{-1},
$$

where G^* is the adjoint of G and $T_q^*(t)$ is the adjoint of $T_q(t)$. It is straightforward that the operator Γ_0^b is a linear bounded operator.

Lemma 4.1 ([24]) *Let Z be a separable reflexive Banach space and let Z^* stands for its dual space. Assume that $\Gamma : Z^* \to Z$ is symmetric. Then the following two conditions are equivalent*:

(i) $\Gamma : Z^* \to Z$ *is positive, that is*, $(z^*, \Gamma z^*) > 0$ *for all nonzero* $z^* \in Z^*$.

(ii) *For all* $h \in Z$, $z_\lambda(h) = \lambda(\lambda I + \Gamma J)^{-1}(h)$ *strongly converges to zero as* $\lambda \to 0^+$. *Here J is the duality mapping of Z into Z^*.*

Lemma 4.2 ([14]) *The linear fractional control system* (2.1) *is approximately controllable on* $[0, b]$ *if and only if* $\lambda R(\lambda, \Gamma_0^b) \to 0$ *as* $\lambda \to 0^+$ *in the strong operator topology.*

Before proving the approximate controllability of (2.1), we impose the following conditions on the data of the problem:

(H_3) $T(t)$ is a compact analytic semigroup in X.

(H_4) $\lambda R(\lambda, \Gamma_0^b) \to 0$ as $\lambda \to 0^+$ in the strong operator topology.

In this section, we will prove that the system (2.1) is approximately controllable, if for any $\lambda > 0$ there exists a continuous function $w(x,t) \in X$ such that

$$w(x,t) = \int_0^t (t-s)^{q-1} T_q(t-s)\big(Gu(x,s) + f\big(s, w(x,s)\big)\big)\, ds, \tag{4.1}$$

where

$$u(x,t) = G^* T_q^*(b-t) R\big(\lambda, \Gamma_0^b\big)\left(w_1 - \int_0^b (b-s)^{q-1} T_q(b-s) f\big(s, w(x,s)\big)\, ds\right). \tag{4.2}$$

Theorem 4.3 *Assume that the conditions* (H_1)-(H_4) *are satisfied and* $f(t,w)$ *is bounded in X. Then the system* (2.1) *is approximate controllable.*

Proof Let $w_\lambda(x,t)$ be a fixed point of N in B_r. Any fixed point of N is a mild solution of (2.1) under the control $u_\lambda(x,t) = G^* T_q^*(b-t) R(\lambda, \Gamma_0^b) p(w_\lambda)$ and satisfies

$$w_\lambda(x,b) = w_1 - \lambda R\big(\lambda, \Gamma_0^b\big) p(w_\lambda),$$

where $p(w_\lambda(x,b)) = w_1 - \int_0^b (b-s)^{q-1} T_q(b-s) f(s, w(x,s))\, ds$.

In fact, we know

$$\begin{aligned}
w_\lambda(x,b) &= \int_0^b (b-s)^{q-1} T_q(b-s)\big(Gu(x,s) + f\big(s, w(x,s)\big)\big)\, ds \\
&= \int_0^b (b-s)^{q-1} T_q(b-s) G\bigg(G^* T_q^*(b-s) R\big(\lambda, \Gamma_0^b\big) \\
&\quad \times \left(w_1 - \int_0^b (b-s)^{q-1} T_q(b-s) f\big(s, w(x,s)\big)\, ds\right)\bigg)\, ds \\
&\quad + \int_0^b (b-s)^{q-1} T_q(b-s) f\big(s, w(x,s)\big)\, ds \\
&= \Gamma_0^b R\big(\lambda, \Gamma_0^b\big) w_1 - \Gamma_0^b R\big(\lambda, \Gamma_0^b\big) \int_0^b (b-s)^{q-1} T_q(b-s) f\big(s, w(x,s)\big)\, ds
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^b (b-s)^{q-1} T_q(b-s) f\bigl(s, w(x,s)\bigr)\, ds \\
&= w_1 - \lambda R\bigl(\lambda, \Gamma_0^b\bigr) w_1 + \lambda R\bigl(\lambda, \Gamma_0^b\bigr) \int_0^b (b-s)^{q-1} T_q(b-s) f\bigl(s, w(x,s)\bigr)\, ds \\
&= w_1 - \lambda R\bigl(\lambda, \Gamma_0^b\bigr)\left(w_1 - \int_0^b (b-s)^{q-1} T_q(b-s) f\bigl(s, w(x,s)\bigr)\, ds\right) \\
&= w_1 - \lambda R\bigl(\lambda, \Gamma_0^b\bigr) p(w_\lambda).
\end{aligned}
$$

Since $f(t,w)$ is bounded in $L^2([0,b], L^2[0,2\pi])$, there exists a subsequence, denoted by $f(t,w)$, that weakly converges to, say, $f(s)$ in $L^2([0,b], L^2[0,2\pi])$. Define $v = w_1 - \int_0^b (b-s)^{q-1} T_q(b-s) f(s)\, ds$. It follows that

$$
\begin{aligned}
&\bigl\| P\bigl(w_\lambda(x,b)\bigr) - v \bigr\| \\
&\quad = \left\| \int_0^b (b-s)^{q-1} T_q(b-s) f\bigl(s, w(x,s)\bigr)\, ds - \int_0^b (b-s)^{q-1} T_q(b-s) f(s)\, ds \right\| \\
&\quad \le \int_0^b (b-s)^{q-1} T_q(b-s) \bigl\| f\bigl(s, w(x,s)\bigr) - f(s) \bigr\|\, ds.
\end{aligned}
$$

Now, by the compactness of an operator $l(\cdot) \to \int_0^{\cdot} (\cdot - s)^{q-1} T_q(\cdot - s) l(s)\, ds : L^2([0,b], L^2[0, 2\pi]) \to C([0,b], L^2[0,2\pi])$, the right-hand side of the above inequality tends to zero as $\lambda \to 0^+$.

Then we obtain

$$
\begin{aligned}
&\bigl\| w_\lambda(x,b) - w_1 \bigr\| \\
&\quad = \bigl\| w_1 - \lambda R\bigl(\lambda, \Gamma_0^b\bigr) p(w_\lambda) - w_1 \bigr\| \\
&\quad = \bigl\| \lambda R\bigl(\lambda, \Gamma_0^b\bigr) p(w_\lambda) \bigr\| \\
&\quad \le \bigl\| \lambda R\bigl(\lambda, \Gamma_0^b\bigr)\bigl(p(w_\lambda) - v\bigr) \bigr\| + \bigl\| \lambda R\bigl(\lambda, \Gamma_0^b\bigr) v \bigr\| \\
&\quad \to 0,
\end{aligned}
$$

as $\lambda \to 0^+$. So the approximate controllability of (2.1) is proved. □

5 Example

As an application of Theorem 4.3, we consider the following system:

$$
\begin{cases}
\frac{\partial^{\frac{2}{3}}}{\partial t^{\frac{2}{3}}} w(x,t) + \frac{\partial^2}{\partial x^2} w(x,t) = G w(x,t) + f(t, w(x,t)), \\
w(x,0) = 0, \\
w(0,t) = w(2\pi, t) = 0.
\end{cases}
\tag{5.1}
$$

To write system (5.1) in the form of (2.1), let $X = L^2(0, 2\pi)$ and A be defined by $Aw = w''$ with domain $D(A) = \{w(\cdot) \in X : w, w' \text{ absolutely continuous}, w'' \in X, w(0) = w(2\pi) = 0\}$.

Then A generates a uniformly bounded analytic semigroup which satisfies the condition (H_3). Furthermore, A has a discrete spectrum, the eigenvalues are $-n^2$, $n \in N$, with the corresponding normalized eigenvectors $w_n(x) = (2/\pi)^{1/2} \sin(nx)$. Then the following properties hold:

(i) If $w \in D(A)$, then

$$Aw = \sum_{n=1}^{\infty} n^2 \langle w, w_n \rangle w_n.$$

(ii) For each $w \in X$,

$$A^{-\frac{1}{2}} w = \sum_{n=1}^{\infty} \frac{1}{n} \langle w, w_n \rangle w_n.$$

Moreover, $\|A^{-\frac{1}{2}}\| = 1$.

(iii) The operator $A^{\frac{1}{2}}$ is given by

$$A^{\frac{1}{2}} w = \sum_{n=1}^{\infty} n \langle w, w_n \rangle w_n$$

on the space $D(A^{\frac{1}{2}}) = \{w(\cdot) \in X, A^{\frac{1}{2}} w \in X\}$.

First of all, if the conditions (H_1) and (H_2) are satisfied, the system (5.1) admits a unique mild solution in X for each control function $u(x,t)$ from Theorem 3.1. Second, if the conditions (H_1)-(H_4) are satisfied and $f(t,w)$ is bounded in X, then the approximate controllability of the system (5.1) follows from Theorem 4.3.

6 Future outlook

The results developed in the present paper can be extended to the case of stochastic fractional partial differential equations with time-delay. We will pay attention to the well posedness and approximate controllability of stochastic fractional partial differential equation with time-delay in the future.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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