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Some new identities of Chebyshev polynomials and their applications

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Abstract

In this paper, we use the properties of Chebyshev polynomials, elementary methods, and combinational techniques to study the computational problem of one kind of convolution sums involving second kind Chebyshev polynomials, and we give an exact computational method, which expresses the sums as second kind Chebyshev polynomials. As some applications of our results, we also obtain several new identities and congruences involving the second kind Chebyshev polynomials, Fibonacci numbers, and Lucas numbers.

MSC: 11B39

Keywords: second kind Chebyshev polynomials; Fibonacci number; Lucas number; identity

1 Introduction

For any integer $n \geq 0$, the famous Chebyshev polynomials of the first and second kind $T_n(x)$ and $U_n(x)$ are defined as follows:

$$T_n(x) = \frac{n}{2}\sum_{k=0}^{[\frac{n}{2}]}(-1)^k \frac{(n-k-1)!}{k!(n-2k)!}(2x)^{n-2k}$$

and

$$U_n(x) = \sum_{k=0}^{[\frac{n}{2}]}(-1)^k \frac{(n-k)!}{k!(n-2k)!}(2x)^{n-2k},$$

where $[m]$ denotes the greatest integer $\leq m$.

It is clear that $T_n(x)$ and $U_n(x)$ are the second-order linear recurrence polynomials, they satisfy the recurrence formulas

$$T_0(x) = 1, \qquad T_1(x) = x \quad \text{and} \quad T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x) \quad \text{for all } n \geq 1,$$
$$U_0(x) = 1, \qquad U_1(x) = 2x \quad \text{and} \quad U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x) \quad \text{for all } n \geq 1.$$

The general formulas of $T_n(x)$ and $U_n(x)$ are

$$T_n(x) = \frac{1}{2}\Big[\big(x+\sqrt{x^2-1}\big)^n + \big(x-\sqrt{x^2-1}\big)^n\Big] \tag{1}$$

and

$$U_n(x) = \frac{1}{2\sqrt{x^2-1}}\left[\left(x+\sqrt{x^2-1}\right)^{n+1} - \left(x-\sqrt{x^2-1}\right)^{n+1}\right]. \tag{2}$$

The generating functions of $T_n(x)$ and $U_n(x)$ are

$$\frac{1-xt}{1-2xt+t^2} = \sum_{n=0}^{\infty} T_n(x)t^n \quad \bigl(|x|<1, |t|<1\bigr)$$

and

$$\frac{1}{1-2xt+t^2} = \sum_{n=0}^{\infty} U_n(x)t^n \quad \bigl(|x|<1, |t|<1\bigr).$$

As regards the elementary properties of Chebyshev polynomials, some authors had studied them, and they obtained many interesting conclusions. For example, Zhang [1] proved that for any positive integer k and nonnegative integer n, one has the identity

$$\sum_{a_1+a_2+\cdots+a_{k+1}=n} U_{a_1}(x)\cdot U_{a_2}(x)\cdots U_{a_{k+1}}(x) = \frac{1}{2^k\cdot k!}U_{n+k}^{(k)}(x), \tag{3}$$

where $U_n^{(k)}(x)$ denotes the kth derivative of $U_n(x)$ with respect to x, the summation is taken over all $k+1$-dimension nonnegative integer coordinates $(a_1, a_2, \ldots, a_{k+1})$ such that $a_1+a_2+\cdots+a_{k+1}=n$.

As some applications of (3), Zhang [1] obtained some identities involving Fibonacci numbers and Lucas numbers.

Ma and Zhang [2], Li [3], Wang and Zhang [4], Cesarano [5], Lee and Wong [6] also proved a series of identities involving Chebyshev polynomials. Bhrawy *et al.* (see [7–10]) and Bircan and Pommerenke [11] obtained many important applications of the Chebyshev polynomials. For an overview of some new work related to the generating functions of Chebyshev polynomials of the first and the second kind, one may refer to Cesarano [12].

It is clear that an interesting problem is whether one can express $U_{n+k}^{(k)}(x)$ by the second kind Chebyshev polynomials.

It seems that none had studied this problem yet, at least we have not seen any related result before. The problem is interesting and important, because it can reveal the inner relations of the second kind Chebyshev polynomials, and it can also express a complex sum in a simple form.

This paper, as a note of [1], we give an exact computational method, which express $U_{n+k}^{(k)}(x)$ by the second Chebyshev polynomials. That is, we shall prove the following main conclusion.

Theorem *For any positive integer k and nonnegative integer n, we have the identity*

$$\begin{aligned}&\sum_{a_1+a_2+\cdots+a_{k+1}=n} U_{a_1}(x)\cdot U_{a_2}(x)\cdots U_{a_{k+1}}(x)\\ &\quad = \frac{1}{2^k\cdot k!}U_{n+k}^{(k)}(x) = \frac{1}{2^k\cdot k!}\cdot\left[\frac{(2k-1)x}{1-x^2}\cdot U_{n+k}^{(k-1)}(x) - \frac{n^2+2n(k+1)+4k}{1-x^2}\cdot U_{n+k}^{(k-2)}(x)\right],\end{aligned}$$

where $(1-x^2)U_n'(x) = (n+1)U_{n-1}(x) - nxU_n(x)$.

It is clear that this theorem gives an exact computational method, which expresses $U_{n+k}^{(k)}(x)$ by Chebyshev polynomials $U_n(x)$. From this theorem we may immediately deduce the following.

Corollary 1 *For any positive integers $n \geq k \geq 2$, we have the identity*

$$\begin{aligned}&\sum_{a_1+a_2+\cdots+a_{k+1}=n} U_{a_1}(x)\cdot U_{a_2}(x)\cdots U_{a_{k+1}}(x)\\&\quad=\frac{1}{2^k\cdot k!\cdot(1-x^2)^k}\cdot\bigl[R(n,k,x)\cdot U_{n+k-1}(x)+S(n,k,x)\cdot U_{n+k}(x)\bigr],\end{aligned}$$

where $R(n,k,x)$ and $S(n,k,x)$ are two computable polynomials of n, k, and x with integral coefficients.

Especially for $k = 2$ and 3, we have the following.

Corollary 2 *For any nonnegative integer n, we have the identity*

$$\begin{aligned}&\sum_{a+b+c=n} U_a(x)\cdot U_b(x)\cdot U_c(x)\\&\quad=\frac{3(n+3)x}{8(1-x^2)^2}\cdot U_{n+1}(x)-\frac{(n+2)(n+4)-(n+1)(n+2)x^2}{8(1-x^2)^2}\cdot U_{n+2}(x).\end{aligned}$$

Corollary 3 *For any nonnegative integer n, we have the identity*

$$\begin{aligned}&\sum_{a+b+c+d=n} U_a(x)\cdot U_b(x)\cdot U_c(x)\cdot U_d(x)\\&\quad=\frac{(n+4)((n^2+8n+27)x^2-(n^2+8n+12))}{48(1-x^2)^3}\cdot U_{n+2}(x)\\&\qquad-\frac{x(n+3)((n^2+3n+2)x^2-(n^2+3n-13))}{48(1-x^2)^3}\cdot U_{n+3}(x).\end{aligned}$$

It is clear that the left-hand side of (3) is a polynomial of x with integral coefficients, so from Corollary 2 and Corollary 3 we can also deduce the following.

Corollary 4 *For any nonnegative integer n, we have the congruence*

$$3(n+3)xU_{n+1}(x)-(n+2)\bigl(n+4-(n+1)x^2\bigr)U_{n+2}(x)\equiv 0 \pmod{8\bigl(1-x^2\bigr)^2}.$$

Corollary 5 *For any nonnegative integer n, we have the congruence*

$$\begin{aligned}&(n+4)\bigl(\bigl(n^2+8n+27\bigr)x^2-\bigl(n^2+8n+12\bigr)\bigr)U_{n+2}(x)\\&\quad\equiv x(n+3)\bigl(\bigl(n^2+3n+2\bigr)x^2-\bigl(n^2+3n-13\bigr)\bigr)U_{n+3}(x) \pmod{48\bigl(1-x^2\bigr)^3}.\end{aligned}$$

As some applications of our results, we find that there are some close relationships among the Chebyshev polynomials, Fibonacci numbers F_n, and Lucas numbers L_n. These sequences are defined as

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

and

$$L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n,$$

for all integers $n \geq 0$.

It is clear that they also satisfy the second-order linear recurrence formulas $F_{n+2} = F_{n+1} + F_n$, $L_{n+2} = L_{n+1} + L_n$ for all $n \geq 0$ with $F_0 = 0$, $F_1 = 1$, $L_0 = 2$, $L_1 = 1$. Some papers related to Fibonacci numbers and Lucas numbers can also be found in [13–17]. From our results we can also deduce the following identities.

Corollary 6 *For any positive integers m and n, we have the identity*

$$\sum_{a+b+c=n} F_{2m(a+1)} \cdot F_{2m(b+1)} \cdot F_{2m(c+1)} = \frac{6(n+3)F_{2m}F_{4m}}{50F_m^4} \cdot F_{2m(n+2)} - \frac{4(n+2)(n+4)F_{2m}^2 - (n+1)(n+2)F_{4m}^2}{50F_m^4} \cdot F_{2m(n+3)}.$$

Corollary 7 *For any positive integers m and n, we have the identity*

$$\begin{aligned}\sum_{a+b+c+d=n} & F_{2m(a+1)} \cdot F_{2m(b+1)} \cdot F_{2m(c+1)} \cdot F_{2m(d+1)} \\ &= \frac{2(n+4)F_{2m}^3((n^2+8n+27)L_{2m}^2 - 4(n^2+8n+12))}{750F_m^6} \cdot F_{2m(n+3)} \\ &\quad - \frac{F_{4m}F_{2m}^2(n+3)((n^2+3n+2)L_{2m}^2 - 4(n^2+3n-13))}{750F_m^6} \cdot F_{2m(n+4)}.\end{aligned}$$

Taking $m = 1$ in Corollaries 4 and 5 we may immediately deduce the following.

Corollary 8 *For any nonnegative integer n, we have the identities*

$$\sum_{a+b+c=n} F_{2(a+1)} \cdot F_{2(b+1)} \cdot F_{2(c+1)} = \frac{18(n+3)}{50} \cdot F_{2(n+2)} + \frac{(n+2)(5n-7)}{50} \cdot F_{2(n+3)}$$

and

$$\sum_{a+b+c+d=n} F_{2(a+1)} \cdot F_{2(b+1)} \cdot F_{2(c+1)} \cdot F_{2(d+1)} = \frac{(n+4)(n^2+8n+39)}{75} \cdot F_{2(n+3)} - \frac{(n+3)(n^2+3n+14)}{50} \cdot F_{2(n+4)}.$$

2 Several simple lemmas

In this section, we shall give several simple lemmas, which are necessary in the proofs of our results. First of all we have the following.

Lemma 1 *For any positive integers $n \geq k > 0$, we have the identity*

$$U_n^{(k)}(x) = \frac{(2k-1)x}{1-x^2} \cdot U_n^{(k-1)}(x) + \frac{(k-2)k - n(n+2)}{1-x^2} \cdot U_n^{(k-2)}(x).$$

Proof It is clear that the second kind Chebyshev polynomials $U_n(x)$ satisfy the differential equation

$$\left(1-x^2\right)\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + n(n+2)y = 0 \quad (n = 0, 1, 2, \ldots).$$

So for any positive integer $n \geq k > 2$, we have

$$\left(1-x^2\right)U_n''(x) = 3xU_n'(x) - n(n+2)U_n(x). \tag{4}$$

Differentiating (4) repeatedly $(k-2)$ times we obtain

$$\begin{aligned}&\left(1-x^2\right)U_n^{(k)}(x) - 2(k-2)xU_n^{(k-1)}(x) - (k-2)(k-3)U_n^{(k-2)}(x)\\ &\quad = 3xU_n^{(k-1)}(x) + 3(k-2)U^{(k-2)}(x) - n(n+2)U_n^{(k-2)}(x)\end{aligned}$$

or

$$U_n^{(k)}(x) = \frac{(2k-1)x}{1-x^2} \cdot U_n^{(k-1)}(x) + \frac{(k-2)k - n(n+2)}{1-x^2} \cdot U_n^{(k-2)}(x).$$

This proves Lemma 1. □

Lemma 2 *For any positive integers $n \geq k \geq 1$, we have the identity*

$$U_n^{(k)}(x) = \frac{1}{(1-x^2)^k} \cdot \left[R(n,k,x) \cdot U_{n-1}(x) + S(n,k,x) \cdot U_n(x)\right],$$

where $R(n,k,x)$ and $S(n,k,x)$ are two computable polynomials of n, k, and x with integral coefficients.

Proof We prove Lemma 2 by complete induction. Note that we have the identity

$$\left(1-x^2\right)U_n'(x) = (n+1)U_{n-1}(x) - nxU_n(x)$$

or

$$U_n'(x) = \frac{1}{(1-x^2)} \cdot \left[(n+1)U_{n-1}(x) - nxU_n(x)\right]. \tag{5}$$

So Lemma 2 holds for $k = 1$.

Assume that Lemma 2 holds for all positive integers $1 \leq k \leq m$. That is, for all positive integers $1 \leq k \leq m$, we have

$$U_n^{(k)}(x) = \frac{1}{(1-x^2)^k} \cdot \left[R_k(n,k,x) \cdot U_{n-1}(x) + S_k(n,k,x) \cdot U_n(x)\right]. \tag{6}$$

Then for $k = m + 1$, from (5), (6), and Lemma 1 we have

$$U_n^{(m+1)}(x) = \frac{(2m+1)x}{1-x^2} \cdot U_n^{(m)}(x) + \frac{(m-1)(m+1) - n(n+2)}{1-x^2} \cdot U_n^{(m-1)}(x)$$
$$= \frac{1}{(1-x^2)^{m+1}} \cdot \bigl[R_{m+1}(n, m+1, x) \cdot U_{n-1}(x) + S_{m+1}(n, m+1, x) \cdot U_n(x)\bigr].$$

This proves Lemma 2 by complete induction. □

Lemma 3 *For any positive integers m and n, we have the identities*

$$T_n\bigl(T_m(x)\bigr) = T_{mn}(x) \quad \textit{and} \quad U_n\bigl(T_m(x)\bigr) = \frac{U_{m(n+1)-1}(x)}{U_{m-1}(x)}.$$

Proof See Lemma 3 in Zhang [1]. □

3 Proof of the theorem

In this section, we shall complete the proofs of our all results. It is clear that our theorem follows from (3) and Lemma 1. In fact, substituting n by $n + k$ in Lemma 1 we have

$$U_{n+k}^{(k)}(x) = \frac{(2k-1)x}{1-x^2} \cdot U_{n+k}^{(k-1)}(x) + \frac{(k-2)k - (n+k)(n+k+2)}{1-x^2} \cdot U_{n+k}^{(k-2)}(x). \tag{7}$$

Combining identities (3) and (7) we may immediately deduce

$$\sum_{a_1+a_2+\cdots+a_{k+1}=n} U_{a_1}(x) \cdot U_{a_2}(x) \cdots U_{a_{k+1}}(x)$$
$$= \frac{1}{2^k \cdot k!} U_{n+k}^{(k)}(x) = \frac{1}{2^k \cdot k!} \cdot \left[\frac{(2k-1)x}{1-x^2} \cdot U_{n+k}^{(k-1)}(x) - \frac{n^2 + 2n(k+1) + 4k}{1-x^2} \cdot U_{n+k}^{(k-2)}(x)\right].$$

This proves our theorem.

It is clear that Corollary 1 follows from our theorem and Lemma 2.

Now we prove Corollary 2. Taking $k = 2$ in our theorem and noting that $(1-x^2)U_n'(x) = (n+1)U_{n-1}(x) - nxU_n(x)$ we have

$$\sum_{a+b+c=n} U_a(x) \cdot U_b(x) \cdot U_c(x)$$
$$= \frac{1}{2^2 \cdot 2!} U_{n+2}''(x) = \frac{1}{8} \cdot \left[\frac{3x}{1-x^2} \cdot U_{n+2}'(x) - \frac{n^2 + 6n + 8}{1-x^2} \cdot U_{n+2}(x)\right]$$
$$= \frac{3x}{8(1-x^2)} \cdot \left[\frac{n+3}{1-x^2} U_{n+1}(x) - \frac{(n+2)x}{1-x^2} U_{n+2}\right] - \frac{(n+2)(n+4)}{8(1-x^2)} U_{n+2}(x)$$
$$= \frac{3(n+3)x}{8(1-x^2)^2} \cdot U_{n+1}(x) - \frac{(n+2)(n+4) - (n+1)(n+2)x^2}{8(1-x^2)^2} \cdot U_{n+2}(x).$$

This proves Corollary 2.

To prove Corollary 3, taking $k = 3$ in our theorem we have

$$\sum_{a+b+c+d=n} U_a(x) \cdot U_b(x) \cdot U_c(x) \cdot U_d(x)$$
$$= \frac{1}{2^3 \cdot 3!} U_{n+3}^{(3)}(x) = \frac{1}{48} \cdot \left[\frac{5x}{1-x^2} \cdot U_{n+3}''(x) - \frac{n^2 + 8n + 12}{1-x^2} \cdot U_{n+3}'(x)\right]$$

$$
\begin{aligned}
&= \frac{5x}{48(1-x^2)} \cdot \left[\frac{3x}{1-x^2}U'_{n+3}(x) - \frac{(n+3)(n+5)}{1-x^2}U_{n+3}(x)\right] - \frac{(n+2)(n+6)}{48(1-x^2)}U'_{n+3}(x) \\
&= \frac{15x^2-(n+2)(n+6)(1-x^2)}{48(1-x^2)^2} \cdot \left[\frac{n+4}{1-x^2}U_{n+2}(x) - \frac{(n+3)x}{1-x^2}U_{n+3}(x)\right] \\
&\quad - \frac{5(n+3)(n+5)x}{48(1-x^2)^2} \cdot U_{n+3}(x) \\
&= \frac{(n+4)((n^2+8n+27)x^2-(n^2+8n+12))}{48(1-x^2)^3} \cdot U_{n+2}(x) \\
&\quad - \frac{x(n+3)((n^2+3n+2)x^2-(n^2+3n-13))}{48(1-x^2)^3} \cdot U_{n+3}(x).
\end{aligned}
$$

This proves Corollary 3.

Now we prove Corollary 6. Taking $x = \frac{3}{2}$ in (1) and (2), we note the identities

$$
\begin{aligned}
T_n\left(\frac{3}{2}\right) &= \frac{1}{2}\left[\left(\frac{3}{2}+\sqrt{\frac{9}{4}-1}\right)^n + \left(\frac{3}{2}-\sqrt{\frac{9}{4}-1}\right)^n\right] \\
&= \frac{1}{2}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{2n} + \left(\frac{1-\sqrt{5}}{2}\right)^{2n}\right] = \frac{1}{2}L_{2n}
\end{aligned}
\tag{8}
$$

and

$$
U_n\left(\frac{3}{2}\right) = \frac{1}{2\sqrt{\frac{9}{4}-1}}\left[\left(\frac{3}{2}+\sqrt{\frac{9}{4}-1}\right)^{n+1} - \left(\frac{3}{2}-\sqrt{\frac{9}{4}-1}\right)^{n+1}\right] = F_{2n+2}. \tag{9}
$$

Applying Lemma 3 and (9) we also have

$$
U_n\left(T_m\left(\frac{3}{2}\right)\right) = \frac{U_{m(n+1)-1}(\frac{3}{2})}{U_{m-1}(\frac{3}{2})} = \frac{F_{2m(n+1)}}{F_{2m}}. \tag{10}
$$

Taking $x = T_m(\frac{3}{2})$ in Corollary 2, applying (8), (9), and (10) we have

$$
\begin{aligned}
&\sum_{a+b+c=n} U_a\left(T_m\left(\frac{3}{2}\right)\right) \cdot U_b\left(T_m\left(\frac{3}{2}\right)\right) \cdot U_c\left(T_m\left(\frac{3}{2}\right)\right) \\
&= \sum_{a+b+c=n} \frac{F_{2m(a+1)}}{F_{2m}} \cdot \frac{F_{2m(b+1)}}{F_{2m}} \cdot \frac{F_{2m(c+1)}}{F_{2m}} \\
&= \frac{3(n+3)L_{2m}}{(4-L_{2m}^2)^2} \cdot \frac{F_{2m(n+2)}}{F_{2m}} - \frac{4(n+2)(n+4)-(n+1)(n+2)L_{2m}^2}{2(4-L_{2m}^2)^2} \cdot \frac{F_{2m(n+3)}}{F_{2m}}
\end{aligned}
$$

or

$$
\begin{aligned}
&\sum_{a+b+c=n} F_{2m(a+1)} \cdot F_{2m(b+1)} \cdot F_{2m(c+1)} \\
&= \frac{6(n+3)F_{2m}F_{4m}}{50F_m^4} \cdot F_{2m(n+2)} - \frac{4(n+2)(n+4)F_{2m}^2-(n+1)(n+2)F_{4m}^2}{50F_m^4} \cdot F_{2m(n+3)},
\end{aligned}
$$

where we have used the identities $F_m \cdot L_m = F_{2m}$ and $L_{2m}^2 - 4 = 5 \cdot F_m^2$.

Similarly, taking $x = T_m(\frac{3}{2})$ in Corollary 3, from (8), (9), and (10) we can also deduce the identity

$$\begin{aligned}
&\sum_{a+b+c+d=n} F_{2m(a+1)} \cdot F_{2m(b+1)} \cdot F_{2m(c+1)} \cdot F_{2m(d+1)} \\
&\quad = \frac{2(n+4)F_{2m}^3((n^2+8n+27)L_{2m}^2 - 4(n^2+8n+12))}{750F_m^6} \cdot F_{2m(n+3)} \\
&\qquad - \frac{F_{4m}F_{2m}^2(n+3)((n^2+3n+2)L_{2m}^2 - 4(n^2+3n-13))}{750F_m^6} \cdot F_{2m(n+4)}.
\end{aligned}$$

This proves Corollaries 6 and 7.

Corollary 8 follows from Corollary 7 with $m = 1$, $L_2 = 3$, $F_1 = F_2 = 1$, $F_4 = 3$.

This completes the proofs of our all results.

Competing interests
The author declares that they have no competing interests.

Author's contributions
WS obtained the main result and completed all the parts of this manuscript. WS read and approved the final manuscript.

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