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The zeros of difference of meromorphic solutions for the difference Riccati equation

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Abstract

In this paper, we mainly investigate some properties of the transcendental meromorphic solution $f(z)$ for the difference Riccati equation $f(z+1)=\frac{p(z)f(z)+q(z)}{f(z)+s(z)}$. We obtain some estimates of the exponents of the convergence of the zeros and poles of $f(z)$ and the difference $\Delta f(z)=f(z+1)-f(z)$.

MSC: 30D35; 39B12

Keywords: difference Riccati equation; Borel exceptional value; admissible meromorphic solution

1 Introduction and main results

Early results for difference equations were largely motivated by the work of Kimura [1] on the iteration of analytic functions. Shimomura [2] and Yanagihara [3] proved the following theorems, respectively.

Theorem A [2] *For any polynomial $P(y)$, the difference equation*

$$y(z+1)=P\bigl(y(z)\bigr)$$

has a non-trivial entire solution.

Theorem B [3] *For any rational function $R(y)$, the difference equation*

$$y(z+1)=R\bigl(y(z)\bigr)$$

has a non-trivial meromorphic solution.

Let f be a function transcendental and meromorphic in the plane. The forward difference is defined in the standard way by $\Delta f(z)=f(z+1)-f(z)$. In what follows, we assume the reader is familiar with the basic notions of Nevanlinna's value distribution theory (see, *e.g.*, [4–6]). In addition, we use the notations $\sigma(f)$ to denote the order of growth of the meromorphic function $f(z)$, and $\lambda(f)$ and $\lambda(\frac{1}{f})$ to denote the exponents of convergence of zeros and poles of $f(z)$, respectively. Moreover, we say that a meromorphic function g is small with respect to f if $T(r,g)=S(r,f)$, where $S(r,f)=o(T(r,f))$ outside of a possible exceptional set of finite logarithmic measure. Denote by $S(f)$ the family of all meromorphic functions which are small compared to $f(z)$. We say that a meromorphic solution f of a difference equation is admissible if all coefficients of the equation are in $S(f)$.

Recently, a number of papers (including [7–21]) focused on complex difference equations and difference analogs of Nevanlinna's theory. As the difference analogs of Nevanlinna's theory were being investigated, many results on the complex difference equations have been got rapidly. Many papers (including [7, 9, 12, 17]) mainly dealt with the growth of meromorphic solutions of difference equations.

In [15], Halburd and Korhonen used value distribution theory to obtain Theorem C.

Theorem C [15] *Let $f(z)$ be an admissible finite order meromorphic solution of the equation*

$$f(z+1)f(z-1)=\frac{c_2(f(z)-c_+)(f(z)-c_-)}{(f(z)-a_+)(f(z)-a_-)}=:R\bigl(z,f(z)\bigr), \tag{1.1}$$

where the coefficients are meromorphic functions, $c_2\not\equiv 0$ and $\deg_f(R)=2$. If the order of the poles of $f(z)$ is bounded, then either $f(z)$ satisfies a difference Riccati equation

$$f(z+1)=\frac{p(z)f(z)+q(z)}{f(z)+s(z)},$$

where $p,q,s\in S(f)$, or (1.1) *can be transformed by a bilinear change in $f(z)$ to one of the equations*

$$f(z+1)f(z-1)=\frac{\underline{\gamma} f^2(z)+\delta\lambda^z f(z)+\gamma\mu\lambda^{2z}}{(f(z)-1)(f(z)-\gamma)},$$

$$f(z+1)f(z-1)=\frac{f^2(z)+\delta e^{i\pi z/2}\lambda^z f(z)+\mu\lambda^{2z}}{f^2(z)-1},$$

where $\lambda\in\mathbb{C}$, and $\delta,\mu,\gamma,\underline{\gamma}=\gamma(z-1)\in S(f)$ are arbitrary finite order periodic functions such that δ and γ have period 2 *and μ has period* 1.

From the above, we see that the difference Riccati equations are an important class of difference equations, they will play an important role in research of difference Painlevé equations. Some papers [9–11, 22, 23] dealt with complex difference Riccati equations.

In this research, we investigate some properties of the difference Riccati equation and prove the following theorems.

Theorem 1.1 *Let $p(z)$, $q(z)$, $s(z)$ be meromorphic functions of finite order, and let $p(z)f(z)+q(z)$, $f(z)+s(z)$ be relatively prime polynomials in f. Suppose that $f(z)$ is a finite order admissible transcendental meromorphic solution of the difference Riccati equation*

$$f(z+1)=\frac{p(z)f(z)+q(z)}{f(z)+s(z)}. \tag{1.2}$$

Then

(i) $\lambda(\frac{1}{f})=\sigma(f)$. *Moreover, if $q(z)\not\equiv 0$, then $\lambda(\frac{1}{f})=\lambda(f)=\sigma(f)$;*

(ii) $\lambda(\frac{1}{\triangle f})=\sigma(\triangle f)=\sigma(f)$; $\lambda(\frac{1}{\frac{\triangle f}{f}})=\sigma(\frac{\triangle f}{f})=\sigma(f)$.

Theorem 1.2 *Let $p(z)$, $q(z)$, $s(z)$ be rational functions, and let $p(z)f(z) + q(z)$, $f(z) + s(z)$ be relatively prime polynomials in f. Suppose that $f(z)$ is a finite order transcendental meromorphic solution of the difference Riccati equation* (1.2). *Then*:

(i) *If $p(z) \equiv s(z)$, and there is a nonconstant rational function $Q(z)$ satisfying $q(z) = Q^2(z)$, then*

$$\lambda(\Delta f) = \lambda\left(\frac{\Delta f}{f}\right) = \sigma(f).$$

(ii) *If $p(z) \equiv -s(z)$, $s(z)$ is a nonconstant rational function, and there is a rational function $h(z)$ satisfying $s^2(z) + q(z) = h^2(z)$, then*

$$\lambda(\Delta f) = \lambda\left(\frac{\Delta f}{f}\right) = \sigma(f).$$

(iii) *If $p(z) \not\equiv \pm s(z)$, and there is a nonconstant rational function $m(z)$ satisfying $(s(z) - p(z))^2 + 4q(z) = m^2(z)$, then*

$$\lambda(\Delta f) = \lambda\left(\frac{\Delta f}{f}\right) = \sigma(f).$$

(iv) *If $p(z)$, $q(z)$, $s(z)$ are polynomials and $\deg p(z)$, $\deg q(z)$, $\deg s(z)$ contain just one maximum, then $f(z)$ has no nonzero Borel exceptional value.*

2 The proof of Theorem 1.1

We need the following lemmas to prove Theorem 1.1.

Lemma 2.1 (see [13, 18]) *Let f be a transcendental meromorphic solution of finite order σ of the difference equation*

$$P(z,f) = 0,$$

where $P(z,f)$ is a difference polynomial in $f(z)$ and its shifts. If $P(z,a) \not\equiv 0$ for a slowly moving target function a, i.e. $T(r,a) = S(r,f)$, then

$$m\left(r, \frac{1}{f-a}\right) = S(r,f)$$

outside of a possible exceptional set of finite logarithmic measure.

Lemma 2.2 (see [18]) *Let f be a transcendental meromorphic solution of finite order σ of a difference equation of the form*

$$H(z,f)P(z,f) = Q(z,f),$$

where $H(z,f)$ is a difference product of total degree n in $f(z)$ and its shifts, and where $P(z,f)$, $Q(z,f)$ are difference polynomials such that the total degree $\deg Q(z,f) \le n$. Then for each $\varepsilon > 0$,

$$m\bigl(r, P(z,f)\bigr) = O\bigl(r^{\sigma-1+\varepsilon}\bigr) + S(r,f)$$

possibly outside of an exceptional set of finite logarithmic measure.

Lemma 2.3 (Valiron-Mohon'ko) (see [5]) *Let $f(z)$ be a meromorphic function. Then for all irreducible rational functions in f,*

$$R(z,f(z)) = \frac{a_0(z) + a_1(z)f(z) + \cdots + a_m(z)f(z)^m}{b_0(z) + b_1(z)f(z) + \cdots + b_n(z)f(z)^n}$$

with meromorphic coefficients $a_i(z)$ $(i = 0,1,\ldots,m)$, $b_j(z)$ $(j = 0,1,\ldots,n)$, the characteristic function of $R(z,f(z))$ satisfies

$$T(r, R(z,f(z))) = dT(r,f) + O(\Psi(r)),$$

where $d = \deg_f R = \max\{m,n\}$ and $\Psi(r) = \max_{i,j}\{T(r,a_i), T(r,b_j)\}$.

In the remark of [15], p.15, it is pointed out that Lemma 2.4 holds.

Lemma 2.4 (see [9]) *Let f be a nonconstant finite order meromorphic function. Then*

$$N(r+1,f) = N(r,f) + S(r,f), \qquad T(r+1,f) = T(r,f) + S(r,f)$$

outside of a possible exceptional set of finite logarithmic measure.

Remark 2.1 In [12], Chiang and Feng proved that if f is a meromorphic function with exponent of convergence of poles $\lambda(\frac{1}{f}) = \lambda < \infty$, $\eta \neq 0$ fixed, then for each $\varepsilon > 0$,

$$N(r,f(z+\eta)) = N(r,f) + O(r^{\lambda-1+\varepsilon}) + O(\log r).$$

Lemma 2.5 (see [12]) *Let $f(z)$ be a meromorphic function with order $\sigma = \sigma(f) < +\infty$, and let η be a fixed non-zero complex number, then for each $\varepsilon > 0$, we have*

$$T(r,f(z+\eta)) = T(r,f) + O(r^{\sigma-1+\varepsilon}) + O(\log r).$$

Lemma 2.6 (see [24]) *Let $g : (0,+\infty) \to R$, $h : (0,+\infty) \to R$ be non-decreasing functions. If* (i) *$g(r) \leq h(r)$ outside of an exceptional set of finite linear measure, or* (ii) *$g(r) \leq h(r)$, $r \notin H \cup (0,1]$, where $H \subset (1,\infty)$ is a set of finite logarithmic measure, then for any $\alpha > 1$, there exists $r_0 > 0$ such that $g(r) \leq h(\alpha r)$ for all $r > r_0$.*

Lemma 2.7 (see [12]) *Let η_1, η_2 be two complex numbers such that $\eta_1 \neq \eta_2$ and let $f(z)$ be a finite order meromorphic function. Let σ be the order of $f(z)$, then for each $\varepsilon > 0$, we have*

$$m\left(r, \frac{f(z+\eta_1)}{f(z+\eta_2)}\right) = O(r^{\sigma-1+\varepsilon}).$$

Proof of Theorem 1.1 (i) Suppose that $f(z)$ is an admissible transcendental meromorphic solution of finite order $\sigma(f)$ of (1.2).

First, we prove $\lambda(\frac{1}{f}) = \sigma(f)$.

By (1.2), we have

$$(f(z) + s(z))f(z+1) = p(z)f(z) + q(z). \tag{2.1}$$

By Lemma 2.2 and (2.1), we get

$$m\bigl(r,f(z+1)\bigr) = O\bigl(r^{\sigma(f)-1+\varepsilon}\bigr) + S(r,f) \tag{2.2}$$

outside of a possible exceptional set of finite logarithmic measure. From (1.2) and Lemma 2.3, we have

$$T\bigl(r,f(z+1)\bigr) = T\bigl(r,f(z)\bigr) + S(r,f). \tag{2.3}$$

Hence, by (2.2) and (2.3), we conclude that

$$N\bigl(r,f(z+1)\bigr) = T\bigl(r,f(z)\bigr) + O\bigl(r^{\sigma(f)-1+\varepsilon}\bigr) + S(r,f) \tag{2.4}$$

outside of a possible exceptional set of finite logarithmic measure. By Lemma 2.4, we get

$$N\bigl(r,f(z+1)\bigr) \le N\bigl(r+1,f(z)\bigr) = N\bigl(r,f(z)\bigr) + S(r,f). \tag{2.5}$$

Hence, by (2.4) and (2.5), we conclude that

$$N\bigl(r,f(z)\bigr) \ge T\bigl(r,f(z)\bigr) + O\bigl(r^{\sigma(f)-1+\varepsilon}\bigr) + S(r,f) \tag{2.6}$$

outside of a possible exceptional set of finite logarithmic measure. By Lemma 2.6 and (2.6), we get $\lambda(\frac{1}{f}) = \sigma(f)$.

Second, we prove $\lambda(\frac{1}{f}) = \lambda(f) = \sigma(f)$ when $q(z) \not\equiv 0$.

By (1.2), we have

$$P\bigl(z,f(z)\bigr) := f(z+1)\bigl(f(z)+s(z)\bigr) - p(z)f(z) - q(z) = 0.$$

Hence, we get

$$P(z,0) = -q(z) \not\equiv 0. \tag{2.7}$$

Thus, by (2.7) and Lemma 2.1, we see that

$$m\left(r,\frac{1}{f}\right) = S(r,f)$$

outside of a possible exceptional set of finite logarithmic measure. Thus, we have

$$N\left(r,\frac{1}{f}\right) = T(r,f) + S(r,f) \tag{2.8}$$

outside of a possible exceptional set of finite logarithmic measure. By Lemma 2.6 and (2.8), we get $\lambda(f) = \sigma(f)$. So $\lambda(\frac{1}{f}) = \lambda(f) = \sigma(f)$.

(ii) Suppose that $f(z)$ is an admissible transcendental meromorphic solution of finite order $\sigma(f)$ of (1.2). By (1.2), we get

$$f(z+1)f(z) = p(z)f(z) - s(z)f(z+1) + q(z). \tag{2.9}$$

By (2.9) and Lemma 2.2, we have

$$m(r,f) = O\left(r^{\sigma(f)-1+\varepsilon}\right) + S(r,f) \tag{2.10}$$

outside of a possible exceptional set of finite logarithmic measure. From Lemma 2.7 and (2.10), we get

$$m\left(r,\triangle f(z)\right) \le m\left(r,f(z)\right) + m\left(r,\frac{\triangle f(z)}{f(z)}\right) = O\left(r^{\sigma(f)-1+\varepsilon}\right) + S(r,f) \tag{2.11}$$

outside of a possible exceptional set of finite logarithmic measure.

By (1.2), we get

$$\triangle f = \frac{p(z)f(z) + q(z)}{f(z) + s(z)} - f(z) = \frac{p(z)f(z) + q(z) - f(z)(f(z) + s(z))}{f(z) + s(z)}. \tag{2.12}$$

Since $p(z)f(z)+q(z)$ and $f(z)+s(z)$ are relatively prime polynomials in f, and $f(z)(f(z)+s(z))$ and $f(z)+s(z)$ have a common factor $f(z)+s(z)$. Therefore $p(z)f(z)+q(z)-f(z)(f(z)+s(z))$ and $f(z)+s(z)$ are relatively prime polynomials in f. By Lemma 2.3 and (2.12), we get

$$T(r,\triangle f) = 2T(r,f) + S(r,f). \tag{2.13}$$

By (2.11) and (2.13), we see that

$$N(r,\triangle f) = 2T(r,f) + O\left(r^{\sigma(f)-1+\varepsilon}\right) + S(r,f) \tag{2.14}$$

outside of a possible exceptional set of finite logarithmic measure. Hence, by Lemma 2.6 and (2.14), we get

$$\lambda\left(\frac{1}{\triangle f}\right) = \sigma(\triangle f) = \sigma(f).$$

By $N(r,\triangle f) = N(r,\frac{\triangle f}{f}\cdot f) \le N(r,\frac{\triangle f}{f}) + N(r,f)$ and (2.14), we get

$$N\left(r,\frac{\triangle f}{f}\right) \ge N(r,\triangle f) - N(r,f) \ge T(r,f) + O\left(r^{\sigma(f)-1+\varepsilon}\right) + S(r,f) \tag{2.15}$$

outside of a possible exceptional set of finite logarithmic measure. Hence, by Lemma 2.6 and (2.15), we have $\lambda(\frac{1}{\frac{\triangle f}{f}}) \ge \sigma(f)$. We have $\sigma(\frac{\triangle f}{f}) \le \sigma(f)$. Thus, we have

$$\lambda\left(\frac{1}{\frac{\triangle f}{f}}\right) = \sigma\left(\frac{\triangle f}{f}\right) = \sigma(f).$$

Theorem 1.1 is proved. □

3 The proof of Theorem 1.2

Suppose that $f(z)$ is a transcendental meromorphic solution of finite order $\sigma(f)$ of (1.2).

(i) By (1.2), we get

$$\Delta f = -\frac{f^2(z) + (s(z) - p(z))f(z) - q(z)}{f(z) + s(z)}. \tag{3.1}$$

Since $s(z) \equiv p(z)$ and $q(z) = Q^2(z)$, by (3.1), we get

$$\Delta f = -\frac{f^2(z) - Q^2(z)}{f(z) + s(z)} = -\frac{(f(z) - Q(z))(f(z) + Q(z))}{f(z) + s(z)}. \tag{3.2}$$

By (1.2), we have

$$P\bigl(z, f(z)\bigr) := f(z+1)f(z) + s(z)f(z+1) - p(z)f(z) - q(z) = 0. \tag{3.3}$$

By (3.3), we see that

$$P\bigl(z, Q(z)\bigr) = Q(z+1)Q(z) + s(z)Q(z+1) - p(z)Q(z) - q(z) \tag{3.4}$$

and

$$P\bigl(z, -Q(z)\bigr) = Q(z+1)Q(z) - s(z)Q(z+1) + p(z)Q(z) - q(z). \tag{3.5}$$

If $p(z) \equiv 0$, then $P(z, Q(z)) = P(z, -Q(z)) = Q(z+1)Q(z) - q(z)$. If $P(z, Q(z)) = P(z, -Q(z)) \equiv 0$, then $Q(z+1)Q(z) = q(z) = Q^2(z)$. Moreover, we get $Q(z+1) \equiv Q(z)$. This is a contradiction since $Q(z)$ is a nonconstant rational function. So $P(z, Q(z)) = P(z, -Q(z)) \not\equiv 0$.

Suppose that $p(z) \not\equiv 0$. If $P(z, Q(z)) \equiv 0$ and $P(z, -Q(z)) \equiv 0$, by (3.4) and (3.5), we get

$$p(z)\bigl(Q(z+1) - Q(z)\bigr) \equiv 0.$$

Thus, we know that $Q(z+1) \equiv Q(z)$. This is a contradiction since $Q(z)$ is a nonconstant rational function. Therefore, we get $P(z, Q(z)) \not\equiv 0$ or $P(z, -Q(z)) \not\equiv 0$. Without loss of generality, we assume that $P(z, Q(z)) \not\equiv 0$. By Lemma 2.1, we get

$$m\left(r, \frac{1}{f(z) - Q(z)}\right) = S(r, f)$$

possibly outside of an exceptional set of finite logarithmic measure. Moreover, we get

$$N\left(r, \frac{1}{f(z) - Q(z)}\right) = T(r, f) + S(r, f) \tag{3.6}$$

possibly outside of an exceptional set of finite logarithmic measure.

If z_0 is a common zero of $f(z) - Q(z)$ and $f(z) + s(z)$, then $Q(z_0) + s(z_0) = 0$. If z_0 is a zero of $f(z) - Q(z)$, and z_0 is a pole of $f(z) + Q(z)$, then z_0 is a pole of $2Q(z)$. Since $p(z)f(z) + q(z)$

and $f(z) + s(z)$ are relatively prime polynomials in f, we see that $p(z)s(z) \not\equiv q(z)$, that is, $Q^2(z) \not\equiv s^2(z)$. Hence, we get $Q(z) + s(z) \not\equiv 0$. Thus, we conclude that

$$\begin{aligned} &N\left(r, -\frac{f(z)+s(z)}{(f(z)-Q(z))(f(z)+Q(z))}\right) \\ &\quad \geq N\left(r, \frac{1}{f(z)-Q(z)}\right) - N\left(r, \frac{1}{Q(z)+s(z)}\right) - N\bigl(r, 2Q(z)\bigr). \end{aligned}$$

Since $Q(z)$ and $s(z)$ are rational functions, we get

$$N\left(r, \frac{1}{\Delta f}\right) \geq N\left(r, \frac{1}{f(z)-Q(z)}\right) + S(r,f). \tag{3.7}$$

By (3.6) and (3.7), we have

$$N\left(r, \frac{1}{\Delta f}\right) \geq T(r,f) + S(r,f)$$

possibly outside of an exceptional set of finite logarithmic measure. Thus, by Lemma 2.6, we see that $\lambda(\Delta f) \geq \sigma(f)$.

By (3.1), we get

$$\frac{\Delta f}{f} = -\frac{(f(z)-Q(z))(f(z)+Q(z))}{f(z)(f(z)+s(z))}.$$

By a similar method to above, we get $\lambda(\frac{\Delta f}{f}) \geq \sigma(f)$. Hence,

$$\lambda(\Delta f) = \lambda\left(\frac{\Delta f}{f}\right) = \sigma(f).$$

(ii) We divide this proof into the following two cases.

Case 1 Suppose that $q(z) \not\equiv 0$. Since $s(z) \equiv -p(z)$ and $h^2(z) = s^2(z) + q(z)$, by (3.1), we get

$$\Delta f = -\frac{(f(z)+s(z))^2 - h^2(z)}{f(z)+s(z)} = -\frac{(f(z)+s(z)+h(z))(f(z)+s(z)-h(z))}{f(z)+s(z)}. \tag{3.8}$$

We affirm $h(z) \not\equiv 0$. In fact, if $h(z) \equiv 0$, then $q(z) + s^2(z) = 0$, that is, $q(z) = -s^2(z)$. Therefore, we get $p(z)f(z) + q(z) = -s(z)(f(z) + s(z))$, and this is a contradiction since $p(z)f(z) + q(z)$ and $(f(z) + s(z))$ are relatively prime polynomials in f.

If $P(z, -(s(z) + h(z))) \equiv 0$ and $P(z, h(z) - s(z)) \equiv 0$, then $s(z + 1)h(z) + p(z)h(z) \equiv 0$. Moreover, $s(z+1) = -p(z) = s(z)$ since $-p(z) = s(z)$. This is a contradiction since $s(z)$ is a nonconstant rational function.

Therefore, we get $P(z, -(s(z) + h(z))) \not\equiv 0$ or $P(z, h(z) - s(z)) \not\equiv 0$. Without loss of generality, we assume that $P(z, h(z) - s(z)) \not\equiv 0$. By a similar method to above, we get

$$N\left(r, \frac{1}{f(z)-(h(z)-s(z))}\right) = T(r,f) + S(r,f) \tag{3.9}$$

possibly outside of an exceptional set of finite logarithmic measure.

If z_0 is a common zero of $f(z)+s(z)-h(z)$ and $f(z)+s(z)$, then $h(z_0)=0$. If z_0 is a zero of $f(z)+s(z)-h(z)$, and z_0 is a pole of $f(z)+s(z)+h(z)$, then z_0 is a pole of $2h(z)$. Thus, we see that

$$\begin{aligned} &N\left(r,-\frac{f(z)+s(z)}{(f(z)+h(z)+s(z))(f(z)-(h(z)-s(z)))}\right) \\ &\quad \geq N\left(r,\frac{1}{f(z)-(h(z)-s(z))}\right)-N\left(r,\frac{1}{h(z)}\right)-N\bigl(r,2h(z)\bigr). \end{aligned}$$

$h(z)$ is a rational function, so we get

$$N\left(r,\frac{1}{\Delta f}\right)\geq N\left(r,\frac{1}{f(z)-(h(z)-s(z))}\right)+S(r,f). \tag{3.10}$$

By (3.9) and (3.10), we see that

$$N\left(r,\frac{1}{\Delta f}\right)\geq T(r,f)+S(r,f)$$

possibly outside of an exceptional set of finite logarithmic measure. Thus, by Lemma 2.6, we see that $\lambda(\Delta f)\geq\sigma(f)$.

By a similar method to above, we get $\lambda(\frac{\Delta f}{f})\geq\sigma(f)$. Hence,

$$\lambda(\Delta f)=\lambda\left(\frac{\Delta f}{f}\right)=\sigma(f).$$

Case 2 Suppose that $q(z)\equiv 0$. Since $s(z)\equiv -p(z)$, by (3.1), we get

$$\Delta f=-\frac{f(z)(f(z)+2s(z))}{f(z)+s(z)}. \tag{3.11}$$

By (3.3), we get

$$P\bigl(z,-2s(z)\bigr)=4s(z+1)s(z)-2s(z)s(z+1)+2p(z)s(z)=2s(z)\bigl(s(z+1)-s(z)\bigr).$$

Since $s(z)$ is a nonconstant rational function, we get $s(z+1)\not\equiv s(z)$. Therefore, we have $P(z,-2s(z))\not\equiv 0$.

By a similar method to above, we have

$$N\left(r,\frac{1}{\Delta f}\right)\geq T(r,f)+S(r,f)$$

possibly outside of an exceptional set of finite logarithmic measure. Thus, by Lemma 2.6, we see that $\lambda(\Delta f)\geq\sigma(f)$.

By (3.11), we get

$$\frac{\Delta f}{f}=-\frac{f(z)(f(z)+2s(z))}{f(z)(f(z)+s(z))}=-\frac{f(z)+2s(z)}{f(z)+s(z)}.$$

By a similar method to above, we get $\lambda(\frac{\Delta f}{f}) \geq \sigma(f)$. Hence,

$$\lambda(\Delta f) = \lambda\left(\frac{\Delta f}{f}\right) = \sigma(f).$$

(iii) First, we prove $\lambda(\Delta f) = \sigma(f)$. We divide this proof into the following two cases.

Case 1 Suppose that $q(z) \not\equiv 0$. Substituting $(s(z) - p(z))^2 + 4q(z) = m^2(z)$ into (3.1), we get

$$\begin{aligned}\Delta f &= -\frac{(f(z) + \frac{s(z)-p(z)}{2})^2 - \frac{(s(z)-p(z))^2+4q(z)}{4}}{f(z) + s(z)}\\ &= -\frac{(f(z) + \frac{s(z)-p(z)}{2} + \frac{m(z)}{2})(f(z) + \frac{s(z)-p(z)}{2} - \frac{m(z)}{2})}{f(z) + s(z)}.\end{aligned} \tag{3.12}$$

From (3.3), we get

$$\begin{aligned}P_1 &:= P\left(z, \frac{m(z)}{2} - \frac{s(z) - p(z)}{2}\right)\\ &= \left(\frac{m(z+1)}{2} - \frac{s(z+1) - p(z+1)}{2}\right) \cdot \left(\frac{m(z)}{2} - \frac{s(z) - p(z)}{2} + s(z)\right)\\ &\quad - p(z)\left(\frac{m(z)}{2} - \frac{s(z) - p(z)}{2}\right) - q(z)\end{aligned} \tag{3.13}$$

and

$$\begin{aligned}P_2 &: = P\left(z, -\frac{m(z)}{2} - \frac{s(z) - p(z)}{2}\right)\\ &= \left(\frac{m(z+1)}{-2} + \frac{s(z+1) - p(z+1)}{-2}\right) \cdot \left(s(z) - \frac{m(z)}{2} - \frac{s(z) - p(z)}{2}\right)\\ &\quad + p(z)\left(\frac{m(z)}{2} + \frac{s(z) - p(z)}{2}\right) - q(z).\end{aligned} \tag{3.14}$$

Since $(s(z) - p(z))^2 + 4q(z) = m^2(z)$, by (3.13) and (3.14), we see that

$$\begin{aligned}P_1 &= \frac{m(z) + s(z) + p(z)}{2}\left(\frac{m(z+1) - s(z+1) + p(z+1)}{2} - \frac{m(z) - s(z) + p(z)}{2}\right),\\ P_2 &= \frac{m(z) - s(z) - p(z)}{2}\left(\frac{s(z+1) - p(z+1) + m(z+1)}{2} - \frac{s(z) - p(z) + m(z)}{2}\right).\end{aligned}$$

We affirm that $m(z) - s(z) - p(z) \not\equiv 0$ and $m(z) + s(z) + p(z) \not\equiv 0$. In fact, if $m(z) - s(z) - p(z) \equiv 0$ or $m(z) + s(z) + p(z) \equiv 0$, then $m(z) = \pm(s(z) + p(z))$. Substituting $m(z) = \pm(s(z) + p(z))$ into $(s(z) - p(z))^2 + 4q(z) = m^2(z)$, we get $q(z) = s(z)p(z)$. This is a contradiction since $p(z)f(z) + q(z)$ and $f(z) + s(z)$ are relatively prime polynomials in f.

We affirm that $s(z) - p(z) + m(z)$ or $s(z) - p(z) - m(z)$ is nonconstant rational function. In fact, if there are two constants c_1 and c_2, such that $s(z) - p(z) + m(z) = c_1$ and $s(z) - p(z) - m(z) = c_2$, then we get $s(z) - p(z) = \frac{c_1+c_2}{2}$. Furthermore, we have $m(z) = \frac{c_1-c_2}{2}$, this is a contradiction since $m(z)$ is a nonconstant rational function. Hence, we conclude that $s(z) - p(z) + m(z)$ or $s(z) - p(z) - m(z)$ is a nonconstant rational function. Thus, we get $s(z +$

$1)-p(z+1)+m(z+1)\not\equiv s(z)-p(z)+m(z)$, or $s(z+1)-p(z+1)-m(z+1)\not\equiv s(z)-p(z)-m(z)$. So, we get $P_1=P(z,\frac{m(z)}{2}-\frac{s(z)-p(z)}{2})\not\equiv 0$, or $P_2=P(z,-\frac{m(z)}{2}-\frac{s(z)-p(z)}{2})\not\equiv 0$.

Without loss of generality, we assume that $P_1=P(z,\frac{m(z)}{2}-\frac{s(z)-p(z)}{2})\not\equiv 0$. By Lemma 2.1, we get

$$m\left(r,\frac{1}{f(z)+\frac{s(z)-p(z)}{2}-\frac{m(z)}{2}}\right)=S(r,f)$$

for all r outside of a possible exceptional set with finite logarithmic measure. Moreover, we get

$$N\left(r,\frac{1}{f(z)+\frac{s(z)-p(z)}{2}-\frac{m(z)}{2}}\right)=T(r,f)+S(r,f) \tag{3.15}$$

for all r outside of a possible exceptional set with finite logarithmic measure.

If z_0 is a common zero of $f(z)+\frac{s(z)-p(z)}{2}-\frac{m(z)}{2}$ and $f(z)+s(z)$, then $-\frac{s(z_0)+p(z_0)+m(z_0)}{2}=0$. If z_0 is a zero of $f(z)+\frac{s(z)-p(z)}{2}-\frac{m(z)}{2}$, and z_0 is a pole of $f(z)+\frac{s(z)-p(z)}{2}+\frac{m(z)}{2}$, then z_0 is a pole of $m(z)$. From above, we know that $-\frac{s(z)+p(z)+m(z)}{2}\not\equiv 0$. Thus, we conclude that

$$\begin{aligned}&N\left(r,-\frac{f(z)+s(z)}{(f(z)+\frac{s(z)-p(z)}{2}+\frac{m(z)}{2})(f(z)+\frac{s(z)-p(z)}{2}-\frac{m(z)}{2})}\right)\\&\quad\geq N\left(r,\frac{1}{f(z)+\frac{s(z)-p(z)}{2}-\frac{m(z)}{2}}\right)-N\left(r,-\frac{2}{s(z)+p(z)+m(z)}\right)-N\big(r,m(z)\big).\end{aligned}$$

Since $s(z)$, $p(z)$ and $m(z)$ are rational functions, we have

$$N\left(r,\frac{1}{\Delta f}\right)\geq N\left(r,\frac{1}{f(z)+\frac{s(z)-p(z)}{2}-\frac{m(z)}{2}}\right)+S(r,f). \tag{3.16}$$

From (3.15) and (3.16), we get

$$N\left(r,\frac{1}{\Delta f}\right)\geq T(r,f)+S(r,f) \tag{3.17}$$

for all r outside of a possible exceptional set with finite logarithmic measure. Hence, by Lemma 2.6 and (3.17), we get $\lambda(\Delta f)\geq\sigma(f)$.

By (3.12), we see that

$$\frac{\Delta f}{f}=-\frac{(f(z)+\frac{s(z)-p(z)}{2}+\frac{m(z)}{2})(f(z)+\frac{s(z)-p(z)}{2}-\frac{m(z)}{2})}{f(z)(f(z)+s(z))}.$$

By a similar method to above, we get $\lambda(\frac{\Delta f}{f})\geq\sigma(f)$. Thus, we get

$$\lambda(\Delta f)=\lambda\left(\frac{\Delta f}{f}\right)=\sigma(f).$$

Case 2 Suppose that $q(z)\equiv 0$. Since $s(z)\not\equiv\pm p(z)$, by (3.1), we get

$$\Delta f=-\frac{f(z)(f(z)+s(z)-p(z))}{f(z)+s(z)}. \tag{3.18}$$

By (3.3), we get

$$P\bigl(z,p(z)-s(z)\bigr) = \bigl(p(z+1)-s(z+1)\bigr)\bigl(p(z)-s(z)+s(z)\bigr) - p(z)\bigl(p(z)-s(z)\bigr).$$

That is,

$$P\bigl(z,p(z)-s(z)\bigr) = p(z)\bigl\{\bigl(p(z+1)-s(z+1)\bigr) - \bigl(p(z)-s(z)\bigr)\bigr\}.$$

By $q(z) \equiv 0$ and $m^2(z) = (p(z)-s(z))^2 + 4q(z)$, we get $m^2(z) = (p(z)-s(z))^2$. Since $m(z)$ is a nonconstant rational function, we see that $p(z)-s(z)$ is a nonconstant rational function too. So $p(z+1)-s(z+1) \not\equiv p(z)-s(z)$. Therefore, we have $P(z,p(z)-s(z)) \not\equiv 0$. By Lemma 2.1, we get

$$m\left(r,\frac{1}{f(z)+s(z)-p(z)}\right) = S(r,f)$$

possibly outside of an exceptional set of finite logarithmic measure. Moreover, we get

$$N\left(r,\frac{1}{f(z)+s(z)-p(z)}\right) = T(r,f) + S(r,f) \tag{3.19}$$

possibly outside of an exceptional set of finite logarithmic measure.

If z_0 is a common zero of $f(z)+s(z)-p(z)$ and $f(z)+s(z)$, then $p(z_0) = 0$. If z_0 is a zero of $f(z)+s(z)-p(z)$, and z_0 is a pole of $f(z)$, then z_0 is a pole of $s(z)-p(z)$. Thus, we see that

$$\begin{aligned} &N\left(r,-\frac{f(z)+s(z)}{f(z)(f(z)+s(z)-p(z))}\right) \\ &\quad \geq N\left(r,\frac{1}{f(z)+s(z)-p(z)}\right) - N\left(r,\frac{1}{p(z)}\right) - N\bigl(r,s(z)-p(z)\bigr). \end{aligned}$$

Since $p(z)$ and $s(z)-p(z)$ are rational functions, we get

$$N\left(r,\frac{1}{\Delta f}\right) \geq N\left(r,\frac{1}{f(z)+s(z)-p(z)}\right) + S(r,f). \tag{3.20}$$

By (3.19) and (3.20), we have

$$N\left(r,\frac{1}{\Delta f}\right) \geq T(r,f) + S(r,f) \tag{3.21}$$

possibly outside of an exceptional set of finite logarithmic measure. Thus, by Lemma 2.6 and (3.21), we see that $\lambda(\Delta f) \geq \sigma(f)$.

By (3.18), we get

$$\frac{\Delta f}{f} = -\frac{f(z)(f(z)+s(z)-p(z))}{f(z)(f(z)+s(z))} = -\frac{f(z)+s(z)-p(z)}{f(z)+s(z)}.$$

By a similar method to above, we get $\lambda(\frac{\Delta f}{f}) \geq \sigma(f)$. Hence,

$$\lambda(\Delta f) = \lambda\left(\frac{\Delta f}{f}\right) = \sigma(f).$$

(iv) Suppose that $f(z)$ is a finite order transcendental meromorphic solution of (1.2).

Without loss of generality, we assume that $\deg p(z) > \max\{\deg q(z), \deg s(z)\}$. Then $\deg p(z) \geq 1$. Set $p(z) = a_k z^k + \cdots + a_0$ ($a_k \neq 0$). Let $a \neq 0$. By (3.3), we have

$$P(z,a) = a^2 + as(z) - ap(z) + q(z) = -aa_k z^k + \cdots \not\equiv 0 \tag{3.22}$$

since $aa_k \neq 0$. By Lemma 2.1 and (3.22), we conclude that

$$m\left(r, \frac{1}{f-a}\right) = S(r,f)$$

possibly outside of an exceptional set of finite logarithmic measure. Thus, we get

$$N\left(r, \frac{1}{f-a}\right) = T(r,f) + S(r,f) \tag{3.23}$$

possibly outside of an exceptional set of finite logarithmic measure. Thus, by Lemma 2.6 and (3.23), we get

$$\lambda(f-a) = \sigma(f).$$

That is, $f(z)$ has no nonzero Borel exceptional value.

Theorem 1.2 is proved.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
CWP completed the main part of this article, CWP and ZXC corrected the main theorems. All authors read and approved the final manuscript.

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