Qiu *Advances in Difference Equations* (2015) 2015:299
DOI 10.1186/s13662-015-0636-y

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH **Open Access**

Oscillation criteria of third-order nonlinear dynamic equations with nonpositive neutral coefficients on time scales

Yang-Cong Qiu[*]

[*]Correspondence:
q840410@qq.com
School of Humanities and Social Science, Shunde Polytechnic, Foshan, Guangdong 528333, P.R. China

Abstract
In this paper, we establish oscillation criteria of third-order nonlinear dynamic equations with nonpositive neutral coefficients on time scales by a generalized Riccati transformation and employing functions in some function classes. Two examples are presented to show the significance of the results.

Keywords: third-order nonlinear dynamic equations; time scales; oscillation criteria; generalized Riccati transformation

1 Introduction

In this paper, we consider third-order nonlinear dynamic equations with nonpositive neutral coefficients of the form

$$\left(r_1(t)\left(\left(r_2(t)\left(z^{\Delta}(t)\right)^{\gamma_2}\right)^{\Delta}\right)^{\gamma_1}\right)^{\Delta} + f\left(t, x\left(h(t)\right)\right) = 0, \tag{1}$$

where $z(t) = x(t) - p(t)x(g(t))$, on a time scale $\mathbb{T}$ satisfying $\inf \mathbb{T} = t_0$ and $\sup \mathbb{T} = \infty$. Throughout this paper we assume that:

(C1) $r_1, r_2 \in C_{\mathrm{rd}}(\mathbb{T}, (0, \infty))$ such that

$$\int_{t_0}^{\infty} \frac{1}{r_1^{1/\gamma_1}(t)} \Delta t = \infty, \qquad \int_{t_0}^{\infty} \frac{1}{r_2^{1/\gamma_2}(t)} \Delta t = \infty;$$

(C2) γ, γ_1, γ_2 are all quotients of odd positive integers, and $\gamma = \gamma_1 \cdot \gamma_2$;

(C3) $p \in C_{\mathrm{rd}}(\mathbb{T}, [0, \infty))$ and there exists a constant p_0 with $0 \le p_0 < 1$ such that

$$\lim_{t \to \infty} p(t) = p_0;$$

(C4) $g \in C_{\mathrm{rd}}(\mathbb{T}, \mathbb{T})$, $g(t) \le t$, $\lim_{t \to \infty} g(t) = \infty$, and there exists a sequence $\{c_k\}_{k \ge 0}$ such that $\lim_{k \to \infty} c_k = \infty$ and $g(c_{k+1}) = c_k$;

(C5) $h \in C_{\mathrm{rd}}(\mathbb{T}, \mathbb{T})$, and for any $t \in \mathbb{T}$,

$$h(t) \ge \begin{cases} \sigma(t), & 0 < \gamma < 1, \\ t, & \gamma \ge 1; \end{cases}$$

(C6) $f \in C(\mathbb{T} \times \mathbb{R}, \mathbb{R})$ and there exists a function $q \in C_{\mathrm{rd}}(\mathbb{T}, (0, \infty))$ such that $uf(t, u) \geq q(t)u^{\gamma+1}$;

(C7) When $0 < \gamma < 1$, it always satisfies

$$\int_{t_0}^{\infty} q(t)\Delta t < \infty.$$

Definition 1.1 A solution x of (1) is said to have a generalized zero at $t^* \in \mathbb{T}$ if $x(t^*)x(\sigma(t^*)) \leq 0$, and it is said to be nonoscillatory on $\mathbb{T}$ if there exists $t_0 \in \mathbb{T}$ such that $x(t)x(\sigma(t)) > 0$ for all $t > t_0$. Otherwise, it is oscillatory. Equation (1) is said to be oscillatory if all solutions of (1) are oscillatory.

In 1988, the theory of time scales was introduced by Hilger in his Ph.D. thesis [1] to unify continuous and discrete analysis; see also [2]. Since then, the theory had received a lot of attention. The details of time scales can be found in [3–6] and are omitted here.

There has been many achievements of the study of oscillation of nonlinear dynamic equations on time scales in the last few years; see [7–16] and the references therein. Hassan [8], Erbe *et al.* [7], and Zhang and Wang [16] gave some oscillation criteria successively for the third-order nonlinear delay dynamic equation

$$\bigl(a(t)\bigl[\bigl(r(t)x^{\Delta}(t)\bigr)^{\Delta}\bigr]^{\gamma}\bigr)^{\Delta} + f\bigl(t, x\bigl(\tau(t)\bigr)\bigr) = 0.$$

Saker *et al.* [13] studied the oscillation of the second-order damped dynamic equation

$$\bigl(a(t)x^{\Delta}(t)\bigr)^{\Delta} + p(t)x^{\Delta^{\sigma}}(t) + q(t)\bigl(f \circ x^{\sigma}\bigr) = 0.$$

Qiu and Wang [10] considered second-order nonlinear dynamic equation

$$\bigl(p(t)\psi\bigl(x(t)\bigr)k \circ x^{\Delta}(t)\bigr)^{\Delta} + f\bigl(t, x\bigl(\sigma(t)\bigr)\bigr) = 0.$$

Employing a generalized Riccati transformation

$$u(t) = A(t)\frac{p(t)\psi(x(t))k \circ x^{\Delta}(t)}{x(t)} + B(t),$$

the authors established some Kamenev-type oscillation criteria. Şenel [14] investigated the oscillation of the second-order nonlinear dynamic equation of the form

$$\bigl(r(t)\bigl(x^{\Delta}(t)\bigr)^{\gamma}\bigr)^{\Delta} + p(t)\bigl(x^{\Delta}(t)\bigr)^{\gamma} + f\bigl(t, x\bigl(g(t)\bigr)\bigr) = 0. \tag{2}$$

Qiu and Wang [11] corrected some mistakes in [14] and established correct oscillation criteria for (2). Yu and Wang [15] considered the third-order nonlinear dynamic equation

$$\left(\frac{1}{a_2(t)}\left(\left(\frac{1}{a_1(t)}\bigl(x^{\Delta}(t)\bigr)^{\alpha_1}\right)^{\Delta}\right)^{\alpha_2}\right)^{\Delta} + q(t)f\bigl(x(t)\bigr) = 0 \tag{3}$$

under the condition $\alpha_1\alpha_2 = 1$, and they established some sufficient conditions which guarantee that every solution x of (3) oscillates or converges to zero on a time scale $\mathbb{T}$. Li *et al.*

[9] studied the second-order neutral delay differential equation

$$\bigl(r(t)\bigl(z'(t)\bigr)^{\alpha}\bigr)' + q(t)f\bigl(x\bigl(\sigma(t)\bigr)\bigr) = 0, \quad t \geq t_0 > 0,$$

where $z(t) = x(t) - p(t)x(\tau(t))$ and $\alpha > 0$ is the ratio of two odd integers. Qiu [12] obtained some significant results for the existence of nonoscillatory solutions to the third-order nonlinear neutral dynamic equation of the form

$$\bigl(r_1(t)\bigl(r_2(t)\bigl(x(t) + p(t)x\bigl(g(t)\bigr)\bigr)^{\Delta}\bigr)^{\Delta}\bigr)^{\Delta} + f\bigl(t, x\bigl(h(t)\bigr)\bigr) = 0,$$

where $\lim_{t\to\infty} p(t) = p_0 \in (-1, 1)$.

In this paper, motivated by [9, 10, 12, 14, 15], we will establish oscillation criteria of (1), which are more general than (3), by a generalized Riccati transformation, and give two examples to show the significance of the results.

For the sake of simplicity, we denote $(a, b) \cap \mathbb{T} = (a, b)_{\mathbb{T}}$ throughout the paper, where $a, b \in \mathbb{R}$, and $[a, b]_{\mathbb{T}}$, $[a, b)_{\mathbb{T}}$, $(a, b]_{\mathbb{T}}$ are similar notations.

2 Preliminary results

To establish the oscillation criteria of (1), we give six lemmas in this section.

Lemma 2.1 *Suppose that $x(t)$ is an eventually positive solution of* (1), *and there exists a constant $a \geq 0$ such that* $\lim_{t\to\infty} z(t) = a$. *Then we have*

$$\lim_{t\to\infty} x(t) = \frac{a}{1 - p_0}.$$

Proof Suppose that $x(t)$ is an eventually positive solution of (1). In view of (C3) and (C5), there exist $T \in [t_0, \infty)_{\mathbb{T}}$ and $p_0 < p_1 < 1$ such that $x(t) > 0$, $x(g(t)) > 0$, and $p(t) \leq p_1$ for $t \in [T, \infty)_{\mathbb{T}}$. We claim that $x(t)$ is bounded on $[T, \infty)_{\mathbb{T}}$. Assume not; then there exists $\{t_n\} \in [T, \infty)_{\mathbb{T}}$ with $t_n \to \infty$ as $n \to \infty$ such that

$$x(t_n) = \max_{t\in[T,t_n]_{\mathbb{T}}} x(t) \quad \text{and} \quad \lim_{n\to\infty} x(t_n) = \infty.$$

Noting that $g(t) \leq t$, we have

$$z(t_n) = x(t_n) - p(t_n)x\bigl(g(t_n)\bigr) \geq (1 - p_1)x(t_n) \to \infty$$

as $n \to \infty$, which contradicts the fact that $\lim_{t\to\infty} z(t) = a$. Therefore, $x(t)$ is bounded. Then assume that

$$\limsup_{t\to\infty} x(t) = \overline{x} \quad \text{and} \quad \liminf_{t\to\infty} x(t) = \underline{x}.$$

Since $0 \leq p_0 < 1$, we have

$$a \geq \overline{x} - p_0\overline{x} \quad \text{and} \quad a \leq \underline{x} - p_0\underline{x},$$

which implies that $\overline{x} \leq \underline{x}$. So $\overline{x} = \underline{x}$, and we see that $\lim_{t\to\infty} x(t)$ exists and $\lim_{t\to\infty} x(t) = a/(1 - p_0)$. The proof is complete. □

Lemma 2.2 *Assume that $x(t)$ is an eventually positive solution of* (1), *then there exists a sufficiently large $T \in [t_0, \infty)_{\mathbb{T}}$ such that, for $t \in [T, \infty)_{\mathbb{T}}$, we have*

$$\left(r_2(t)\left(z^{\Delta}(t)\right)^{\gamma_2}\right)^{\Delta} > 0$$

and

$$z^{\Delta}(t) > 0 \quad or \quad z^{\Delta}(t) < 0.$$

Proof Suppose that $x(t)$ is an eventually positive solution of (1). From (C3) and (C5), there exist $t_1 \in [t_0, \infty)_{\mathbb{T}}$ and $p_0 < p_1 < 1$ such that $x(t) > 0$, $x(g(t)) > 0$, $x(h(t)) > 0$, and $p(t) \leq p_1$ for $t \in [t_1, \infty)_{\mathbb{T}}$. By (1) and (C6), it follows that, for $t \in [t_1, \infty)_{\mathbb{T}}$,

$$\left(r_1(t)\left(\left(r_2(t)\left(z^{\Delta}(t)\right)^{\gamma_2}\right)^{\Delta}\right)^{\gamma_1}\right)^{\Delta} = -f\left(t, x\left(h(t)\right)\right) < 0. \tag{4}$$

Hence, $r_1(t)((r_2(t)(z^{\Delta}(t))^{\gamma_2})^{\Delta})^{\gamma_1}$ is strictly decreasing on $[t_1, \infty)_{\mathbb{T}}$. We claim that

$$r_1(t)\left(\left(r_2(t)\left(z^{\Delta}(t)\right)^{\gamma_2}\right)^{\Delta}\right)^{\gamma_1} > 0, \quad t \in [t_1, \infty)_{\mathbb{T}}. \tag{5}$$

Assume not; then there exists $t_2 \in [t_1, \infty)_{\mathbb{T}}$ such that

$$r_1(t)\left(\left(r_2(t)\left(z^{\Delta}(t)\right)^{\gamma_2}\right)^{\Delta}\right)^{\gamma_1} < 0$$

for $t \in [t_2, \infty)_{\mathbb{T}}$. So there exists a constant $c < 0$ and we have $t_3 \in [t_2, \infty)_{\mathbb{T}}$ such that $r_1(t)((r_2(t)(z^{\Delta}(t))^{\gamma_2})^{\Delta})^{\gamma_1} \leq c$ for $t \in [t_3, \infty)_{\mathbb{T}}$, which means that

$$\left(r_2(t)\left(z^{\Delta}(t)\right)^{\gamma_2}\right)^{\Delta} \leq \left(\frac{c}{r_1(t)}\right)^{1/\gamma_1}, \quad t \in [t_3, \infty)_{\mathbb{T}}. \tag{6}$$

Substituting s for t, and integrating (6) from t_3 to $t \in [\sigma(t_3), \infty)_{\mathbb{T}}$, we obtain

$$r_2(t)\left(z^{\Delta}(t)\right)^{\gamma_2} \leq r_2(t_3)\left(z^{\Delta}(t_3)\right)^{\gamma_2} + c^{1/\gamma_1} \int_{t_3}^{t} \frac{\Delta s}{r_1^{1/\gamma_1}(s)}.$$

Letting $t \to \infty$, by (C1) we have $r_2(t)(z^{\Delta}(t))^{\gamma_2} \to -\infty$. Then there exists $t_4 \in [t_3, \infty)_{\mathbb{T}}$ such that $r_2(t)(z^{\Delta}(t))^{\gamma_2} \leq r_2(t_4)(z^{\Delta}(t_4))^{\gamma_2} < 0$ for $t \in [t_4, \infty)_{\mathbb{T}}$, which implies that

$$z^{\Delta}(t) \leq r_2^{1/\gamma_2}(t_4) z^{\Delta}(t_4) \cdot \frac{1}{r_2^{1/\gamma_2}(t)}. \tag{7}$$

Substituting s for t, and integrating (7) from t_4 to $t \in [\sigma(t_4), \infty)_{\mathbb{T}}$, we obtain

$$z(t) - z(t_4) \leq r_2^{1/\gamma_2}(t_4) z^{\Delta}(t_4) \int_{t_4}^{t} \frac{\Delta s}{r_2^{1/\gamma_2}(s)}.$$

Letting $t \to \infty$, by (C1) we have $z(t) \to -\infty$. Then there exists $t_5 \in [t_4, \infty)_{\mathbb{T}}$ such that $z(t) < 0$ or

$$x(t) < p(t) x\left(g(t)\right) \leq p_1 x\left(g(t)\right), \quad t \in [t_5, \infty)_{\mathbb{T}}.$$

By (C4), we can choose some positive integer k_0 such that $c_k \in [t_5, \infty)_{\mathbb{T}}$ for all $k \geq k_0$. Then for any $k \geq k_0 + 1$, we have

$$\begin{aligned} x(c_k) &< p_1 x\big(g(c_k)\big) = p_1 x(c_{k-1}) < p_1^2 x\big(g(c_{k-1})\big) = p_1^2 x(c_{k-2}) < \cdots \\ &< p_1^{k-k_0} x\big(g(c_{k_0+1})\big) = p_1^{k-k_0} x(c_{k_0}). \end{aligned}$$

The inequality above implies that $\lim_{k\to\infty} x(c_k) = 0$. It follows that

$$\lim_{k\to\infty} z(c_k) = 0,$$

and this contradicts $\lim_{t\to\infty} z(t) = -\infty$. So (5) holds, which implies that

$$\big(r_2(t)\big(z^{\Delta}(t)\big)^{\gamma_2}\big)^{\Delta} > 0, \quad t \in [t_1, \infty)_{\mathbb{T}}.$$

Therefore, $r_2(t)(z^{\Delta}(t))^{\gamma_2}$ is strictly increasing on $[t_1, \infty)_{\mathbb{T}}$. It follows that $r_2(t)(z^{\Delta}(t))^{\gamma_2}$ is eventually positive or $r_2(t)(z^{\Delta}(t))^{\gamma_2} < 0$ on $[t_1, \infty)_{\mathbb{T}}$. Lemma 2.2 is proved. □

Lemma 2.3 *Assume that $x(t)$ is an eventually positive solution of* (1), *then $z(t)$ is eventually positive or* $\lim_{t\to\infty} x(t) = 0$.

Proof Suppose that $x(t)$ is an eventually positive solution of (1), by Lemma 2.2 there exists $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $z^{\Delta}(t) > 0$ or $z^{\Delta}(t) < 0$, $t \in [t_1, \infty)_{\mathbb{T}}$.

(i) $z^{\Delta}(t) > 0$, $t \in [t_1, \infty)_{\mathbb{T}}$. Then it follows that $z(t)$ is eventually positive or eventually negative. If $z(t)$ is eventually positive, the lemma is proved. If $z(t)$ is eventually negative, we see that $\lim_{t\to\infty} z(t)$ exists. Assume that $\lim_{t\to\infty} z(t) < 0$. Similarly as in the proof of Lemma 2.2, we will have the contradiction. Hence, $\lim_{t\to\infty} z(t) = 0$. Then it follows that $\lim_{t\to\infty} x(t) = 0$ by Lemma 2.1.

(ii) $z^{\Delta}(t) < 0$, $t \in [t_1, \infty)_{\mathbb{T}}$. Similarly, we see that $z(t)$ is eventually positive or eventually negative. Assume that $z(t)$ is eventually negative, there exists a constant $c < 0$ and we have $t_2 \in [t_1, \infty)_{\mathbb{T}}$ such that $z(t) < c$, $t \in [t_2, \infty)_{\mathbb{T}}$. It will cause a similar contradiction as in the proof of Lemma 2.2. Hence, $z(t)$ is eventually positive and the lemma is proved.

The proof is complete. □

Lemma 2.4 *For* $0 < \gamma < 1$, *assume that $x(t)$ is an eventually positive solution of* (1), *and $z(t)$, $z^{\Delta}(t)$ are both eventually positive. Then there exists $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that*

$$\left(\frac{z^{\Delta}(t)}{z^{\sigma}(t)}\right)^{1-\gamma} \geq \alpha(t) = \left(\frac{\delta(t)}{r_2(t)}\right)^{(1-\gamma)/\gamma_2} \left(\int_t^{\infty} q(s)\Delta s\right)^{(1-\gamma)/\gamma}, \quad t \in [t_1, \infty)_{\mathbb{T}},$$

where

$$\delta(t) = \int_{t_1}^{t} \frac{\Delta s}{r_1^{1/\gamma_1}(s)}.$$

Proof Suppose that $x(t)$ is an eventually positive solution of (1), and $z(t)$, $z^{\Delta}(t)$ are both eventually positive, then there exists $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $x(t) > 0$, $x(g(t)) > 0$, $x(h(t)) > 0$,

$z(t) > 0$, and $z^{\Delta}(t) > 0$ for $t \in [t_1, \infty)_{\mathbb{T}}$. By Lemma 2.2 we have

$$\big(r_2(t)\big(z^{\Delta}(t)\big)^{\gamma_2}\big)^{\Delta} > 0, \quad t \in [t_1, \infty)_{\mathbb{T}}.$$

By $z^{\Delta}(t) > 0$ and $z(t) = x(t) - p(t)x(g(t)) \le x(t)$, it follows that, for $t \in [t_1, \infty)_{\mathbb{T}}$,

$$\begin{aligned}&\big(r_1(t)\big(\big(r_2(t)\big(z^{\Delta}(t)\big)^{\gamma_2}\big)^{\Delta}\big)^{\gamma_1}\big)^{\Delta}\\ &\quad = -f\big(t, x\big(h(t)\big)\big) \le -q(t)x^{\gamma}\big(h(t)\big) \le -q(t)z^{\gamma}\big(h(t)\big) \le -q(t)z^{\gamma}\big(\sigma(t)\big) < 0.\end{aligned} \tag{8}$$

Substituting s for t, and integrating (8) from $t \in [t_1, \infty)_{\mathbb{T}}$ to ∞, we obtain

$$r_1(t)\big(\big(r_2(t)\big(z^{\Delta}(t)\big)^{\gamma_2}\big)^{\Delta}\big)^{\gamma_1} \ge \int_t^{\infty} q(s)z^{\gamma}\big(\sigma(s)\big)\Delta s \ge z^{\gamma}\big(\sigma(t)\big)\int_t^{\infty} q(s)\Delta s.$$

As $r_1(t)((r_2(t)(z^{\Delta}(t))^{\gamma_2})^{\Delta})^{\gamma_1}$ is strictly decreasing on $[t_1, \infty)_{\mathbb{T}}$, we have, for $t \in [\sigma(t_1), \infty)_{\mathbb{T}}$,

$$\begin{aligned} r_2(t)\big(z^{\Delta}(t)\big)^{\gamma_2} &= r_2(t_1)\big(z^{\Delta}(t_1)\big)^{\gamma_2} + \int_{t_1}^{t} \frac{r_1^{1/\gamma_1}(s)(r_2(s)(z^{\Delta}(s))^{\gamma_2})^{\Delta}}{r_1^{1/\gamma_1}(s)}\Delta s \\ &\ge r_1^{1/\gamma_1}(t)\big(r_2(t)\big(z^{\Delta}(t)\big)^{\gamma_2}\big)^{\Delta}\int_{t_1}^{t}\frac{1}{r_1^{1/\gamma_1}(s)}\Delta s \\ &\ge \delta(t)\bigg(z^{\gamma}\big(\sigma(t)\big)\int_t^{\infty} q(s)\Delta s\bigg)^{1/\gamma_1} = \delta(t)z^{\gamma_2}\big(\sigma(t)\big)\bigg(\int_t^{\infty} q(s)\Delta s\bigg)^{1/\gamma_1}.\end{aligned}$$

Hence, when $0 < \gamma < 1$, we have

$$\frac{z^{\Delta}(t)}{z^{\sigma}(t)} \ge \bigg(\frac{\delta(t)}{r_2(t)}\bigg)^{1/\gamma_2}\bigg(\int_t^{\infty} q(s)\Delta s\bigg)^{1/\gamma}, \quad t \in [t_1, \infty)_{\mathbb{T}},$$

which implies that

$$\bigg(\frac{z^{\Delta}(t)}{z^{\sigma}(t)}\bigg)^{1-\gamma} \ge \alpha(t), \quad t \in [t_1, \infty)_{\mathbb{T}}.$$

Lemma 2.4 is proved. □

Lemma 2.5 *For $\gamma \ge 1$, assume that $x(t)$ is an eventually positive solution of* (1), *and $z^{\Delta}(t)$ is eventually negative. If it satisfies*

$$\int_{t_0}^{\infty} q(t)\Delta t = \infty, \tag{9}$$

then $\lim_{t\to\infty} x(t) = 0$.

Proof Suppose that $x(t)$ is an eventually positive solution of (1) and $z^{\Delta}(t)$ is eventually negative. By the proof of Lemma 2.3, we see that $z(t)$ is eventually positive. Then there exists $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $x(t) > 0$, $x(g(t)) > 0$, $x(h(t)) > 0$, $z(t) > 0$, and $z^{\Delta}(t) < 0$ for $t \in [t_1, \infty)_{\mathbb{T}}$. By Lemma 2.2 we have

$$\big(r_2(t)\big(z^{\Delta}(t)\big)^{\gamma_2}\big)^{\Delta} > 0, \quad t \in [t_1, \infty)_{\mathbb{T}}.$$

By $z^{\Delta}(t) < 0$, we claim that there exists $b \geq 0$ such that $\lim_{t\to\infty} z(t) = b$. Assume not; then there exists $t_2 \in [t_1,\infty)_{\mathbb{T}}$ such that $z(t) < 0$ for $t \in [t_2,\infty)_{\mathbb{T}}$. It will cause a similar contradiction as in the proof of Lemma 2.2. Then assuming $b > 0$, by (8) and $z(\sigma(t)), z(g(t)) > b$, we obtain

$$\bigl(r_1(t)\bigl(\bigl(r_2(t)\bigl(z^{\Delta}(t)\bigr)^{\gamma_2}\bigr)^{\Delta}\bigr)^{\gamma_1}\bigr)^{\Delta} \leq -q(t)z^{\gamma}\bigl(\sigma(t)\bigr) < -b^{\gamma}q(t). \tag{10}$$

Letting $\nu(t) = r_1(t)((r_2(t)(z^{\Delta}(t))^{\gamma_2})^{\Delta})^{\gamma_1}$, $t \in [t_1,\infty)_{\mathbb{T}}$, we have $\nu(t) > 0$, and

$$\nu^{\Delta}(t) < -b^{\gamma}q(t), \quad t \in [t_1,\infty)_{\mathbb{T}}. \tag{11}$$

Substituting s for t, and integrating (11) from t_1 to $t \in [\sigma(t_1),\infty)_{\mathbb{T}}$, we obtain

$$\nu(t) < \nu(t_1) - b^{\gamma}\int_{t_1}^{t} q(s)\Delta s.$$

By (9), there exists a sufficiently large $t_2 \in [t_1,\infty)_{\mathbb{T}}$ such that $\nu(t) < 0$, $t \in [t_2,\infty)_{\mathbb{T}}$, which contradicts $\nu(t) > 0$. So $b = 0$, and Lemma 2.5 is proved. □

Lemma 2.6 *Assume that $x(t)$ is an eventually positive solution of* (1), *and there exists $t_1 \in [t_0,\infty)_{\mathbb{T}}$ such that $x(t) > 0$, $x(g(t)) > 0$, $x(h(t)) > 0$, $z(t) > 0$, and $z^{\Delta}(t) > 0$ for $t \in [t_1,\infty)_{\mathbb{T}}$. For $t \in [t_1,\infty)_{\mathbb{T}}$, define*

$$u(t) = A(t)\frac{r_1(t)((r_2(t)(z^{\Delta}(t))^{\gamma_2})^{\Delta})^{\gamma_1}}{z^{\gamma}(t)} + B(t), \tag{12}$$

where $A \in C^1_{\mathrm{rd}}(\mathbb{T},(0,\infty))$, $B \in C^1_{\mathrm{rd}}(\mathbb{T},\mathbb{R})$. Then $u(t)$ satisfies

$$u^{\Delta}(t) + A(t)q(t) - B^{\Delta}(t) - \Phi_0(t) \leq 0, \tag{13}$$

where

$$\Phi_0(t) = \begin{cases} A^{\Delta}(t)(\frac{u(t)-B(t)}{A(t)})^{\sigma} - \gamma A(t)\alpha(t)(\frac{\delta(t)}{r_2(t)})^{\gamma_1}[(\frac{u(t)-B(t)}{A(t)})^{\sigma}]^2, & 0 < \gamma < 1, \\ A^{\Delta}(t)(\frac{u(t)-B(t)}{A(t)})^{\sigma} - \gamma A(t)(\frac{\delta(t)}{r_2(t)})^{1/\gamma_2}[(\frac{u(t)-B(t)}{A(t)})^{\sigma}]^{(1+\gamma)/\gamma}, & \gamma \geq 1. \end{cases}$$

Proof Since $x(t)$ is an eventually positive solution of (1), and there exists $t_1 \in [t_0,\infty)_{\mathbb{T}}$ such that $x(t) > 0$, $x(g(t)) > 0$, $x(h(t)) > 0$, $z(t) > 0$, and $z^{\Delta}(t) > 0$ for $t \in [t_1,\infty)_{\mathbb{T}}$, Lemmas 2.2 and 2.4 hold. Let $u(t)$ be defined by (12). Then, differentiating (12) and using (1), it follows that

$$\begin{aligned} u^{\Delta}(t) = {} & \frac{A(t)}{z^{\gamma}(t)}\bigl(r_1(t)\bigl(\bigl(r_2(t)\bigl(z^{\Delta}(t)\bigr)^{\gamma_2}\bigr)^{\Delta}\bigr)^{\gamma_1}\bigr)^{\Delta} \\ & + \left(\frac{A(t)}{z^{\gamma}(t)}\right)^{\Delta}\bigl(r_1(t)\bigl(\bigl(r_2(t)\bigl(z^{\Delta}(t)\bigr)^{\gamma_2}\bigr)^{\Delta}\bigr)^{\gamma_1}\bigr)^{\sigma} + B^{\Delta}(t) \\ = {} & -\frac{A(t)}{z^{\gamma}(t)}\cdot f\bigl(t,x\bigl(h(t)\bigr)\bigr) + B^{\Delta}(t) \\ & + \frac{A^{\Delta}(t)z^{\gamma}(t) - A(t)(z^{\gamma}(t))^{\Delta}}{z^{\gamma}(t)z^{\gamma}(\sigma(t))}\bigl(r_1(t)\bigl(\bigl(r_2(t)\bigl(z^{\Delta}(t)\bigr)^{\gamma_2}\bigr)^{\Delta}\bigr)^{\gamma_1}\bigr)^{\sigma}. \end{aligned}$$

Using the fact that

$$f\big(t,x\big(h(t)\big)\big) \geq q(t)x^{\gamma}\big(h(t)\big) \geq q(t)z^{\gamma}\big(h(t)\big) \geq q(t)z^{\gamma}(t),$$

we obtain

$$\begin{aligned} u^{\Delta}(t) &\leq -A(t)q(t) + B^{\Delta}(t) + A^{\Delta}(t)\left(\frac{u(t)-B(t)}{A(t)}\right)^{\sigma} \\ &\quad - A(t)\frac{(z^{\gamma}(t))^{\Delta}}{z^{\gamma}(t)}\left(\frac{u(t)-B(t)}{A(t)}\right)^{\sigma}. \end{aligned} \tag{14}$$

When $0 < \gamma < 1$, using the Pötzsche chain rule (see [5]), we have

$$\big(z^{\gamma}(t)\big)^{\Delta} = \gamma \int_0^1 \big(z(t) + h\mu(t)z^{\Delta}(t)\big)^{\gamma-1}\,dh \cdot z^{\Delta}(t) \geq \gamma\big(z^{\sigma}(t)\big)^{\gamma-1}z^{\Delta}(t),$$

and it follows that

$$\frac{(z^{\gamma}(t))^{\Delta}}{z^{\gamma}(t)} \geq \frac{\gamma(z^{\sigma}(t))^{\gamma-1}z^{\Delta}(t)}{z^{\gamma}(t)} = \gamma\frac{z^{\Delta}(t)}{z^{\sigma}(t)}\left(\frac{z^{\sigma}(t)}{z(t)}\right)^{\gamma}.$$

By Lemmas 2.2 and 2.4, for $t \in [t_1,\infty)_{\mathbb{T}}$, we obtain

$$\begin{aligned} \frac{z^{\Delta}(t)}{z^{\sigma}(t)} &= \frac{1}{r_2^{\gamma_1}(t)}\frac{r_2^{\gamma_1}(t)(z^{\Delta}(t))^{\gamma}}{(z^{\sigma}(t))^{\gamma}}\left(\frac{z^{\Delta}(t)}{z^{\sigma}(t)}\right)^{1-\gamma} \\ &\geq \alpha(t)\left(\frac{\delta(t)}{r_2(t)}\right)^{\gamma_1}\frac{r_1(t)((r_2(t)(z^{\Delta}(t))^{\gamma_2})^{\Delta})^{\gamma_1}}{(z^{\gamma}(t))^{\sigma}} \\ &\geq \alpha(t)\left(\frac{\delta(t)}{r_2(t)}\right)^{\gamma_1}\frac{(r_1(t)((r_2(t)(z^{\Delta}(t))^{\gamma_2})^{\Delta})^{\gamma_1})^{\sigma}}{(z^{\gamma}(t))^{\sigma}} \\ &= \alpha(t)\left(\frac{\delta(t)}{r_2(t)}\right)^{\gamma_1}\left(\frac{u(t)-B(t)}{A(t)}\right)^{\sigma} \end{aligned}$$

and

$$\frac{z^{\sigma}(t)}{z(t)} \geq 1.$$

So (14) becomes

$$\begin{aligned} u^{\Delta}(t) &\leq -A(t)q(t) + B^{\Delta}(t) + A^{\Delta}(t)\left(\frac{u(t)-B(t)}{A(t)}\right)^{\sigma} \\ &\quad - \gamma A(t)\alpha(t)\left(\frac{\delta(t)}{r_2(t)}\right)^{\gamma_1}\left[\left(\frac{u(t)-B(t)}{A(t)}\right)^{\sigma}\right]^2. \end{aligned} \tag{15}$$

When $\gamma \geq 1$, we have

$$\big(z^{\gamma}(t)\big)^{\Delta} = \gamma \int_0^1 \big(z(t) + h\mu(t)z^{\Delta}(t)\big)^{\gamma-1}\,dh \cdot z^{\Delta}(t) \geq \gamma z^{\gamma-1}(t)z^{\Delta}(t),$$

and it follows that

$$\frac{(z^{\gamma}(t))^{\Delta}}{z^{\gamma}(t)} \geq \frac{\gamma z^{\gamma-1}(t)z^{\Delta}(t)}{z^{\gamma}(t)} = \frac{\gamma z^{\Delta}(t)}{z(t)}.$$

By Lemmas 2.2 and 2.4, for $t \in [t_1, \infty)_{\mathbb{T}}$, we obtain

$$\begin{aligned}
\left(\frac{z^{\Delta}(t)}{z(t)}\right)^{\gamma} &= \frac{1}{r_2^{\gamma_1}(t)} \frac{r_2^{\gamma_1}(t)(z^{\Delta}(t))^{\gamma}}{z^{\gamma}(t)} \\
&\geq \left(\frac{\delta(t)}{r_2(t)}\right)^{\gamma_1} \frac{r_1(t)((r_2(t)(z^{\Delta}(t))^{\gamma_2})^{\Delta})^{\gamma_1}}{z^{\gamma}(t)} \\
&\geq \left(\frac{\delta(t)}{r_2(t)}\right)^{\gamma_1} \frac{(r_1(t)((r_2(t)(z^{\Delta}(t))^{\gamma_2})^{\Delta})^{\gamma_1})^{\sigma}}{(z^{\gamma}(t))^{\sigma}} \\
&= \left(\frac{\delta(t)}{r_2(t)}\right)^{\gamma_1} \left(\frac{u(t) - B(t)}{A(t)}\right)^{\sigma},
\end{aligned}$$

which implies that

$$\frac{z^{\Delta}(t)}{z(t)} \geq \left(\frac{\delta(t)}{r_2(t)}\right)^{1/\gamma_2} \left[\left(\frac{u(t) - B(t)}{A(t)}\right)^{\sigma}\right]^{1/\gamma}.$$

So (14) becomes

$$\begin{aligned}
u^{\Delta}(t) \leq{}& -A(t)q(t) + B^{\Delta}(t) + A^{\Delta}(t)\left(\frac{u(t) - B(t)}{A(t)}\right)^{\sigma} \\
&- \gamma A(t)\left(\frac{\delta(t)}{r_2(t)}\right)^{1/\gamma_2} \left[\left(\frac{u(t) - B(t)}{A(t)}\right)^{\sigma}\right]^{(1+\gamma)/\gamma}.
\end{aligned} \tag{16}$$

By (15) and (16), (13) holds. Lemma 2.6 is proved. □

3 Main results

In this section, we establish oscillation criteria of (1) by a generalized Riccati transformation. Firstly, we give some definitions as follows.

Let $D_0 = \{s \in \mathbb{T} : s \geq 0\}$ and $D = \{(t,s) \in \mathbb{T}^2 : t \geq s \geq 0\}$. For any function $f(t,s)$: $\mathbb{T}^2 \to \mathbb{R}$, denote by f_2^{Δ} the partial derivative of f with respect to s. Define

$$\begin{aligned}
(\mathscr{A}, \mathscr{B}) &= \bigl\{(A,B) : A(s) \in C_{\mathrm{rd}}^1\bigl(D_0, (0,\infty)\bigr), B(s) \in C_{\mathrm{rd}}^1(D_0, \mathbb{R}), s \in D_0\bigr\}; \\
\mathscr{H} &= \bigl\{H(t,s) \in C^1\bigl(D, [0,\infty)\bigr) : H(t,t) = 0, H(t,s) > 0, H_2^{\Delta}(t,s) \leq 0, t > s \geq 0\bigr\}.
\end{aligned}$$

These function classes will be used in the sequel. Now, we give our first theorem.

Theorem 3.1 *Assume that there exist $(A,B) \in (\mathscr{A}, \mathscr{B})$ and $H \in \mathscr{H}$ such that, for any $t_1 \in [t_0, \infty)_{\mathbb{T}}$,*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_1)} \int_{t_1}^{t} \bigl[H(t,s)\bigl(A(s)q(s) - B^{\Delta}(s)\bigr) - H_2^{\Delta}(t,s)B^{\sigma}(s) - \Phi_1(s)\bigr]\Delta s = \infty, \tag{17}$$

where

$$\Phi_1(s) = \begin{cases} (\frac{r_2(s)}{\delta(s)})^{\gamma_1} \frac{(H_2^\Delta(t,s)A^\sigma(s)+H(t,s)A^\Delta(s))^2}{4\gamma H(t,s)A(s)\alpha(s)}, & 0 < \gamma < 1, \\ (\frac{r_2(s)}{\delta(s)})^{\gamma_1} \frac{1}{(H(t,s)A(s))^\gamma} (\frac{H_2^\Delta(t,s)A^\sigma(s)+H(t,s)A^\Delta(s)}{1+\gamma})^{1+\gamma}, & \gamma \geq 1. \end{cases}$$

Then (1) *is oscillatory or* $\lim_{t\to\infty} x(t)$ *exists.*

Proof Assume that (1) is not oscillatory. Without loss of generality, we may suppose that $x(t)$ is an eventually positive solution of (1). Then by Lemma 2.3, we have $z(t)$ is eventually positive or $\lim_{t\to\infty} x(t) = 0$.

If $\lim_{t\to\infty} x(t) = 0$, the theorem is proved. While $z(t)$ is eventually positive, it follows that there exists $T \in [t_0, \infty)_\mathbb{T}$ such that $z(t) > 0$ for $t \in [T, \infty)_\mathbb{T}$. By Lemma 2.2, there exists $t_1 \in [T, \infty)_\mathbb{T}$ such that either $z^\Delta(t) > 0$ or $z^\Delta(t) < 0$ holds for $t \in [t_1, \infty)_\mathbb{T}$. Assume that $z^\Delta(t) > 0$, $t \in [t_1, \infty)_\mathbb{T}$. Let $u(t)$ be defined by (12). Then by Lemma 2.6, (13) holds.

Multiplying (13), where t is replaced by s, by H, and integrating it with respect to s from t_1 to t with $t \in [\sigma(t_1), \infty)_\mathbb{T}$, we obtain

$$\begin{aligned} &\int_{t_1}^t H(t,s)\big(A(s)q(s) - B^\Delta(s)\big)\Delta s \\ &\quad \leq -\int_{t_1}^t H(t,s)u^\Delta(s)\Delta s + \int_{t_1}^t H(t,s)\Phi_0(s)\Delta s. \end{aligned}$$

Noting that $H(t,t) = 0$, by the integration by parts formula we have

$$\begin{aligned} &\int_{t_1}^t H(t,s)\big(A(s)q(s) - B^\Delta(s)\big)\Delta s \\ &\quad \leq H(t,t_1)u(t_1) + \int_{t_1}^t \big(H_2^\Delta(t,s)u^\sigma(s) + H(t,s)\Phi_0(s)\big)\Delta s \\ &\quad = H(t,t_1)u(t_1) + \int_{t_1}^t H_2^\Delta(t,s)B^\sigma(s)\Delta s \\ &\qquad + \int_{t_1}^t \left(H_2^\Delta(t,s)A^\sigma(s)\left(\frac{u(s) - B(s)}{A(s)}\right)^\sigma + H(t,s)\Phi_0(s)\right)\Delta s. \end{aligned} \tag{18}$$

When $0 < \gamma < 1$, we have

$$\begin{aligned} &H_2^\Delta(t,s)A^\sigma(s)\left(\frac{u(s) - B(s)}{A(s)}\right)^\sigma + H(t,s)\Phi_0(s) \\ &\quad = \big(H_2^\Delta(t,s)A^\sigma(s) + H(t,s)A^\Delta(s)\big)\left(\frac{u(s) - B(s)}{A(s)}\right)^\sigma \\ &\qquad - \gamma H(t,s)A(s)\alpha(s)\left(\frac{\delta(s)}{r_2(s)}\right)^{\gamma_1}\left[\left(\frac{u(s) - B(s)}{A(s)}\right)^\sigma\right]^2 \\ &\quad = \left(\frac{r_2(s)}{\delta(s)}\right)^{\gamma_1} \frac{(H_2^\Delta(t,s)A^\sigma(s) + H(t,s)A^\Delta(s))^2}{4\gamma H(t,s)A(s)\alpha(s)} \\ &\qquad - \gamma H(t,s)A(s)\alpha(s)\left(\frac{\delta(s)}{r_2(s)}\right)^{\gamma_1}\left[\left(\frac{u(s) - B(s)}{A(s)}\right)^\sigma\right. \end{aligned}$$

$$
\begin{aligned}
&\quad - \left(\frac{r_2(s)}{\delta(s)}\right)^{\gamma_1} \frac{H_2^\Delta(t,s)A^\sigma(s) + H(t,s)A^\Delta(s)}{2\gamma H(t,s)A(s)\alpha(s)}\Bigg]^2 \\
&\le \left(\frac{r_2(s)}{\delta(s)}\right)^{\gamma_1} \frac{(H_2^\Delta(t,s)A^\sigma(s) + H(t,s)A^\Delta(s))^2}{4\gamma H(t,s)A(s)\alpha(s)}.
\end{aligned}
$$

When $\gamma \ge 1$, we have

$$
\begin{aligned}
&H_2^\Delta(t,s)A^\sigma(s)\left(\frac{u(s)-B(s)}{A(s)}\right)^\sigma + H(t,s)\Phi_0(s) \\
&\quad = \big(H_2^\Delta(t,s)A^\sigma(s) + H(t,s)A^\Delta(s)\big)\left(\frac{u(s)-B(s)}{A(s)}\right)^\sigma \\
&\qquad - \gamma H(t,s)A(s)\left(\frac{\delta(s)}{r_2(s)}\right)^{1/\gamma_2}\left[\left(\frac{u(s)-B(s)}{A(s)}\right)^\sigma\right]^{(1+\gamma)/\gamma}.
\end{aligned}
$$

Using the inequality

$$
\lambda a b^{\lambda-1} - a^\lambda \le (\lambda-1)b^\lambda,
$$

let $\lambda = \frac{1+\gamma}{\gamma}$, and

$$
\begin{aligned}
a^\lambda &= a^{(1+\gamma)/\gamma} = \gamma H(t,s)A(s)\left(\frac{\delta(s)}{r_2(s)}\right)^{1/\gamma_2}\left[\left(\frac{u(s)-B(s)}{A(s)}\right)^\sigma\right]^{(1+\gamma)/\gamma}, \\
b^{\lambda-1} &= b^{1/\gamma} = \frac{\gamma}{1+\gamma}\left(\frac{r_2(s)}{\delta(s)}\right)^{\gamma_1/(1+\gamma)} \frac{H_2^\Delta(t,s)A^\sigma(s) + H(t,s)A^\Delta(s)}{(\gamma H(t,s)A(s))^{\gamma/(1+\gamma)}},
\end{aligned}
$$

then we have

$$
\begin{aligned}
&H_2^\Delta(t,s)A^\sigma(s)\left(\frac{u(s)-B(s)}{A(s)}\right)^\sigma + H(t,s)\Phi_0(s) \\
&\quad \le \left(\frac{r_2(s)}{\delta(s)}\right)^{\gamma_1} \frac{1}{(H(t,s)A(s))^\gamma}\left(\frac{H_2^\Delta(t,s)A^\sigma(s) + H(t,s)A^\Delta(s)}{1+\gamma}\right)^{1+\gamma}.
\end{aligned}
$$

Therefore, for all $\gamma > 0$, by (18) we have

$$
\begin{aligned}
&\int_{t_1}^t H(t,s)\big(A(s)q(s) - B^\Delta(s)\big)\Delta s \\
&\quad \le H(t,t_1)u(t_1) + \int_{t_1}^t H_2^\Delta(t,s)B^\sigma(s)\Delta s + \int_{t_1}^t \Phi_1(s)\Delta s,
\end{aligned}
$$

which implies that

$$
\int_{t_1}^t \big[H(t,s)\big(A(s)q(s) - B^\Delta(s)\big) - H_2^\Delta(t,s)B^\sigma(s) - \Phi_1(s)\big]\Delta s \le H(t,t_1)u(t_1).
$$

Hence,

$$
\frac{1}{H(t,t_1)}\int_{t_1}^t \big[H(t,s)\big(A(s)q(s) - B^\Delta(s)\big) - H_2^\Delta(t,s)B^\sigma(s) - \Phi_1(s)\big]\Delta s \le u(t_1) < \infty,
$$

which contradicts (17). So $z^{\Delta}(t) < 0$, $t \in [t_1, \infty)_{\mathbb{T}}$, and it is clear that $\lim_{t\to\infty} z(t)$ exists. By Lemma 2.1 we see that $\lim_{t\to\infty} x(t)$ exists. The proof is completed. □

When $\gamma \geq 1$, if (9) holds, we have the following corollary on the basis of Lemma 2.5 and Theorem 3.1.

Corollary 3.2 *When* $\gamma \geq 1$, *assume that* (9) *holds and there exist* $(A, B) \in (\mathscr{A}, \mathscr{B})$ *and* $H \in \mathscr{H}$ *such that, for any* $t_1 \in [t_0, \infty)_{\mathbb{T}}$,

$$\limsup_{t\to\infty} \frac{1}{H(t,t_1)} \int_{t_1}^{t} \bigl[H(t,s)\bigl(A(s)q(s) - B^{\Delta}(s)\bigr) - H_2^{\Delta}(t,s)B^{\sigma}(s) - \Phi_1(s)\bigr]\Delta s = \infty, \tag{19}$$

where

$$\Phi_1(s) = \left(\frac{r_2(s)}{\delta(s)}\right)^{\gamma_1} \frac{1}{(H(t,s)A(s))^{\gamma}} \left(\frac{H_2^{\Delta}(t,s)A^{\sigma}(s) + H(t,s)A^{\Delta}(s)}{1+\gamma}\right)^{1+\gamma}.$$

Then (1) *is oscillatory or* $\lim_{t\to\infty} x(t) = 0$.

Remark 3.3 In Corollary 3.2, letting $(A, B) = (1, 0)$, we can simplify (19) as

$$\limsup_{t\to\infty} \frac{1}{H(t,t_1)} \int_{t_1}^{t} \left[H(t,s)q(s) - \left(\frac{r_2(s)}{\delta(s)}\right)^{\gamma_1} \frac{1}{H^{\gamma}(t,s)} \left(\frac{H_2^{\Delta}(t,s)}{1+\gamma}\right)^{1+\gamma}\right]\Delta s = \infty.$$

When $B = 0$, (12) is simplified as

$$u(t) = A(t)\frac{r_1(t)((r_2(t)(z^{\Delta}(t))^{\gamma_2})^{\Delta})^{\gamma_1}}{z^{\gamma}(t)}, \quad t \in [t_1, \infty)_{\mathbb{T}}. \tag{20}$$

Now we have the following theorem.

Theorem 3.4 *Assume that there exists* $A \in C^1_{\mathrm{rd}}(D_0, (0,\infty))$ *such that, for any* $t_1 \in [t_0, \infty)_{\mathbb{T}}$,

$$\limsup_{t\to\infty} \int_{t_1}^{t} \bigl[A(s)q(s) - \Phi_2(s)\bigr]\Delta s = \infty, \tag{21}$$

where

$$\Phi_2(s) = \begin{cases} (\frac{r_2(s)}{\delta(s)})^{\gamma_1} \frac{(A^{\Delta}(s))^2}{4\gamma A(s)\alpha(s)}, & 0 < \gamma < 1, \\ (\frac{r_2(s)}{\delta(s)})^{\gamma_1} \frac{1}{A^{\gamma}(s)} (\frac{A^{\Delta}(s)}{1+\gamma})^{1+\gamma}, & \gamma \geq 1. \end{cases}$$

Then (1) *is oscillatory or* $\lim_{t\to\infty} x(t)$ *exists.*

Proof Assume that (1) is not oscillatory. Without loss of generality, we may suppose that $x(t)$ is an eventually positive solution of (1). Similarly as in the proof of Theorem 3.1, we have $z(t)$ is eventually positive or $\lim_{t\to\infty} x(t) = 0$.

If $\lim_{t\to\infty} x(t) = 0$, the theorem is proved. If $z(t)$ is eventually positive, there exists $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $z(t) > 0$, and either $z^{\Delta}(t) > 0$ or $z^{\Delta}(t) < 0$ holds for $t \in [t_1, \infty)_{\mathbb{T}}$

by Lemma 2.2. Assume that $z^{\Delta}(t) > 0$, $t \in [t_1, \infty)_{\mathbb{T}}$. Let $u(t)$ be defined by (20). Then by Lemma 2.6, we have

$$u^{\Delta}(t) + A(t)q(t) - \Phi_0(t) \le 0,$$

where $\Phi_0(t)$ is simplified as

$$\Phi_0(t) = \begin{cases} A^{\Delta}(t)(\frac{u(t)}{A(t)})^{\sigma} - \gamma A(t)\alpha(t)(\frac{\delta(t)}{r_2(t)})^{\gamma_1}[(\frac{u(t)}{A(t)})^{\sigma}]^2, & 0 < \gamma < 1, \\ A^{\Delta}(t)(\frac{u(t)}{A(t)})^{\sigma} - \gamma A(t)(\frac{\delta(t)}{r_2(t)})^{1/\gamma_2}[(\frac{u(t)}{A(t)})^{\sigma}]^{(1+\gamma)/\gamma}, & \gamma \ge 1. \end{cases}$$

When $0 < \gamma < 1$, we have

$$\begin{aligned} u^{\Delta}(t) &\le -A(t)q(t) + A^{\Delta}(t)\left(\frac{u(t)}{A(t)}\right)^{\sigma} - \gamma A(t)\alpha(t)\left(\frac{\delta(t)}{r_2(t)}\right)^{\gamma_1}\left[\left(\frac{u(t)}{A(t)}\right)^{\sigma}\right]^2 \\ &= -A(t)q(t) + \left(\frac{r_2(t)}{\delta(t)}\right)^{\gamma_1} \frac{(A^{\Delta}(t))^2}{4\gamma A(t)\alpha(t)} \\ &\quad - \gamma A(t)\alpha(t)\left(\frac{\delta(t)}{r_2(t)}\right)^{\gamma_1}\left[\left(\frac{u(t)}{A(t)}\right)^{\sigma} - \left(\frac{r_2(t)}{\delta(t)}\right)^{\gamma_1}\frac{A^{\Delta}(t)}{2\gamma A(t)\alpha(t)}\right]^2 \\ &\le -A(t)q(t) + \left(\frac{r_2(t)}{\delta(t)}\right)^{\gamma_1} \frac{(A^{\Delta}(t))^2}{4\gamma A(t)\alpha(t)}. \end{aligned}$$

When $\gamma \ge 1$, we have

$$u^{\Delta}(t) \le -A(t)q(t) + A^{\Delta}(t)\left(\frac{u(t)}{A(t)}\right)^{\sigma} - \gamma A(t)\left(\frac{\delta(t)}{r_2(t)}\right)^{1/\gamma_2}\left[\left(\frac{u(t)}{A(t)}\right)^{\sigma}\right]^{(1+\gamma)/\gamma}.$$

Using the inequality

$$\lambda a b^{\lambda-1} - a^{\lambda} \le (\lambda - 1)b^{\lambda},$$

let $\lambda = \frac{1+\gamma}{\gamma}$, and

$$\begin{aligned} a^{\lambda} &= a^{(1+\gamma)/\gamma} = \gamma A(t)\left(\frac{\delta(t)}{r_2(t)}\right)^{1/\gamma_2}\left[\left(\frac{u(t)}{A(t)}\right)^{\sigma}\right]^{(1+\gamma)/\gamma}, \\ b^{\lambda-1} &= b^{1/\gamma} = \frac{\gamma}{1+\gamma}\left(\frac{r_2(t)}{\delta(t)}\right)^{\gamma_1/(1+\gamma)} \frac{A^{\Delta}(t)}{(\gamma A(t))^{\gamma/(1+\gamma)}}, \end{aligned}$$

then we have

$$u^{\Delta}(t) \le -A(t)q(t) + \left(\frac{r_2(t)}{\delta(t)}\right)^{\gamma_1} \frac{1}{A^{\gamma}(t)}\left(\frac{A^{\Delta}(t)}{1+\gamma}\right)^{1+\gamma}.$$

Therefore, for all $\gamma > 0$, we always have

$$u^{\Delta}(t) \le -A(t)q(t) + \Phi_2(t),$$

which implies that

$$A(t)q(t) - \Phi_2(t) \le -u^{\Delta}(t). \tag{22}$$

Letting t be replaced by s, and integrating (22) with respect to s from t_1 to $t \in [\sigma(t_1), \infty)_{\mathbb{T}}$, we obtain

$$\int_{t_1}^{t} \bigl[A(s)q(s) - \Phi_2(s)\bigr]\Delta s \leq -\int_{t_1}^{t} u^{\Delta}(s)\Delta s = u(t_1) - u(t) < u(t_1) < \infty,$$

which is a contradiction of (21). So $z^{\Delta}(t) < 0$, $t \in [t_1, \infty)_{\mathbb{T}}$, and as before, $\lim_{t\to\infty} z(t)$ and $\lim_{t\to\infty} x(t)$ exist. The proof is completed. □

When $\gamma \geq 1$, if (9) holds, from Lemma 2.5 and Theorem 3.4 we have the following result.

Corollary 3.5 *When* $\gamma \geq 1$, *assume that* (9) *holds and there exists* $A \in C_{\mathrm{rd}}^{1}(D_0, (0,\infty))$ *such that, for any* $t_1 \in [t_0, \infty)_{\mathbb{T}}$,

$$\limsup_{t\to\infty} \int_{t_1}^{t} \biggl[A(s)q(s) - \biggl(\frac{r_2(s)}{\delta(s)}\biggr)^{\gamma_1} \frac{1}{A^{\gamma}(s)} \biggl(\frac{A^{\Delta}(s)}{1+\gamma}\biggr)^{1+\gamma}\biggr]\Delta s = \infty. \tag{23}$$

Then (1) *is oscillatory or* $\lim_{t\to\infty} x(t) = 0$.

Remark 3.6 It is not difficult to satisfy the conditions in Corollary 3.5. Indeed, letting $A = 1$, by (9) we have (23). The condition (23) can be deleted in Corollary 3.5. Therefore, when $\gamma \geq 1$, assume that (9) holds, then it follows that (1) is oscillatory or $\lim_{t\to\infty} x(t) = 0$.

Remark 3.7 Take $r_1(t) = 1/a_2(t)$, $r_2(t) = 1/a_1(t)$, $\gamma_1 = \alpha_2$, $\gamma_2 = \alpha_1$, $\gamma = 1$, $p(t) = 0$, $h(t) = t$, and $f(t,x) = q(t)f_0(x)$, where f_0 is equivalent to f in Yu and Wang [15]. It is obvious that the conclusions in this paper extend the ones in [15]. Meanwhile, the proofs and results above may provide some enlightenment to the study of oscillation of higher-order nonlinear dynamic equations with nonpositive neutral coefficients on time scales.

4 Examples

In this section, the application of our oscillation criteria will be shown in two examples. Now we give the first example to demonstrate Theorem 3.1 (or Corollary 3.2).

Example 4.1 Let $\mathbb{T} = \bigcup_{n=1}^{\infty}[2n-1, 2n]$. Consider the equation

$$\biggl(t\biggl(\biggl(\frac{1}{t}\biggl(\biggl(x(t) - \frac{t-1}{2t}x(t-2)\biggr)^{\Delta}\biggr)^{1/3}\biggr)^{\Delta}\biggr)^{5}\biggr)^{\Delta} + \frac{2+\sin t}{t}x^{5/3}\bigl(h(t)\bigr) = 0, \tag{24}$$

where $r_1(t) = t$, $r_2(t) = 1/t$, $p(t) = (t-1)/2t$, $g(t) = t-2$, $\gamma_1 = 5$, $\gamma_2 = 1/3$, $\gamma = 5/3$, $h(t) \geq t$, and $t_0 = 1$. By (C3) we have $p_0 = 1/2$, and by (C6) we take $q(t) = 1/t$. Since

$$\int_{t_0}^{\infty} \frac{1}{r_1^{1/\gamma_1}(t)}\Delta t = \int_{1}^{\infty} \frac{1}{t^{1/5}}\Delta t = \infty, \qquad \int_{t_0}^{\infty} \frac{1}{r_2^{1/\gamma_2}(t)}\Delta t = \int_{1}^{\infty} t^3\Delta t = \infty$$

and

$$\int_{t_0}^{\infty} q(t)\Delta t = \int_{1}^{\infty} \frac{1}{t}\Delta t = \infty,$$

it is obvious that the coefficients of (24) satisfy (C1)-(C6) and (9). Letting $H(t,s) = (t-s)^2$ and $(A,B) = (s,0)$, we have

$$\delta(t) = \int_{t_1}^{t} \frac{\Delta s}{r_1^{1/\gamma_1}(s)} = \int_{t_1}^{t} \frac{\Delta s}{s^{1/5}} = O\bigl(t^{4/5}\bigr)$$

and

$$\begin{aligned}\Phi_1(s) &= \left(\frac{r_2(s)}{\delta(s)}\right)^{\gamma_1} \frac{1}{(H(t,s)A(s))^{\gamma}} \left(\frac{H_2^{\Delta}(t,s)A^{\sigma}(s) + H(t,s)A^{\Delta}(s)}{1+\gamma}\right)^{1+\gamma} \\ &= \left(\frac{s^{-1}}{O(s^{4/5})}\right)^{5} \frac{1}{((t-s)^2 s)^{5/3}} \left(\frac{O(s)\cdot O(s) + (t-s)^2}{8/3}\right)^{8/3} = O\bigl(s^{-26/3}\bigr).\end{aligned}$$

Hence,

$$\begin{aligned}&\limsup_{t\to\infty} \frac{1}{H(t,t_1)} \int_{t_1}^{t} \bigl[H(t,s)\bigl(A(s)q(s) - B^{\Delta}(s)\bigr) - H_2^{\Delta}(t,s)B^{\sigma}(s) - \Phi_1(s)\bigr]\Delta s \\ &\quad = \limsup_{t\to\infty} \frac{1}{(t-t_1)^2} \int_{t_1}^{t} \bigl[(t-s)^2 - O\bigl(s^{-26/3}\bigr)\bigr]\Delta s = \infty.\end{aligned}$$

That is, (19) holds. By Theorem 3.1 (or Corollary 3.2) we see that (24) is oscillatory or $\lim_{t\to\infty} x(t) = 0$.

The second example illustrates Theorem 3.4.

Example 4.2 Let $\mathbb{T} = \bigcup_{n=0}^{\infty}[3^n, 2\cdot 3^n]$. Consider the equation

$$\left(\frac{1}{t^2}\left(\left(\sqrt{t}\left(\left(x(t) - \frac{1}{t}x\left(\frac{t}{3}\right)\right)\right)^{\Delta}\right)^{5/3}\right)^{\Delta}\right)^{1/5}\Biggr)^{\Delta} + \frac{2+t^2}{t^2(1+t^2)} x^{1/3}\bigl(h(t)\bigr) = 0, \tag{25}$$

where $r_1(t) = 1/t^2$, $r_2(t) = \sqrt{t}$, $p(t) = 1/t$, $g(t) = t/3$, $\gamma_1 = 1/5$, $\gamma_2 = 5/3$, $\gamma = 1/3$, $h(t) \geq \sigma(t)$, and $t_0 = 1$. By (C3) we have $p_0 = 0$, and by (C6) we take $q(t) = 1/t^2$. Since

$$\int_{t_0}^{\infty} \frac{1}{r_1^{1/\gamma_1}(t)} \Delta t = \int_1^{\infty} t^{10}\,\Delta t = \infty, \qquad \int_{t_0}^{\infty} \frac{1}{r_2^{1/\gamma_2}(t)} \Delta t = \int_1^{\infty} \frac{1}{t^{3/10}} \Delta t = \infty$$

and

$$\int_{t_0}^{\infty} q(t)\Delta t = \int_1^{\infty} \frac{1}{t^2} \Delta t < \infty,$$

it is obvious that the coefficients of (25) satisfy (C1)-(C7). Then, letting $(A,B) = (s^2,0)$, we obtain

$$\begin{aligned}\delta(t) &= \int_{t_1}^{t} \frac{\Delta s}{r_1^{1/\gamma_1}(s)} = \int_{t_1}^{t} s^{10}\,\Delta s = O\bigl(t^{11}\bigr), \\ \alpha(t) &= \left(\frac{\delta(t)}{r_2(t)}\right)^{(1-\gamma)/\gamma_2} \left(2\int_t^{\infty} q(s)\Delta s\right)^{(1-\gamma)/\gamma} \\ &= \left(\frac{O(t^{11})}{\sqrt{t}}\right)^{2/5} \bigl(O\bigl(t^{-1}\bigr)\bigr)^2 = O\bigl(t^{11/5}\bigr),\end{aligned}$$

and

$$\Phi_2(s) = \left(\frac{r_2(s)}{\delta(s)}\right)^{\gamma_1} \frac{(A^{\Delta}(s))^2}{4\gamma A(s)\alpha(s)}$$
$$= \left(\frac{\sqrt{s}}{O(s^{11})}\right)^{1/5} \frac{O(s^2)}{4/3 \cdot s^2 \cdot O(s^{11/5})} = O\bigl(s^{-43/10}\bigr).$$

Therefore,

$$\limsup_{t\to\infty} \int_{t_1}^{t} \bigl[A(s)q(s) - \Phi_2(s)\bigr]\Delta s = \limsup_{t\to\infty} \int_{t_1}^{t} \bigl[1 - O\bigl(s^{-43/10}\bigr)\bigr]\Delta s = \infty.$$

That is, (21) holds. By Theorem 3.4 we see that (25) is oscillatory or $\lim_{t\to\infty} x(t)$ exists.

Competing interests
The author declares that he has no competing interests.

Acknowledgements
This project was supported by the NNSF of China (no. 11271379).

Received: 18 May 2015 Accepted: 6 September 2015 Published online: 18 September 2015

References
1. Hilger, S: Ein Maßkettenkalkül mit Anwendung auf Zentrumsmannigfaltigkeiten. Ph.D. thesis, Universität Würzburg (1988)
2. Hilger, S: Analysis on measure chain - a unified approach to continuous and discrete calculus. Results Math. **18**, 18-56 (1990)
3. Agarwal, RP, Bohner, M: Basic calculus on time scales and some of its applications. Results Math. **35**, 3-22 (1999)
4. Agarwal, RP, Bohner, M, O'Regan, D, Peterson, A: Dynamic equations on time scales: a survey. J. Comput. Appl. Math. **141**, 1-26 (2002)
5. Bohner, M, Peterson, A: Dynamic Equations on Time Scales: An Introduction with Applications. Birkhäuser, Boston (2001)
6. Bohner, M, Peterson, A (eds.): Advances in Dynamic Equations on Time Scales. Birkhäuser, Boston (2003)
7. Erbe, L, Hassan, TS, Peterson, A: Oscillation of third order nonlinear functional dynamic equations on time scales. Differ. Equ. Dyn. Syst. **18**(1), 199-227 (2010)
8. Hassan, TS: Oscillation of third order nonlinear delay dynamic equations on time scales. Math. Comput. Model. **49**, 1573-1586 (2009)
9. Li, Q, Wang, R, Chen, F, Li, TX: Oscillation of second-order nonlinear delay differential equations with nonpositive neutral coefficients. Adv. Differ. Equ. **2015**, Article ID 35 (2015)
10. Qiu, YC, Wang, QR: Kamenev-type oscillation criteria of second-order nonlinear dynamic equations on time scales. Discrete Dyn. Nat. Soc. **2013**, Article ID 315158 (2013). doi:10.1155/2013/315158
11. Qiu, YC, Wang, QR: Oscillation criteria of second-order dynamic equations with damping on time scales. Abstr. Appl. Anal. **2014**, Article ID 964239 (2014). doi:10.1155/2014/964239
12. Qiu, YC: Nonoscillatory solutions to third-order neutral dynamic equations on time scales. Adv. Differ. Equ. **2014**, Article ID 309 (2014)
13. Saker, SH, Agarwal, RP, O'Regan, D: Oscillation of second-order damped dynamic equations on time scales. J. Math. Anal. Appl. **330**(2), 1317-1337 (2007)
14. Şenel, MT: Kamenev-type oscillation criteria for the second-order nonlinear dynamic equations with damping on time scales. Abstr. Appl. Anal. **2012**, Article ID 253107 (2012). doi:10.1155/2012/253107
15. Yu, ZH, Wang, QR: Asymptotic behavior of solutions of third-order nonlinear dynamic equations on time scales. J. Comput. Appl. Math. **225**, 531-540 (2009)
16. Zhang, SY, Wang, QR: Oscillation for third-order nonlinear dynamic equations on time scales. Acta Sci. Natur. Univ. Sunyatseni **51**(4), 50-55 (2012) (in Chinese)