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Existence of periodic solutions for differential equations with multiple delays under dichotomy condition

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Abstract

Using Krasnoselskii's fixed point theorem and dichotomy theory, we prove the existence of periodic solutions for differential equations with multiple delays of the form $x'(t) + cx'(t-\tau) = A(t)x(t) + f(t, x(t-\alpha_1(t)), \ldots, x(t-\alpha_m(t)))$, where the parameter $c \ll 1$ is a small perturbation for a delayed forced term. Moreover, we discuss the convergence of these solutions to a solution of the unperturbed problem as $c \to 0$.

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1 Introduction

Delay differential equations are of interest in many areas of applications, such as population dynamics, drug administration, automatic control, laser optics, neural networks, economics and others (see for example [1–5]). There are significant theoretical researches on delay differential equation addressing many aspects of the dynamics, for example, stability of equilibria, existence of periodic solutions, complicated behavior, and chaos. Several methods were developed to obtain periodic solutions of autonomous delay differential equations, both for equations with time-invariant delay and with state-dependent delay. For references, see [6–8].

One of the main tools that has been extensively used when studying bounded solutions of differential equations is the concept of exponential dichotomy of the associated linear system

$$x'(t) = A(t)x(t). \tag{1.1}$$

Several results on the existence and uniqueness of bounded solutions, periodic solutions and almost periodic solutions of both linear and nonlinear differential equations are obtained under the assumption that the associated homogeneous linear equation (1.1) has an exponential dichotomy (see, for example, [9–11] and references therein). However, there are similar results on the existence and uniqueness of bounded solutions under a more general condition such as the (h,k)-dichotomy or the integrable dichotomy [12, 13].

In [13], the existence and uniqueness of periodic solutions of an integro-differential equations with bounded and unbounded delays were proved under the integrable dichotomy condition on system (1.1).

For the second order differential equation of motion

$$mx''(t) + bx'(t) + kx(t) = 0,$$

it has been studied that a time lag thats could result in the force represented by the term $cx'(t-\tau)$ so that the equation becomes

$$mx''(t) + bx'(t) + cx'(t-\tau) + kx(t) = 0$$

(see [14], p.236). Motivated by this, we will consider the first order linear system (1.1) with a small delayed forced term $cx'(t-\tau)$. Our aim is to establish the existence of periodic solutions under periodic perturbation and multiple variable lags. More precisely, we consider the differential system (1.1) with periodic coefficients under the integrable dichotomy condition and a periodic nonlinear perturbation with several delays,

$$f\bigl(t, x\bigl(t-\alpha_1(t)\bigr), \dots, x\bigl(t-\alpha_m(t)\bigr)\bigr),$$

together with a small delayed forced term $cx'(t-\tau)$, which results in the following system:

$$x'(t) + cx'(t-\tau) = A(t)x(t) + f\bigl(t, x\bigl(t-\alpha_1(t)\bigr), \dots, x\bigl(t-\alpha_m(t)\bigr)\bigr), \tag{1.2}$$

where we consider the parameter $c \ll 1$ as a small perturbation. A similar problem was considered in [15] under an exponential-typed condition on the corresponding linear system. Our new result on the existence of periodic solutions of (1.2) is based on a more general condition of an integrable dichotomy and Krasnoselskii's fixed point theorem.

It should also be noted that there are recent works in [16] and [17] on the existence of periodic solutions of similar systems of differential equations with multiple delays of the form

$$x'(t) = Ax(t) + \sum_{j=1}^{k} B_j x(t-j) + \sum_{j=1}^{k} C_j x'(t-j) + f(t)$$

and

$$x'(t) = Ax(t) + Bx(t-1) + Cx'(t-1) + F\bigl(t, x(t), x(t-1), \dots, x(t-k)\bigr),$$

respectively. In those works, the periodic solutions are constructed as an infinite series $x(t) = \sum_{i=0}^{\infty} x_i(t)$ of solutions of system of delay equations. In our work, the coefficients $A(t)$ in system (1.2) is non-autonomous and the delay term $x(t-\alpha_i(t))$ is more general. Moreover, we apply a different approach to obtain the existence of periodic solutions.

The paper is organized as follows. In the next section, some preliminary results on integrable dichotomy based on the result of [13], and the frameworks of our problem are introduced. Section 3 is devoted to establishing some criteria for the existence of periodic solutions of system (1.2). Finally, in Section 4, we discuss the convergence of these solutions to a solution of unperturbed problem as $c \to 0$.

2 Preliminaries and frameworks

The concept of an exponential dichotomy has been extensively used when studying bounded solutions of differential equations. Several results on the existence and uniqueness of bounded solutions, periodic solutions and almost periodic solutions of both linear and nonlinear differential equations are obtained under the assumption that the associated homogeneous linear equation satisfies the exponential dichotomy condition. However, there are similar results on the existence and uniqueness of bounded solutions under more general conditions such as the (h,k)-dichotomy, integrable dichotomy, and integrable (h,k)-dichotomy.

2.1 Periodic solutions for linear differential systems

Consider a linear differential system of the form (1.1), where $A(t)$ is a continuous $N \times N$ matrix function. Denote by $\Phi(t)$ the fundamental matrix solution of system (1.1), that is, $\Phi(t)$ is a solution matrix of (1.1) with $\Phi(0) = I$. Let P be a projection matrix. We define a Green matrix $G = G_P$ associated with P by

$$G(t,s) = \begin{cases} \Phi(t)P\Phi^{-1}(s) & \text{for } t \geq s, \\ -\Phi(t)(I-P)\Phi^{-1}(s) & \text{for } t < s. \end{cases} \tag{2.1}$$

We take the following definitions from [13].

Definition 2.1 We say that the linear differential system (1.1) has an *integrable dichotomy* if there exist a projection matrix P and a positive constant μ such that the associated Green matrix $G = G_P$ satisfies

$$\sup_{t\in\mathbb{R}} \int_{-\infty}^{\infty} \| G(t,s) \| \, ds = \mu. \tag{2.2}$$

A special case of an integrable dichotomy includes the following class of integrable (h,k)-dichotomies.

Definition 2.2 Let $h,k : \mathbb{R} \to \mathbb{R}^+$ be two positive continuous functions. We say that the linear differential system (1.1) has an *(h,k)-dichotomy* if there exist a projection matrix P and a positive constant c such that the associated Green matrix $G = G_P$ satisfies

$$\| G(t,s) \| \leq g_{h,k}(t,s),$$

for all $t,s \in \mathbb{R}$ where

$$g_{h,k}(t,s) = \begin{cases} c\frac{h(t)}{h(s)} & \text{for } t \geq s, \\ c\frac{k(s)}{k(t)} & \text{for } t \leq s. \end{cases} \tag{2.3}$$

In addition, if there exists $\mu_{h,k} > 0$ such that

$$\sup_{t\in\mathbb{R}} \int_{-\infty}^{\infty} g_{h,k}(t,s) \, ds = \mu_{h,k}$$

then we say that the linear differential system (1.1) has an *integrable (h,k)-dichotomy*.

Remark If the differential system (1.1) has an integrable (h,k)-dichotomy with projection P, then (1.1) has an integrable dichotomy for which $\Phi(t)P\Phi^{-1}(t)$ is bounded.

We review the following result about bounded solutions of linear differential system (1.1).

Proposition 2.1 ([13]) *Suppose that a linear differential system* (1.1) *has an integrable dichotomy. Then $x(t) = 0$ is the unique bounded solution of* (1.1).

Under an integrable dichotomy condition for system (1.1), we consider the solution of the corresponding non-homogeneous linear system

$$x'(t) = A(t)x(t) + f(t). \tag{2.4}$$

We denote by $BC(\mathbb{R},\mathbb{R}^N)$ the set of all bounded and continuous functions defined on $\mathbb{R}$ to $\mathbb{R}^N$.

Theorem 2.1 ([13]) *Suppose that the homogeneous system* (1.1) *has an integrable dichotomy. If f is a function in $BC(\mathbb{R},\mathbb{R}^N)$, then system* (2.4) *has a unique bounded solution $x \in BC(\mathbb{R},\mathbb{R}^N)$. Moreover, we have*

$$x(t) = \int_{-\infty}^{\infty} G(t,s)f(s)\,ds. \tag{2.5}$$

In addition, if the differential operator $A(t)$ and the non-homogeneous term $f(t)$ are T-periodic, we can obtain periodic solutions of (2.4).

Proposition 2.2 ([13]) *Suppose that the homogeneous system* (1.1) *has an integrable dichotomy for which $\Phi(t)P\Phi^{-1}(t)$ is bounded. If $A(t+T) = A(t)$, then $\Phi(t)P\Phi^{-1}(t)$ is also T-periodic.*

Theorem 2.2 ([13]) *Suppose that the homogeneous system* (1.1) *has an integrable dichotomy for which $\Phi(t)P\Phi^{-1}(t)$ is bounded. If $A(t+T) = A(t)$ and $f \in BC(\mathbb{R},\mathbb{R}^N)$ is T-periodic, then* (2.4) *has a unique periodic solution satisfying* (2.5).

2.2 Frameworks

In this paper, we will investigate the existence of a periodic solution of a delay differential equation of the form (1.2), where τ and c are constants with $|c| \ll 1$ sufficiently small perturbation and $\alpha_i(t)$, $i = 1,2,\ldots,m$, are real continuous functions on $\mathbb{R}$ with period $T > 0$.

Denote

$$BC^1(\mathbb{R},\mathbb{R}^N) = \bigl\{u : \mathbb{R} \to \mathbb{R}^N \mid u, u' \text{ are bounded and continuous}\bigr\}.$$

It is easily seen that $BC^1(\mathbb{R},\mathbb{R}^N)$ is a Banach space when equipped with the norm

$$\|u\| = \|u\|_\infty + \bigl\|u'\bigr\|_\infty,$$

where $\|u\|_\infty = \sup_{t\in\mathbb{R}} |u(t)|$ and $\|u'\|_\infty = \sup_{t\in\mathbb{R}} |u'(t)|$.

Let $u \in BC^1(\mathbb{R}, \mathbb{R}^N)$ be a T-periodic function. Consider the linear periodic systems (1.1) and the corresponding non-homogeneous system

$$x'(t) = A(t)x(t) + f\bigl(t, u\bigl(t - \alpha_1(t)\bigr), \dots, u\bigl(t - \alpha_m(t)\bigr)\bigr) - cu'(t - \tau). \tag{2.6}$$

We assume the following conditions.

Assumption 1 We assume that $A(t)$ is an $N \times N$ real continuous matrix function defined on $\mathbb{R}$ and T-periodic, that is, $A(t + T) = A(t)$ for all $t \in \mathbb{R}$.

Note that by the periodicity and continuity of $A(t)$, we have

$$L := \sup_{t \in \mathbb{R}} \bigl\| A(t) \bigr\| < \infty. \tag{2.7}$$

Assumption 2 System (1.1) has an integrable dichotomy for which $\Phi(t)P\Phi^{-1}(t)$ is bounded.

Here, $\Phi(t)$ is the fundamental matrix solution of (1.1). Hence, the associated Green matrix $G(t,s)$ given by (2.1) satisfies (2.2) for some positive constant μ.

In addition, we impose the following condition on f.

Assumption 3 We assume that $f(t, u_1, \dots, u_k)$ is a real continuous vector function defined on $\mathbb{R} \times \mathbb{R}^N \times \cdots \times \mathbb{R}^N$ such that

(i) $f(t + T, u_1, \dots, u_m) = f(t, u_1, \dots, u_m)$ for all $(t, u_1, \dots, u_m) \in \mathbb{R} \times \mathbb{R}^N \times \cdots \times \mathbb{R}^N$.

(ii) There exists a positive constant $r < \frac{1}{m(1+L\mu+\mu)}$ such that

$$\bigl|f(t, u_1, \dots, u_m) - f(t, v_1, \dots, v_m)\bigr| < r\bigl(|u_1 - v_1| + |u_2 - v_2| + \cdots + |u_m - v_m|\bigr),$$

for every $u_1, u_2, \dots, u_m, v_1, v_2, \dots, v_m \in \mathbb{R}^N$ and $t \in \mathbb{R}$,

where the constants L and μ are given by (2.7) and (2.2), respectively.

We prove the existence of a periodic solution of (1.2) under an integrable dichotomy condition.

By Theorem 2.2, system (2.6) has a unique periodic solution satisfying the integral equation

$$x(t) = \int_{-\infty}^{\infty} G(t,s)\bigl[f\bigl(s, u\bigl(s - \alpha_1(s)\bigr), \dots, u\bigl(s - \alpha_m(s)\bigr)\bigr) - cu'(s - \tau)\bigr]\, ds. \tag{2.8}$$

3 Existence of periodic solutions

In this section, we prove our main result on the existence of periodic solutions to system (1.2).

Theorem 3.1 *Suppose that Assumptions* 1, 2, *and* 3 *are satisfied. For every* $|c| \ll 1$ *sufficiently small, there exists at least a* T*-periodic solution of system* (1.2).

To establish the existence result, we will apply Krasnoselskii's fixed point theorem [18] as stated below.

Theorem 3.2 *Let K be a bounded nonempty closed and convex subset of a Banach space X. Suppose that Γ_1 and Γ_2 are maps on K into X such that*

(i) *$\Gamma_1 x + \Gamma_2 y \in K$ for all $x, y \in K$;*

(ii) *Γ_1 is a contraction on K;*

(iii) *Γ_2 is completely continuous on K.*

Then there exists $x^* \in K$ such that $\Gamma_1 x^* + \Gamma_2 x^* = x^*$.

For $u \in BC^1(\mathbb{R}, \mathbb{R}^N)$, we define the operators V and W by

$$(Vu)(t) := -cu(t-\tau) \tag{3.1}$$

and

$$(Wu)(t) := \int_{-\infty}^{\infty} G(t,s)\bigl[f\bigl(s, u\bigl(s-\alpha_1(s)\bigr), \ldots, u\bigl(s-\alpha_m(s)\bigr)\bigr) - cu'(s-\tau)\bigr]\,ds + cu(t-\tau). \tag{3.2}$$

Lemma 3.1 *The operators V and W defined above are operators from $BC^1(\mathbb{R}, \mathbb{R}^N)$ into itself, that is, $V, W : BC^1(\mathbb{R}, \mathbb{R}^N) \to BC^1(\mathbb{R}, \mathbb{R}^N)$.*

Proof The statement is clear for V. By Assumption 1, we see that

$$\sup_{t\in[0,T]} \bigl\|A(t)\bigr\| := L < \infty. \tag{3.3}$$

Denoted by $\nu := \sup_{t\in[0,T]} |f(t, 0, \ldots, 0)|$. Let $u \in BC^1(\mathbb{R}, \mathbb{R}^N)$ be arbitrary, we have from Assumption 3

$$\begin{aligned}
&\bigl|f\bigl(t, u\bigl(t-\alpha_1(t)\bigr), \ldots, u\bigl(t-\alpha_m(t)\bigr)\bigr)\bigr| \\
&\quad\le \bigl|f\bigl(t, u\bigl(t-\alpha_1(t)\bigr), \ldots, u\bigl(t-\alpha_m(t)\bigr)\bigr) - f(t, 0, \ldots, 0)\bigr| + \bigl|f(t, 0, \ldots, 0)\bigr| \\
&\quad\le r\bigl(\bigl|u\bigl(t-\alpha_1(t)\bigr)\bigr| + \cdots + \bigl|u\bigl(t-\alpha_m(t)\bigr)\bigr|\bigr) + \bigl|f(t, 0, \ldots, 0)\bigr| \\
&\quad\le rm\|u\| + \sup_{t\in[0,T]} \bigl|f(t, 0, \ldots, 0)\bigr| \\
&\quad\le rm\|u\| + \nu
\end{aligned} \tag{3.4}$$

for all $t \in \mathbb{R}$. It follows that

$$\begin{aligned}
\bigl|(Wu)(t)\bigr| &\le \int_{-\infty}^{\infty} \bigl|G(t,s)\bigl[f\bigl(s, u\bigl(s-\alpha_1(s)\bigr), \ldots, u\bigl(s-\alpha_m(s)\bigr)\bigr) - cu'(s-\tau)\bigr]\bigr|\,ds \\
&\quad + |c|\bigl|u(t-\tau)\bigr| \\
&\le \mu\bigl(rm\|u\| + \nu\bigr) + \mu|c|\bigl\|u'\bigr\|_\infty + |c|\|u\|_\infty \\
&\le \mu\bigl(rm\|u\| + \nu\bigr) + \mu|c|\|u\| + |c|\|u\|
\end{aligned}$$

for all $t \in \mathbb{R}$. Hence, $\|Wu\|_\infty \le \mu(rm\|u\| + \nu) + \mu|c|\|u\| + |c|\|u\|$. In addition,

$$\begin{aligned}
(Wu)'(t) &= \frac{d}{dt}\biggl(\int_{-\infty}^{\infty} G(t,s)\bigl[f\bigl(s, u\bigl(s-\alpha_1(s)\bigr), \ldots, u\bigl(s-\alpha_m(s)\bigr)\bigr) - cu'(s-\tau)\bigr]\,ds\biggr) \\
&\quad + cu'(t-\tau)
\end{aligned}$$

$$\begin{aligned}
&= A(t)\int_{-\infty}^{\infty} G(t,s)\bigl[f\bigl(s,u\bigl(s-\alpha_1(s)\bigr),\ldots,u\bigl(s-\alpha_m(s)\bigr)\bigr)-cu'(s-\tau)\bigr]\,ds \\
&\quad + f\bigl(t,u\bigl(t-\alpha_1(t)\bigr),\ldots,u\bigl(t-\alpha_m(t)\bigr)\bigr)-cu'(t-\tau)+cu'(t-\tau) \\
&= A(t)\int_{-\infty}^{\infty} G(t,s)\bigl[f\bigl(s,u\bigl(s-\alpha_1(s)\bigr),\ldots,u\bigl(s-\alpha_m(s)\bigr)\bigr)-cu'(s-\tau)\bigr]\,ds \\
&\quad + f\bigl(t,u\bigl(t-\alpha_1(t)\bigr),\ldots,u\bigl(t-\alpha_m(t)\bigr)\bigr).
\end{aligned} \tag{3.5}$$

Thus,

$$\begin{aligned}
\bigl|(Wu)'(t)\bigr| &\le \bigl\|A(t)\bigr\|\left|\int_{-\infty}^{\infty} G(t,s)\bigl[f\bigl(s,u\bigl(s-\alpha_1(s)\bigr),\ldots,u\bigl(s-\alpha_m(s)\bigr)\bigr)-cu'(s-\tau)\bigr]\,ds\right| \\
&\quad + \bigl|f\bigl(t,u\bigl(t-\alpha_1(t)\bigr),\ldots,u\bigl(t-\alpha_m(t)\bigr)\bigr)\bigr| \\
&\le L\mu\bigl[\bigl(rm\|u\|+\nu\bigr)+|c|\|u\|\bigr]+rm\|u\|+\nu
\end{aligned}$$

for all $t\in\mathbb{R}$. Hence,

$$\bigl\|(Wu)'\bigr\|_{\infty} \le L\mu\bigl[\bigl(rm\|u\|+\nu\bigr)+|c|\|u\|\bigr]+rm\|u\|+\nu.$$

Consequently, we have

$$\begin{aligned}
\|Wu\| &\le \mu\bigl(rm\|u\|+\nu\bigr)+\mu|c|\|u\|+|c|\|u\| \\
&\quad + L\mu\bigl[\bigl(rm\|u\|+\nu\bigr)+|c|\|u\|\bigr]+rm\|u\|+\nu \\
&= (L+1)\mu\bigl[\bigl(rm\|u\|+\nu\bigr)+|c|\|u\|\bigr]+\bigl(|c|+rm\bigr)\|u\|+\nu<\infty.
\end{aligned} \tag{3.6}$$

The lemma follows. □

It is clear that if $V+W$ has a fixed point, then the fixed point is a periodic solution of (1.2). Hence, we will turn to the problem of establishing a fixed point of the operator $V+W$.

Let $M>0$ be a positive constant. Denote

$$K_M=\bigl\{u\in BC^1\bigl(\mathbb{R},\mathbb{R}^N\bigr):\|u\|\le M \text{ and } u(t+T)=u(t) \text{ for all } t\in\mathbb{R}\bigr\}$$

and

$$B_M=\bigl\{u\in BC^1\bigl(\mathbb{R},\mathbb{R}^N\bigr):\|u\|<M \text{ and } u(t+T)=u(t) \text{ for all } t\in\mathbb{R}\bigr\}.$$

Clearly, the set K_M is a bounded nonempty closed and convex subset of $BC^1(\mathbb{R},\mathbb{R}^N)$.

Lemma 3.2 *The operator* $V:BC^1(\mathbb{R},\mathbb{R}^N)\to BC^1(\mathbb{R},\mathbb{R}^N)$ *defined by* (3.1) *is a contraction.*

Proof The result is clear since $\|Vu\|=|c|\|u\|$ for $u\in BC^1(\mathbb{R},\mathbb{R}^N)$ and $|c|<1$. □

Lemma 3.3 *There exists* $M>0$ *such that, for any* $v,w\in K_M$, *we have* $Vv+Ww\in K_M$ *whenever* $|c|\ll 1$ *is sufficiently small.*

Proof Let $v, w \in K_M$. Then $\|v\|, \|w\| \le M$. By (3.6), we have

$$\begin{aligned}
\|Vv + Ww\| &\le |c|M + (L+1)\mu\big[(rmM + \nu) + |c|M\big] + \big(|c| + rm\big)M + \nu \\
&= \big[|c| + (L+1)\mu rm + (L+1)\mu|c| + |c| + rm\big]M + (L+1)\mu\nu + \nu \\
&= \big[2|c| + rm + (L+1)\mu_{h,k}\big(rm + |c|\big)\big]M + (L+1)\mu_{h,k}\nu + \nu \\
&= \left[2|c| + rm + (L+1)\mu\big(rm + |c|\big) + \frac{(L+1)\mu\nu + \nu}{M}\right]M \\
&\le \left[\big(2|c| + rm\big)\big(1 + (L+1)\mu\big) + \frac{(L+1)\mu\nu + \nu}{M}\right]M \\
&= \left[2|c|\big(1 + (L+1)\mu\big) + rm\big(1 + (L+1)\mu\big) + \frac{(L+1)\mu\nu + \nu}{M}\right]M. \qquad (3.7)
\end{aligned}$$

By Assumption 3, we have

$$rm\big(1 + (L+1)\mu\big) < \frac{m(1 + (L+1)\mu)}{m(1 + L\mu + \mu)} = 1. \qquad (3.8)$$

We can choose $|c| \ll 1$ sufficiently small so that

$$2|c|\big(1 + (L+1)\mu\big) + rm\big(1 + (L+1)\mu\big) < 1. \qquad (3.9)$$

In addition, since $\frac{(L+1)\mu\nu+\nu}{M} \to 0$ as $M \to \infty$, we can choose $M > 0$ sufficiently large so that

$$\frac{(L+1)\mu\nu + \nu}{M} < 1 - 2|c|\big(1 + (L+1)\mu\big) - rm\big(1 + (L+1)\mu\big). \qquad (3.10)$$

It follows from (3.7), (3.9), and (3.10) that

$$\|Vv + Ww\| < M,$$

for all $v, w \in K_M$. □

We next prove that W is a completely continuous operator on K_M, which is a consequence of the following lemma.

Lemma 3.4 *The set $W(B_M)$ is relatively compact in $BC^1(\mathbb{R}, \mathbb{R}^N)$.*

Proof Since $W(B_M) \subset K_M$, we see that $\{Wu : u \in B_M\}$ is bounded in $BC^1(\mathbb{R}, \mathbb{R}^N)$. In particular, we have

$$\|Wu\|_\infty < M$$

and

$$\big\|(Wu)'\big\|_\infty < M$$

for all $u \in B_M$. Therefore, $\{Wu : u \in B_M\}$ is equicontinuous. Let u_n be arbitrary sequence in B_M. It follows from the Arzela-Ascoli theorem that there is a subsequence, denoted again by u_n, such that

$$\|Wu_n - v\|_\infty \to 0$$

as $n \to \infty$ with the limit $v \in BC(\mathbb{R}, \mathbb{R}^N)$. Moreover, for any $u \in B_M$, we have from (3.5)

$$\begin{aligned}(Wu)'(t) &= A(t) \int_{-\infty}^{\infty} G(t,s)\bigl[f\bigl(s, u\bigl(s - \alpha_1(s)\bigr), \dots, u\bigl(s - \alpha_m(s)\bigr)\bigr) - cu'(s - \tau)\bigr]\,ds \\ &\quad + f\bigl(t, u\bigl(t - \alpha_1(t)\bigr), \dots, u\bigl(t - \alpha_m(t)\bigr)\bigr) \\ &= A(t)\bigl[(Vu)(t) + (Wu)(t)\bigr] + f\bigl(t, u\bigl(t - \alpha_1(t)\bigr), \dots, u\bigl(t - \alpha_m(t)\bigr)\bigr).\end{aligned}$$

Notice that $A(t)$ is uniformly continuous on $[0, T]$ and $f(t, u_1, \dots, u_m)$ is uniformly continuous on $[0, T] \times \{x \in \mathbb{R}^N : |x| \leq M\} \times \cdots \times \{x \in \mathbb{R}^N : |x| \leq M\}$. In addition, the families $\{u \in B_M\}$, $\{Vu : u \in B_M\}$, and $\{Wu : u \in B_M\}$ are equicontinuous. Therefore, $\{(Wu)' : u \in B_M\}$ is equicontinuous. It follows from the Arzela-Ascoli theorem that there is a further subsequence u_{n_k} such that

$$\bigl\|(Wu_{n_k})' - w\bigr\|_\infty \to 0$$

as $k \to \infty$ with the limit $w \in BC(\mathbb{R}, \mathbb{R}^N)$. Since the convergence $Wu_{n_k} \to v$ and the convergence $(Wu_{n_k})' \to w$ are uniform, we have $w = v'$ and thus $Wu_{n_k} \to v$ in $BC^1(\mathbb{R}, \mathbb{R}^N)$. This shows that W is relatively compact. □

It follows from the above lemmas and Krasnoselskii's fixed point theorem (Theorem 3.2) that there exists a T-periodic solution of (1.2). Hence, we have proved Theorem 3.1.

Remark We give the following remarks on our main result.

(1) The result in Theorem 3.1 can be applied to the case when $c = 0$, however, we indeed obtain a unique periodic solution (see Lemma 4.1).

(2) We may consider an alternative approach to establish the existence of periodic solutions of (1.2) by using the transformation $y(t) = x(t) + cx(t - \tau)$. Hence, system (1.2) can be written as

$$y'(t) = A(t)y(t) + g(t),$$

where $g(t) := f(t, x(t - \alpha_1(t)), \dots, x(t - \alpha_m(t))) - A(t)cx(t - \tau)$. By Theorem 2.1, we know that the solution of the above system satisfies the integral equation

$$y(t) = \int_{-\infty}^{\infty} G(t,s)g(s)\,ds.$$

Hence, the problem is reduced to showing the existence of periodic solution of the following delay integral equation

$$x(t) = -cx(t - \tau) \int_{-\infty}^{\infty} G(t,s)\bigl[f\bigl(s, x\bigl(s - \alpha_1(s)\bigr), \dots, x\bigl(s - \alpha_m(s)\bigr)\bigr) - A(s)cx(s - \tau)\bigr]\,ds.$$

4 Continuity of periodic solutions in a neighborhood of $c = 0$

In this section, we show that periodic solutions for the systems with periodic perturbation (1.2) converge to a solution of the following unperturbed system:

$$x'(t) = A(t)x(t) + f\big(t, x\big(t - \alpha_1(t)\big), \dots, x\big(t - \alpha_m(t)\big)\big) \tag{4.1}$$

when $c \to 0$.

Lemma 4.1 *Suppose that Assumptions* 1, 2, *and* 3 *are satisfied. Then the T-periodic solution of* (4.1) *exists and is unique.*

Proof Since a solution of (1.2) is a fixed point of S defined by

$$(Su)(t) := \int_{-\infty}^{\infty} G(t,s)\big[f\big(s, u\big(s - \alpha_1(s)\big), \dots, u\big(s - \alpha_m(s)\big)\big)\big]\, ds.$$

By our assumptions, the map S is a contraction since

$$\begin{aligned}
&\big|(Su)(t) - (Sv)(t)\big| \\
&\quad = \bigg|\int_{-\infty}^{\infty} G(t,s)\big[f\big(s, u\big(s - \alpha_1(s)\big), \dots, u\big(s - \alpha_m(s)\big)\big) \\
&\qquad - f\big(s, v\big(s - \alpha_1(s)\big), \dots, v\big(s - \alpha_m(s)\big)\big)\big]\, ds\bigg| \\
&\quad \le rm\|u - v\|_\infty \int_{-\infty}^{\infty} \big\|G(t,s)\big\|\, ds \\
&\quad \le rm\mu\|u - v\|_\infty \\
&\quad < \frac{1}{m(1 + L\mu + \mu)} m\mu\|u - v\|_\infty \\
&\quad < \|u - v\|_\infty,
\end{aligned}$$

for all T-periodic function u, v. By the contraction mapping theorem, (4.1) has a unique T-periodic solution. □

Theorem 4.1 *Suppose that Assumptions* 1, 2, *and* 3 *are satisfied. For $|c| \ll 1$ sufficiently small, a sequence of T-periodic solutions u_c of system* (1.2) *converges to the T-periodic solution of system* (4.1) *as $c \to 0$.*

Proof From the proof of Theorem 3.1, we see that periodic solutions $u_c \in BC^1(\mathbb{R}, \mathbb{R}^N)$ of system (1.2) are uniformly bounded and equicontinuous. By the Arzela-Ascoli theorem, we can extract a subsequence u_{c_n} that converges uniformly to u. Since u_{c_n} is a fixed point of $V + W$, we have

$$u_{c_n}(t) = \int_{-\infty}^{\infty} G(t,s)\big[f\big(s, u_{c_n}\big(s - \alpha_1(s)\big), \dots, u_{c_n}\big(s - \alpha_m(s)\big)\big) - c_n u'_{c_n}(s - \tau)\big]\, ds.$$

As f is continuous, we obtain from the dominated convergence theorem

$$u(t) = \int_{-\infty}^{\infty} G(t,s)\big[f\big(s, u\big(s - \alpha_1(s)\big), \dots, u\big(s - \alpha_m(s)\big)\big)\big]\, ds.$$

So the limit u is a T-periodic solution of (4.1). Since the T-periodic solution of (4.1) is unique, we conclude that the whole sequence u_c converges to the solution u of (4.1) as $c \to 0$. □

Competing interests
The author declares that he has no competing interests.

Author's contributions
The author is solely contributed in this article. The author read and approved the final manuscript.

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