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On a close to symmetric system of difference equations of second order

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Abstract

Closed form formulas of the solutions to the following system of difference equations:

$$x_n = \frac{y_{n-1}y_{n-2}}{x_{n-1}(a_n + b_n y_{n-1}y_{n-2})}, \qquad y_n = \frac{x_{n-1}x_{n-2}}{y_{n-1}(\alpha_n + \beta_n x_{n-1}x_{n-2})}, \qquad n \in \mathbb{N}_0,$$

where a_n, b_n, α_n, β_n, $n \in \mathbb{N}_0$, and initial values x_{-i}, y_{-i}, $i \in \{1,2\}$ are real numbers, are found. The domain of undefinable solutions to the system is described. The long-term behavior of its solutions is studied in detail for the case of constant a_n, b_n, α_n and β_n, $n \in \mathbb{N}_0$.

MSC: Primary 39A10; 39A20

Keywords: system of difference equations; closed form solution; long-term behavior; periodic solutions

1 Introduction

Studying concrete nonlinear difference equations and systems is a topic of a great recent interest (see, *e.g.*, [1–46] and the references therein). Studying systems of difference equations, especially symmetric and close to symmetric ones, is a topic of considerable interest (see, *e.g.*, [2, 6, 7, 10, 12–16, 18, 19, 23, 24, 26–29, 31–38, 40, 41, 44, 46]). Another topic of interest is solvable difference equations and systems and their applications (see, *e.g.*, [1–5, 7, 17, 20, 21, 23–27, 29–37, 39–46]). Renewed interest in the area started after the publication of [20] where a formula for a solution of a difference equation was theoretically explained. The most interesting thing in [20] was a change of variables which reduced the equation to a linear one with constant coefficients. Related ideas were later used, *e.g.*, in [1, 4, 7, 17, 21, 23–27, 29–37, 39–45].

Quite recently in [2] the following systems of difference equations were presented:

$$\begin{aligned} x_n &= \frac{y_{n-1}y_{n-2}}{x_{n-1}(\pm 1 \pm y_{n-1}y_{n-2})}, \\ y_n &= \frac{x_{n-1}x_{n-2}}{y_{n-1}(\pm 1 \pm x_{n-1}x_{n-2})}, \qquad n \in \mathbb{N}_0, \end{aligned} \tag{1}$$

where x_{-i}, y_{-i}, $i \in \{1,2\}$ are real numbers, and some formulas for their solutions are given, some of which are proved by induction.

The next system of difference equations

$$x_n = \frac{y_{n-1}y_{n-2}}{x_{n-1}(a_n + b_n y_{n-1}y_{n-2})},$$
$$y_n = \frac{x_{n-1}x_{n-2}}{y_{n-1}(\alpha_n + \beta_n x_{n-1}x_{n-2})}, \quad n \in \mathbb{N}_0, \tag{2}$$

where a_n, b_n, α_n, β_n, $n \in \mathbb{N}_0$, and initial values x_{-i}, y_{-i}, $i \in \{1,2\}$, are real numbers, is a generalization of the system in (1). Our aim is to show that more general system (2) is solvable by giving a natural method for getting its solutions. The domain of undefinable solutions to the system is also described. For the case when a_n, b_n, α_n, β_n, $n \in \mathbb{N}_0$, are constant, the long-term behavior of its solutions is investigated in detail.

A solution $(x_n, y_n)_{n\geq -2}$ of system (2) is called *periodic*, or *eventually periodic*, with period p if there is $n_0 \geq -2$ such that

$$x_{n+p} = x_n \quad \text{and} \quad y_{n+p} = y_n \quad \text{for } n \geq n_0.$$

For some results in the area, see, *e.g.*, [6, 9–11, 19, 21, 22, 28].

2 Solutions to system (2) in closed form

Assume first that $x_{-i} \neq 0$, $y_{-i} \neq 0$, $i \in \{1,2\}$. Then, by the method of induction and the equations in (2), it follows that for every well-defined solution to system (2), $x_n \neq 0$ and $y_n \neq 0$, for every $n \in \mathbb{N}_0$. On the other hand, if $x_{n_0} = 0$ for some $n_0 \in \mathbb{N}$, then the first equation in (2) implies that $y_{n_0-1} = 0$ or $y_{n_0-2} = 0$. If $y_{n_0-1} = 0$, then $x_{n_0-2} = 0$ or $x_{n_0-3} = 0$, while if $y_{n_0-2} = 0$, then $x_{n_0-3} = 0$ or $x_{n_0-4} = 0$. Repeating this procedure, we get that $x_{-i} = 0$ or $y_{-i} = 0$ for some $i \in \{1,2\}$. Similarly, if $y_{n_1} = 0$ for some $n_1 \in \mathbb{N}$, we get $x_{-i} = 0$ or $y_{-i} = 0$ for some $i \in \{1,2\}$. Hence, for a well-defined solution $(x_n, y_n)_{n\geq -2}$ of system (2), we have that

$$x_n y_n \neq 0, \quad n \geq -2 \tag{3}$$

if and only if $x_{-i}y_{-i} \neq 0$, $i \in \{1,2\}$.

Assume now that $(x_n, y_n)_{n\geq -2}$ is a solution to system (2) such that (3) holds. Then, by multiplying the first equation in (2) by x_{n-1} and the second one by y_{n-1}, and using the following changes of variables

$$u_n = \frac{1}{x_n x_{n-1}}, \qquad v_n = \frac{1}{y_n y_{n-1}}, \tag{4}$$

$n \geq -1$, system (2) is transformed in the following one:

$$u_n = a_n v_{n-1} + b_n, \qquad v_n = \alpha_n u_{n-1} + \beta_n, \quad n \in \mathbb{N}_0. \tag{5}$$

From (5) it follows that

$$u_n = a_n \alpha_{n-1} u_{n-2} + a_n \beta_{n-1} + b_n, \tag{6}$$

$$v_n = \alpha_n a_{n-1} v_{n-2} + \alpha_n b_{n-1} + \beta_n, \quad n \in \mathbb{N}. \tag{7}$$

This means that $(u_{2n})_{n\in\mathbb{N}_0}$, $(u_{2n-1})_{n\in\mathbb{N}_0}$, $(v_{2n})_{n\in\mathbb{N}_0}$, and $(v_{2n-1})_{n\in\mathbb{N}_0}$ are solutions to two linear first-order difference equations, which are solvable.

Solving these equations, we get

$$u_{2n} = u_0 \prod_{j=1}^{n} a_{2j}\alpha_{2j-1} + \sum_{i=1}^{n}(a_{2i}\beta_{2i-1} + b_{2i}) \prod_{s=i+1}^{n} a_{2s}\alpha_{2s-1}, \tag{8}$$

$$u_{2n-1} = u_{-1} \prod_{j=1}^{n} a_{2j-1}\alpha_{2j-2} + \sum_{i=1}^{n}(a_{2i-1}\beta_{2i-2} + b_{2i-1}) \prod_{s=i+1}^{n} a_{2s-1}\alpha_{2s-2}, \tag{9}$$

$$v_{2n} = v_0 \prod_{j=1}^{n} \alpha_{2j}a_{2j-1} + \sum_{i=1}^{n}(\alpha_{2i}b_{2i-1} + \beta_{2i}) \prod_{s=i+1}^{n} \alpha_{2s}a_{2s-1}, \tag{10}$$

$$v_{2n-1} = v_{-1} \prod_{j=1}^{n} \alpha_{2j-1}a_{2j-2} + \sum_{i=1}^{n}(\alpha_{2i-1}b_{2i-2} + \beta_{2i-1}) \prod_{s=i+1}^{n} \alpha_{2s-1}a_{2s-2}. \tag{11}$$

Using (4) we obtain

$$x_{2n+i} = \frac{1}{u_{2n+i}x_{2n+i-1}} = \frac{u_{2n+i-1}}{u_{2n+i}} x_{2(n-1)+i}, \quad i \in \{0,1\},$$

and

$$y_{2n+i} = \frac{1}{v_{2n+i}y_{2n+i-1}} = \frac{v_{2n+i-1}}{v_{2n+i}} y_{2(n-1)+i}, \quad i \in \{0,1\},$$

for $2n + i \geq 0$, from which it follows that

$$x_{2m+i} = x_{i-2} \prod_{j=0}^{m} \frac{u_{2j+i-1}}{u_{2j+i}}, \tag{12}$$

$$y_{2m+i} = y_{i-2} \prod_{j=0}^{m} \frac{v_{2j+i-1}}{v_{2j+i}} \tag{13}$$

for every $m \in \mathbb{N}_0$, $i \in \{0,1\}$.

3 Case of constant coefficients

In this section we consider the case when all the coefficients in system (2) are constant, that is, when

$$a_n = a, \qquad b_n = b, \qquad \alpha_n = \alpha, \qquad \beta_n = \beta, \quad n \in \mathbb{N}_0.$$

Then (2) is

$$\begin{aligned} x_n &= \frac{y_{n-1}y_{n-2}}{x_{n-1}(a + by_{n-1}y_{n-2})}, \\ y_n &= \frac{x_{n-1}x_{n-2}}{y_{n-1}(\alpha + \beta x_{n-1}x_{n-2})}, \quad n \in \mathbb{N}_0. \end{aligned} \tag{14}$$

Assume that $(x_n, y_n)_{n\geq -2}$ is a solution to system (2) such that (3) holds. Then we have

$$u_n = av_{n-1} + b, \qquad v_n = \alpha u_{n-1} + \beta, \quad n \in \mathbb{N}_0, \tag{15}$$

and

$$u_n = a\alpha u_{n-2} + a\beta + b, \tag{16}$$

$$v_n = a\alpha v_{n-2} + \alpha b + \beta, \quad n \in \mathbb{N}. \tag{17}$$

From (8)-(11), we obtain

$$\begin{aligned} u_{2n-l} &= u_{-l}(a\alpha)^n + (a\beta + b)\frac{1-(a\alpha)^n}{1-a\alpha} \\ &= \frac{a\beta + b + (a\alpha)^n(u_{-l}(1-a\alpha) - a\beta - b)}{1-a\alpha} \end{aligned} \tag{18}$$

for $n \in \mathbb{N}_0$, $l \in \{0,1\}$ when $a\alpha \neq 1$, while if $a\alpha = 1$, we have

$$u_{2n-l} = u_{-l} + (a\beta + b)n, \quad n \in \mathbb{N}_0, l \in \{0,1\}, \tag{19}$$

and we also have

$$\begin{aligned} v_{2n-l} &= v_{-l}(a\alpha)^n + (\alpha b + \beta)\frac{1-(a\alpha)^n}{1-a\alpha} \\ &= \frac{\alpha b + \beta + (a\alpha)^n(v_{-l}(1-a\alpha) - \alpha b - \beta)}{1-a\alpha}, \end{aligned} \tag{20}$$

$n \in \mathbb{N}_0$, $l \in \{0,1\}$ if $a\alpha \neq 1$, while if $a\alpha = 1$, we have

$$v_{2n-l} = v_{-l} + (\alpha b + \beta)n, \quad n \in \mathbb{N}_0, l \in \{0,1\}. \tag{21}$$

Now we present formulae for solutions to system (14).

Case $a\alpha \neq 1$. We have

$$x_{2m} = x_{-2}\prod_{j=0}^{m}\frac{u_{2j-1}}{u_{2j}} = x_{-2}\prod_{j=0}^{m}\frac{a\beta + b + (a\alpha)^j(u_{-1}(1-a\alpha) - a\beta - b)}{a\beta + b + (a\alpha)^j(u_0(1-a\alpha) - a\beta - b)}, \tag{22}$$

$$x_{2m+1} = x_{-1}\prod_{j=0}^{m}\frac{u_{2j}}{u_{2j+1}} = x_{-1}\prod_{j=0}^{m}\frac{a\beta + b + (a\alpha)^j(u_0(1-a\alpha) - a\beta - b)}{a\beta + b + (a\alpha)^{j+1}(u_{-1}(1-a\alpha) - a\beta - b)}, \tag{23}$$

$$y_{2m} = y_{-2}\prod_{j=0}^{m}\frac{v_{2j-1}}{v_{2j}} = y_{-2}\prod_{j=0}^{m}\frac{\alpha b + \beta + (a\alpha)^j(v_{-1}(1-a\alpha) - \alpha b - \beta)}{\alpha b + \beta + (a\alpha)^j(v_0(1-a\alpha) - \alpha b - \beta)}, \tag{24}$$

$$y_{2m+1} = y_{-1}\prod_{j=0}^{m}\frac{v_{2j}}{v_{2j+1}} = y_{-1}\prod_{j=0}^{m}\frac{\alpha b + \beta + (a\alpha)^j(v_0(1-a\alpha) - \alpha b - \beta)}{\alpha b + \beta + (a\alpha)^{j+1}(v_{-1}(1-a\alpha) - \alpha b - \beta)} \tag{25}$$

for every $m \in \mathbb{N}_0$.

Case $a\alpha = 1$. We have

$$x_{2m} = x_{-2}\prod_{j=0}^{m}\frac{u_{2j-1}}{u_{2j}} = x_{-2}\prod_{j=0}^{m}\frac{u_{-1}+(a\beta+b)j}{u_0+(a\beta+b)j}, \tag{26}$$

$$x_{2m+1} = x_{-1}\prod_{j=0}^{m}\frac{u_{2j}}{u_{2j+1}} = x_{-1}\prod_{j=0}^{m}\frac{u_0+(a\beta+b)j}{u_{-1}+(a\beta+b)(j+1)}, \tag{27}$$

$$y_{2m} = y_{-2}\prod_{j=0}^{m}\frac{v_{2j-1}}{v_{2j}} = y_{-2}\prod_{j=0}^{m}\frac{v_{-1}+(\alpha b+\beta)j}{v_0+(\alpha b+\beta)j}, \tag{28}$$

$$y_{2m+1} = y_{-1}\prod_{j=0}^{m}\frac{v_{2j}}{v_{2j+1}} = y_{-1}\prod_{j=0}^{m}\frac{v_0+(\alpha b+\beta)j}{v_{-1}+(\alpha b+\beta)(j+1)} \tag{29}$$

for every $m \in \mathbb{N}_0$.

4 Long-term behavior of solutions to system (14)

Before we formulate and prove the main results regarding the long-term behavior of well-defined solutions to system (14), we quote the following well-known asymptotic formula which will be used in the proofs of the main results:

$$(1+x)^{-1} = 1 - x + O(x^2), \quad \text{as } x \to 0. \tag{30}$$

We also define the following quantities:

$$L_1 := \frac{u_{-1}(1-a\alpha)-a\beta-b}{u_0(1-a\alpha)-a\beta-b}, \qquad L_2 := \frac{u_0(1-a\alpha)-a\beta-b}{a\alpha(u_{-1}(1-a\alpha)-a\beta-b)},$$
$$L_3 := \frac{v_{-1}(1-a\alpha)-\alpha b-\beta}{v_0(1-a\alpha)-\alpha b-\beta}, \qquad L_4 := \frac{v_0(1-a\alpha)-\alpha b-\beta}{a\alpha(v_{-1}(1-a\alpha)-\alpha b-\beta)}.$$

Finally, we give another auxiliary result.

Lemma 1 *If $a\alpha \neq 1$, $a\beta + b \neq 0 \neq \alpha b + \beta$. Then system* (14) *has two-periodic solutions.*

Proof The equilibrium solution to system (15) is

$$u_n = \bar{u} = \frac{a\beta+b}{1-a\alpha} \neq 0, \qquad v_n = \bar{v} = \frac{\alpha b+\beta}{1-a\alpha} \neq 0, \quad n \in \mathbb{N}_0. \tag{31}$$

From (4) and (31) it follows that

$$x_n = \frac{1-a\alpha}{(a\beta+b)x_{n-1}} = x_{n-2}, \quad n \in \mathbb{N}_0, \tag{32}$$

and

$$y_n = \frac{1-a\alpha}{(\alpha b+\beta)y_{n-1}} = y_{n-2}, \quad n \in \mathbb{N}_0, \tag{33}$$

as desired. □

The next three results are devoted to the long-term behavior of well-defined solutions to system (14).

Theorem 1 *Assume that* $|a\alpha| \neq 1$ *and* $(x_n, y_n)_{n\geq -2}$ *is a well-defined solution to system* (14). *Then the following statements are true.*

(a) *If* $a\beta + b \neq 0 \neq \alpha b + \beta$ *and* $|a\alpha| < 1$, *then* (x_n, y_n) *converges to a, not necessarily prime, two-periodic solution.*
(b) *If* $u_{-1} = u_0 = (a\beta + b)/(1 - a\alpha)$, *then the sequences* $(x_{2m})_{m\geq -1}$ *and* $(x_{2m+1})_{m\geq -1}$ *are constant.*
(c) *If* $v_{-1} = v_0 = (\alpha b + \beta)/(1 - a\alpha)$, *then the sequences* $(y_{2m})_{m\geq -1}$ *and* $(y_{2m+1})_{m\geq -1}$ *are constant.*
(d) *If* $|a\alpha| > 1$ *and* $u_{-1} = (a\beta + b)/(1 - a\alpha) \neq u_0$, *then* $x_{2m} \to 0$ *and* $|x_{2m+1}| \to \infty$, *as* $m \to \infty$.
(e) *If* $|a\alpha| > 1$ *and* $u_{-1} \neq (a\beta + b)/(1 - a\alpha) = u_0$, *then* $x_{2m+1} \to 0$ *and* $|x_{2m}| \to \infty$, *as* $m \to \infty$.
(f) *If* $|a\alpha| > 1$ *and* $v_{-1} = (a\beta + b)/(1 - a\alpha) \neq v_0$, *then* $y_{2m} \to 0$ *and* $|y_{2m+1}| \to \infty$, *as* $m \to \infty$.
(g) *If* $|a\alpha| > 1$ *and* $v_{-1} \neq (a\beta + b)/(1 - a\alpha) = v_0$, *then* $y_{2m+1} \to 0$ *and* $|y_{2m}| \to \infty$, *as* $m \to \infty$.
(h) *If* $|a\alpha| > 1$, $u_{-1} \neq (a\beta + b)/(1 - a\alpha) \neq u_0$ *and* $|L_1| < 1$, *then* $x_{2m} \to 0$, *as* $m \to \infty$.
(i) *If* $|a\alpha| > 1$, $u_{-1} \neq (a\beta + b)/(1 - a\alpha) \neq u_0$ *and* $|L_1| > 1$, *then* $|x_{2m}| \to \infty$, *as* $m \to \infty$.
(j) *If* $|a\alpha| > 1$, $u_{-1} \neq (a\beta + b)/(1 - a\alpha) \neq u_0$ *and* $L_1 = 1$, *then* $(x_{2m})_{m\geq -1}$ *is constant.*
(k) *If* $|a\alpha| > 1$, $u_{-1} \neq (a\beta + b)/(1 - a\alpha) \neq u_0$ *and* $L_1 = -1$, *then* $(x_{4m})_{m\geq -1}$ *and* $(x_{4m+2})_{m\geq -1}$ *are convergent.*
(l) *If* $|a\alpha| > 1$, $u_{-1} \neq (a\beta + b)/(1 - a\alpha) \neq u_0$ *and* $|L_2| < 1$, *then* $x_{2m+1} \to 0$, *as* $m \to \infty$.
(m) *If* $|a\alpha| > 1$, $u_{-1} \neq (a\beta + b)/(1 - a\alpha) \neq u_0$ *and* $|L_2| > 1$, *then* $|x_{2m+1}| \to \infty$, *as* $m \to \infty$.
(n) *If* $|a\alpha| > 1$, $u_{-1} \neq (a\beta + b)/(1 - a\alpha) \neq u_0$ *and* $L_2 = 1$, *then* $(x_{2m+1})_{m\geq -1}$ *is constant.*
(o) *If* $|a\alpha| > 1$, $u_{-1} \neq (a\beta + b)/(1 - a\alpha) \neq u_0$ *and* $L_2 = -1$, *then* $(x_{4m+1})_{m\geq -1}$ *and* $(x_{4m+3})_{m\geq -1}$ *are convergent.*
(p) *If* $|a\alpha| > 1$, $v_{-1} \neq (\alpha b + \beta)/(1 - a\alpha) \neq v_0$ *and* $|L_3| < 1$, *then* $y_{2m} \to 0$, *as* $m \to \infty$.
(q) *If* $|a\alpha| > 1$, $v_{-1} \neq (\alpha b + \beta)/(1 - a\alpha) \neq v_0$ *and* $|L_3| > 1$, *then* $|y_{2m}| \to \infty$, *as* $m \to \infty$.
(r) *If* $|a\alpha| > 1$, $v_{-1} \neq (\alpha b + \beta)/(1 - a\alpha) \neq v_0$ *and* $L_3 = 1$, *then* $(y_{2m})_{m\geq -1}$ *is constant.*
(s) *If* $|a\alpha| > 1$, $v_{-1} \neq (\alpha b + \beta)/(1 - a\alpha) \neq v_0$ *and* $L_3 = -1$, *then* $(y_{4m})_{m\geq -1}$ *and* $(y_{4m+2})_{m\geq -1}$ *are convergent.*
(t) *If* $|a\alpha| > 1$, $v_{-1} \neq (\alpha b + \beta)/(1 - a\alpha) \neq v_0$ *and* $|L_4| < 1$, *then* $y_{2m+1} \to 0$, *as* $m \to \infty$.
(u) *If* $|a\alpha| > 1$, $v_{-1} \neq (\alpha b + \beta)/(1 - a\alpha) \neq v_0$ *and* $|L_4| > 1$, *then* $|y_{2m+1}| \to \infty$, *as* $m \to \infty$.
(v) *If* $|a\alpha| > 1$, $v_{-1} \neq (\alpha b + \beta)/(1 - a\alpha) \neq v_0$ *and* $L_4 = 1$, *then* $(y_{2m+1})_{m\geq -1}$ *is constant.*
(w) *If* $|a\alpha| > 1$, $v_{-1} \neq (\alpha b + \beta)/(1 - a\alpha) \neq v_0$ *and* $L_4 = -1$, *then* $(y_{4m+1})_{m\geq -1}$ *and* $(y_{4m+3})_{m\geq -1}$ *are convergent.*

Proof Let

$$p_m = \frac{a\beta + b + (a\alpha)^m(u_{-1}(1 - a\alpha) - a\beta - b)}{a\beta + b + (a\alpha)^m(u_0(1 - a\alpha) - a\beta - b)},$$

$$\hat{p}_m = \frac{a\beta + b + (a\alpha)^m(u_0(1 - a\alpha) - a\beta - b)}{a\beta + b + (a\alpha)^{m+1}(u_{-1}(1 - a\alpha) - a\beta - b)},$$

$$q_m = \frac{\alpha b + \beta + (a\alpha)^m(v_{-1}(1-a\alpha) - \alpha b - \beta)}{\alpha b + \beta + (a\alpha)^m(v_0(1-a\alpha) - \alpha b - \beta)},$$
$$\hat{q}_m = \frac{\alpha b + \beta + (a\alpha)^m(v_0(1-a\alpha) - \alpha b - \beta)}{\alpha b + \beta + (a\alpha)^{m+1}(v_{-1}(1-a\alpha) - \alpha b - \beta)}$$

for $m \in \mathbb{N}_0$.

(a) By using (30) we have

$$\begin{aligned} p_m &= \frac{1 + (a\alpha)^m(u_{-1}(1-a\alpha) - a\beta - b)(a\beta + b)^{-1}}{1 + (a\alpha)^m(u_0(1-a\alpha) - a\beta - b)(a\beta + b)^{-1}} \\ &= 1 + (u_{-1} - u_0)(1-a\alpha)(a\beta + b)^{-1}(a\alpha)^m + o\big((a\alpha)^m\big), \end{aligned} \tag{34}$$

$$\begin{aligned} \hat{p}_m &= \frac{1 + (a\alpha)^m(u_0(1-a\alpha) - a\beta - b)(a\beta + b)^{-1}}{1 + (a\alpha)^{m+1}(u_{-1}(1-a\alpha) - a\beta - b)(a\beta + b)^{-1}} \\ &= 1 + \frac{(1-a\alpha)(u_0 - a\alpha u_{-1} - a\beta - b)}{a\beta + b}(a\alpha)^m + o\big((a\alpha)^m\big), \end{aligned} \tag{35}$$

$$\begin{aligned} q_m &= \frac{1 + (a\alpha)^m(v_{-1}(1-a\alpha) - \alpha b - \beta)(\alpha b + \beta)^{-1}}{1 + (a\alpha)^m(v_0(1-a\alpha) - \alpha b - \beta)(\alpha b + \beta)^{-1}} \\ &= 1 + (v_{-1} - v_0)(1-a\alpha)(\alpha b + \beta)^{-1}(a\alpha)^m + o\big((a\alpha)^m\big), \end{aligned} \tag{36}$$

$$\begin{aligned} \hat{q}_m &= \frac{1 + (a\alpha)^m(v_0(1-a\alpha) - \alpha b - \beta)(\alpha b + \beta)^{-1}}{1 + (a\alpha)^{m+1}(v_{-1}(1-a\alpha) - \alpha b - \beta)(\alpha b + \beta)^{-1}} \\ &= 1 + \frac{(1-a\alpha)(v_0 - a\alpha v_{-1} - \alpha b - \beta)}{\alpha b + \beta}(a\alpha)^m + o\big((a\alpha)^m\big) \end{aligned} \tag{37}$$

for sufficiently large m.

From (34)-(37), by using the condition $|a\alpha| < 1$ and a well-known criterion for the convergence of products, the statement easily follows.

(b) By using the condition $u_{-1} = u_0 = (a\beta + b)/(1 - a\alpha)$ in (22) and (23), the statement immediately follows.

(c) By using the condition $v_{-1} = v_0 = (\alpha b + \beta)/(1 - a\alpha)$ in (24) and (25), the statement immediately follows.

(d) By using the condition $u_{-1} = (a\beta + b)/(1 - a\alpha) \neq u_0$, we get

$$p_m = \frac{a\beta + b}{a\beta + b + (a\alpha)^m(u_0(1-a\alpha) - a\beta - b)}, \tag{38}$$

$$\hat{p}_m = \frac{a\beta + b + (a\alpha)^m(u_0(1-a\alpha) - a\beta - b)}{a\beta + b}. \tag{39}$$

Letting $m \to \infty$ in (38) and (39) and using the condition $|a\alpha| > 1$, we have $p_m \to 0$ and $|\hat{p}_m| \to \infty$, from which along with (22) and (23) the statement easily follows.

(e) By using the condition $u_{-1} \neq (a\beta + b)/(1 - a\alpha) = u_0$, we get

$$p_m = \frac{a\beta + b + (a\alpha)^m(u_{-1}(1-a\alpha) - a\beta - b)}{a\beta + b}, \tag{40}$$

$$\hat{p}_m = \frac{a\beta + b}{a\beta + b + (a\alpha)^{m+1}(u_{-1}(1-a\alpha) - a\beta - b)}. \tag{41}$$

Letting $m \to \infty$ in (40) and (41) and using the condition $|a\alpha| > 1$, we have $|p_m| \to \infty$ and $\hat{p}_m \to 0$, from which along with (22) and (23) the statement easily follows.

(f) By using the condition $v_{-1} = (a\beta + b)/(1 - a\alpha) \neq v_0$, we get

$$q_m = \frac{\alpha b + \beta}{\alpha b + \beta + (a\alpha)^m (v_0(1 - a\alpha) - \alpha b - \beta)}, \tag{42}$$

$$\hat{q}_m = \frac{\alpha b + \beta + (a\alpha)^m (v_0(1 - a\alpha) - \alpha b - \beta)}{\alpha b + \beta}. \tag{43}$$

Letting $m \to \infty$ in (42) and (43) and using the condition $|a\alpha| > 1$, we have $q_m \to 0$ and $|\hat{q}_m| \to \infty$, from which along with (24) and (25) the statement easily follows.

(g) By using the condition $v_{-1} \neq (a\beta + b)/(1 - a\alpha) = v_0$, we get

$$q_m = \frac{\alpha b + \beta + (a\alpha)^m (v_{-1}(1 - a\alpha) - \alpha b - \beta)}{\alpha b + \beta}, \tag{44}$$

$$\hat{q}_m = \frac{\alpha b + \beta}{\alpha b + \beta + (a\alpha)^{m+1} (v_{-1}(1 - a\alpha) - \alpha b - \beta)}. \tag{45}$$

Letting $m \to \infty$ in (44) and (45) and using the condition $|a\alpha| > 1$, we have $|q_m| \to \infty$ and $\hat{q}_m \to 0$, from which along with (24) and (25) the statement easily follows.

(h), (i) Note that $\lim_{m\to\infty} p_m = L_1$. Hence, from the assumptions $|L_1| < 1$, that is, $|L_1| > 1$ along with (22), the statements easily follow.

(j) The statement immediately follows by using the condition $L_1 = 1$ in (22).

(k) Since $L_1 = -1$ and by using (30), we have that

$$\begin{aligned} p_m &= \frac{a\beta + b + (a\alpha)^m (u_{-1}(1 - a\alpha) - a\beta - b)}{a\beta + b - (a\alpha)^m (u_{-1}(1 - a\alpha) - a\beta - b)} \\ &= -\frac{1 + \frac{a\beta + b}{(a\alpha)^m (u_{-1}(1 - a\alpha) - a\beta - b)}}{1 - \frac{a\beta + b}{(a\alpha)^m (u_{-1}(1 - a\alpha) - a\beta - b)}} \\ &= -\left(1 + \frac{2(a\beta + b)}{(a\alpha)^m (u_{-1}(1 - a\alpha) - a\beta - b)} + o\left(\frac{1}{(a\alpha)^m}\right)\right). \end{aligned} \tag{46}$$

From (46), by using the condition $|a\alpha| > 1$ and a well-known criterion for the convergence of products, the statement easily follows.

(l), (m) Note that $\lim_{m\to\infty} \hat{p}_m = L_2$. Hence, from the assumptions $|L_2| < 1$, that is, $|L_2| > 1$ along with (23), the statements easily follow.

(n) The statement immediately follows by using the condition $L_2 = 1$ in (23).

(o) Since $L_2 = -1$ and by using (30), we have that

$$\begin{aligned} \hat{p}_m &= \frac{a\beta + b + (a\alpha)^m (u_0(1 - a\alpha) - a\beta - b)}{a\beta + b - (a\alpha)^m (u_0(1 - a\alpha) - a\beta - b)} \\ &= -\frac{1 + \frac{a\beta + b}{(a\alpha)^m (u_0(1 - a\alpha) - a\beta - b)}}{1 - \frac{a\beta + b}{(a\alpha)^m (u_0(1 - a\alpha) - a\beta - b)}} \\ &= -\left(1 + \frac{2(a\beta + b)}{(a\alpha)^m (u_0(1 - a\alpha) - a\beta - b)} + o\left(\frac{1}{(a\alpha)^m}\right)\right). \end{aligned} \tag{47}$$

From (47), by using the condition $|a\alpha| > 1$ and a well-known criterion for the convergence of products, the statement easily follows.

(p), (q) Note that $\lim_{m\to\infty} q_m = L_3$. Hence, from the assumptions $|L_3| < 1$, that is, $|L_3| > 1$ along with (24), the statements easily follow.

(r) The statement immediately follows by using the condition $L_3 = 1$ in (24).

(s) Since $L_3 = -1$ and by using (30), we have that

$$\begin{aligned} q_m &= \frac{\alpha b+\beta+(a\alpha)^m(v_{-1}(1-a\alpha)-\alpha b-\beta)}{\alpha b+\beta-(a\alpha)^m(v_{-1}(1-a\alpha)-\alpha b-\beta)} \\ &= -\frac{1+\frac{\alpha b+\beta}{(a\alpha)^m(v_{-1}(1-a\alpha)-\alpha b-\beta)}}{1-\frac{\alpha b+\beta}{(a\alpha)^m(v_{-1}(1-a\alpha)-\alpha b-\beta)}} \\ &= -\left(1+\frac{2(\alpha b+\beta)}{(a\alpha)^m(v_{-1}(1-a\alpha)-\alpha b-\beta)}+o\left(\frac{1}{(a\alpha)^m}\right)\right). \end{aligned} \tag{48}$$

From (48), by using the condition $|a\alpha| > 1$ and a well-known criterion for the convergence of products, the statement easily follows.

(t), (u) Note that $\lim_{m\to\infty} \hat{q}_m = L_4$. Hence, from the assumptions $|L_4| < 1$, that is, $|L_4| > 1$ along with (25), the statements easily follow.

(v) The statement immediately follows by using the condition $L_4 = 1$ in (25).

(w) Since $L_4 = -1$ and by using (30), we have that

$$\begin{aligned} \hat{q}_m &= \frac{\alpha b+\beta+(a\alpha)^m(v_0(1-a\alpha)-\alpha b-\beta)}{\alpha b+\beta-(a\alpha)^m(v_0(1-a\alpha)-\alpha b-\beta)} \\ &= -\frac{1+\frac{\alpha b+\beta}{(a\alpha)^m(v_0(1-a\alpha)-\alpha b-\beta)}}{1-\frac{\alpha b+\beta}{(a\alpha)^m(v_0(1-a\alpha)-\alpha b-\beta)}} \\ &= -\left(1+\frac{2(\alpha b+\beta)}{(a\alpha)^m(v_0(1-a\alpha)-\alpha b-\beta)}+o\left(\frac{1}{(a\alpha)^m}\right)\right). \end{aligned} \tag{49}$$

From (49), by using the condition $|a\alpha| > 1$ and a well-known criterion for the convergence of products, the statement easily follows. □

Let

$$M_1 := \frac{u_{-1}(u_{-1}-b-a\beta)}{u_0(u_0-b-a\beta)}, \qquad M_2 := \frac{v_{-1}(v_{-1}-\beta-\alpha b)}{v_0(v_0-\beta-\alpha b)}.$$

Theorem 2 *Assume that $a\alpha = -1$ and $(x_n, y_n)_{n\ge -2}$ is a well-defined solution to system* (14). *Then the following statements are true.*

(a) *If $|M_1| < 1$, then $x_{2m} \to 0$ and $|x_{2m+1}| \to \infty$, as $m \to \infty$.*
(b) *If $|M_1| > 1$, then $x_{2m+1} \to 0$ and $|x_{2m}| \to \infty$, as $m \to \infty$.*
(c) *If $M_1 = 1$, then $(x_n)_{n\ge -2}$ is four-periodic.*
(d) *If $M_1 = -1$, then $(x_n)_{n\ge -2}$ is eight-periodic.*
(e) *If $|M_2| < 1$, then $y_{2m} \to 0$ and $|y_{2m+1}| \to \infty$, as $m \to \infty$.*
(f) *If $|M_2| > 1$, then $y_{2m+1} \to 0$ and $|y_{2m}| \to \infty$, as $m \to \infty$.*
(g) *If $M_2 = 1$, then $(y_n)_{n\ge -2}$ is four-periodic.*
(h) *If $M_2 = -1$, then $(y_n)_{n\ge -2}$ is eight-periodic.*

Proof First, note that since $a\alpha = -1$, from (22)-(25) we have

$$x_{4m} = x_0 M_1^m, \qquad x_{4m+2} = x_{-2} M_1^{m+1}, \qquad x_{4m+1} = \frac{x_1}{M_1^m}, \qquad x_{4m+3} = \frac{x_{-1}}{M_1^{m+1}}, \tag{50}$$

$$y_{4m} = y_0 M_2^m, \qquad y_{4m+2} = y_{-2} M_2^{m+1}, \qquad y_{4m+1} = \frac{y_1}{M_2^m}, \qquad y_{4m+3} = \frac{y_{-1}}{M_2^{m+1}}, \tag{51}$$

for $m \in \mathbb{N}_0$. From (50) and (51) all the statements easily follow. □

Let

$$N_1 := \frac{u_{-1}}{u_0}, \qquad N_2 := \frac{v_{-1}}{v_0}.$$

Theorem 3 *Assume that $a\alpha = 1$ and $(x_n, y_n)_{n\ge -2}$ is a well-defined solution to system* (14). *Then the following statements hold true.*

(a) *If $a\beta + b = 0$ and $|N_1| < 1$, then $x_{2m} \to 0$ and $|x_{2m+1}| \to \infty$, as $m \to \infty$;*
(b) *If $a\beta + b = 0$ and $|N_1| > 1$, then $|x_{2m}| \to \infty$ and $x_{2m+1} \to 0$, as $m \to \infty$;*
(c) *If $a\beta + b = 0$ and $N_1 = 1$, then $(x_{2m})_{m\ge -1}$ and $(x_{2m+1})_{m\ge -1}$ are constant;*
(d) *If $a\beta + b = 0$ and $N_1 = -1$, then $(x_{4m+i})_{m\ge -1}$, $i = \overline{0,3}$, are constant.*
(e) *If $a\beta + b \ne 0$ and $(u_{-1} - u_0)/(a\beta + b) > 0$, then $|x_{2m}| \to \infty$, as $m \to \infty$;*
(f) *If $a\beta + b \ne 0$ and $(u_{-1} - u_0)/(a\beta + b) < 0$, then $x_{2m} \to 0$, as $m \to \infty$;*
(g) *If $a\beta + b \ne 0$ and $u_{-1} = u_0$, then $(x_{2m})_{m\ge -1}$ is constant;*
(h) *If $a\beta + b \ne 0$ and $(u_0 - u_{-1})/(a\beta + b) > 1$, then $|x_{2m+1}| \to \infty$, as $m \to \infty$;*
(i) *If $a\beta + b \ne 0$ and $(u_0 - u_{-1})/(a\beta + b) < 1$, then $x_{2m+1} \to 0$, as $m \to \infty$;*
(j) *If $a\beta + b \ne 0$ and $u_{-1} - u_0 = a\beta + b$, then $(x_{2m+1})_{m\ge -1}$ is constant;*
(k) *If $\alpha b + \beta = 0$ and $|N_2| < 1$, then $y_{2m} \to 0$ and $|y_{2m+1}| \to \infty$, as $m \to \infty$;*
(l) *If $\alpha b + \beta = 0$ and $|N_2| > 1$, then $|y_{2m}| \to \infty$ and $y_{2m+1} \to 0$, as $m \to \infty$;*
(m) *If $\alpha b + \beta = 0$ and $N_2 = 1$, then $(y_{2m})_{m\ge -1}$ and $(y_{2m+1})_{m\ge -1}$ are constant;*
(n) *If $\alpha b + \beta = 0$ and $N_2 = -1$, then $(y_{4m+i})_{m\ge -1}$, $i = \overline{0,3}$, are constant.*
(o) *If $\alpha b + \beta \ne 0$ and $(v_{-1} - v_0)/(\alpha b + \beta) > 0$, then $|y_{2m}| \to \infty$, as $m \to \infty$;*
(p) *If $\alpha b + \beta \ne 0$ and $(v_{-1} - v_0)/(\alpha b + \beta) < 0$, then $y_{2m} \to 0$, as $m \to \infty$;*
(q) *If $\alpha b + \beta \ne 0$ and $v_{-1} = v_0$, then $(y_{2m})_{m\ge -1}$ is constant.*
(r) *If $\alpha b + \beta \ne 0$ and $(v_0 - v_{-1})/(\alpha b + \beta) < 1$, then $y_{2m+1} \to 0$, as $m \to \infty$;*
(s) *If $\alpha b + \beta \ne 0$ and $(v_0 - v_{-1})/(\alpha b + \beta) > 1$, then $|y_{2m+1}| \to \infty$, as $m \to \infty$;*
(t) *If $\alpha b + \beta \ne 0$ and $v_{-1} - v_0 = \alpha b + \beta$, then $(y_{2m+1})_{m\ge -1}$ is constant.*

Proof Let

$$r_m = \frac{u_{-1} + (a\beta + b)m}{u_0 + (a\beta + b)m}, \qquad \hat{r}_m = \frac{u_0 + (a\beta + b)m}{u_{-1} + (a\beta + b)(m+1)},$$
$$s_m = \frac{v_{-1} + (\alpha b + \beta)m}{v_0 + (\alpha b + \beta)m}, \qquad \hat{s}_m = \frac{v_0 + (\alpha b + \beta)m}{v_{-1} + (\alpha b + \beta)(m+1)}, \quad m \in \mathbb{N}_0.$$

(a)-(d) Since in this case we have

$$x_{2m} = x_{-2}\left(\frac{u_{-1}}{u_0}\right)^{m+1}, \qquad x_{2m+1} = x_{-1}\left(\frac{u_0}{u_{-1}}\right)^{m+1}, \quad m \in \mathbb{N}_0,$$

these statements easily follow.

(e), (f) By using (30) we have

$$\begin{aligned} r_m &= \frac{u_{-1} + (a\beta + b)m}{u_0 + (a\beta + b)m} = \left(1 + \frac{u_{-1}}{(a\beta + b)m}\right)\left(1 + \frac{u_0}{(a\beta + b)m}\right)^{-1} \\ &= \left(1 + \frac{u_{-1}}{(a\beta + b)m} + O\left(\frac{1}{m^2}\right)\right)\left(1 - \frac{u_0}{(a\beta + b)m} + O\left(\frac{1}{m^2}\right)\right) \\ &= 1 + \frac{u_{-1} - u_0}{(a\beta + b)m} + O\left(\frac{1}{m^2}\right) \end{aligned} \tag{52}$$

for sufficiently large m.

From (52), by using the fact that for every $k \in \mathbb{N}$

$$\sum_{j=k}^{m} \frac{1}{j} \to \infty, \quad \text{as } m \to \infty, \tag{53}$$

and a known criterion for convergence of products, the statements easily follow.

(g) Using the condition $u_{-1} = u_0$ in (26), the statement immediately follows.

(h), (i) By using (30) we have

$$\begin{aligned} \hat{r}_m &= \frac{u_0 + (a\beta + b)m}{u_{-1} + (a\beta + b)(m + 1)} = \left(1 + \frac{u_0}{(a\beta + b)m}\right)\left(1 + \frac{u_{-1} + a\beta + b}{(a\beta + b)m}\right)^{-1} \\ &= \left(1 + \frac{u_0}{(a\beta + b)m}\right)\left(1 - \frac{u_{-1} + a\beta + b}{(a\beta + b)m} + O\left(\frac{1}{m^2}\right)\right) \\ &= 1 + \frac{u_0 - u_{-1} - a\beta - b}{(a\beta + b)m} + O\left(\frac{1}{m^2}\right) \end{aligned} \tag{54}$$

for sufficiently large m.

From (54), (53), (27) and a known criterion for convergence of products, the statements easily follow.

(j) Using the condition $u_0 = u_{-1} + a\beta + b$ in (27), the statement immediately follows.

(k)-(n) Since in this case we have

$$y_{2m} = y_{-2}\left(\frac{v_{-1}}{v_0}\right)^{m+1}, \qquad y_{2m+1} = y_{-1}\left(\frac{v_0}{v_{-1}}\right)^{m+1}, \quad m \in \mathbb{N}_0,$$

these statements easily follow.

(o), (p) By using (30) we have

$$\begin{aligned} s_m &= \frac{v_{-1} + (\alpha b + \beta)m}{v_0 + (\alpha b + \beta)m} = \left(1 + \frac{v_{-1}}{(\alpha b + \beta)m}\right)\left(1 + \frac{v_0}{(\alpha b + \beta)m}\right)^{-1} \\ &= \left(1 + \frac{v_{-1}}{(\alpha b + \beta)m}\right)\left(1 - \frac{v_0}{(\alpha b + \beta)m} + O\left(\frac{1}{m^2}\right)\right) \\ &= 1 + \frac{v_{-1} - v_0}{(\alpha b + \beta)m} + O\left(\frac{1}{m^2}\right) \end{aligned} \tag{55}$$

for sufficiently large m.

From (55), (53), (28) and a known criterion for convergence of products, the statements easily follow.

(q) Using the condition $\nu_0 = \nu_{-1}$ in (28), the statement immediately follows.

(r), (s) By using (30) we have

$$\begin{aligned}\hat{s}_m &= \frac{\nu_0 + (\alpha b + \beta)m}{\nu_{-1} + (\alpha b + \beta)(m+1)} = \left(1 + \frac{\nu_0}{(\alpha b + \beta)m}\right)\left(1 + \frac{\nu_{-1} + \alpha b + \beta}{(\alpha b + \beta)m}\right)^{-1} \\ &= \left(1 + \frac{\nu_0}{(\alpha b + \beta)m}\right)\left(1 - \frac{\nu_{-1} + \alpha b + \beta}{(\alpha b + \beta)m} + O\left(\frac{1}{m^2}\right)\right) \\ &= 1 + \frac{\nu_0 - \nu_{-1} - \alpha b - \beta}{(\alpha b + \beta)m} + O\left(\frac{1}{m^2}\right) \qquad (56)\end{aligned}$$

for sufficiently large m.

From (56), (53), (29) and a known criterion for convergence of products, the statements easily follow.

(t) Using the condition $\nu_0 = \nu_{-1} + \alpha b + \beta$ in (29), the statement immediately follows. □

5 Domain of undefinable solutions to system (2)

In Section 2 we proved that solutions to system (2), for which $x_{-j} = 0$ or $y_{-j} = 0$ for some $j \in \{1, 2\}$, are not defined. The set of all such initial values is characterized here.

Definition 1 Consider the system of difference equations

$$\begin{aligned}x_n &= f(x_{n-1}, \dots, x_{n-s}, y_{n-1}, \dots, y_{n-s}, n), \\ y_n &= g(x_{n-1}, \dots, x_{n-s}, y_{n-1}, \dots, y_{n-s}, n), \quad n \in \mathbb{N}_0,\end{aligned} \qquad (57)$$

where $s \in \mathbb{N}$, and $x_{-i}, y_{-i} \in \mathbb{R}$, $i = \overline{1, s}$. The string of vectors

$$(x_{-s}, y_{-s}), \dots, (x_{-1}, y_{-1}), (x_0, y_0), \dots, (x_{n_0}, y_{n_0}),$$

where $n_0 \geq -1$, is called an *undefined solution* of system (57) if

$$x_j = f(x_{j-1}, \dots, x_{j-s}, y_{j-1}, \dots, y_{j-s}, j)$$

and

$$y_j = g(x_{j-1}, \dots, x_{j-s}, y_{j-1}, \dots, y_{j-s}, j)$$

for $0 \leq j < n_0 + 1$, and x_{n_0+1} or y_{n_0+1} is not a defined number, that is, the quantity

$$f(x_{n_0}, \dots, x_{n_0-s+1}, y_{n_0}, \dots, y_{n_0-s+1}, n_0 + 1)$$

or

$$g(x_{n_0}, \dots, x_{n_0-s+1}, y_{n_0}, \dots, y_{n_0-s+1}, n_0 + 1)$$

is not defined.

The set of all initial values $(x_{-s}, y_{-s}), \dots, (x_{-1}, y_{-1})$ which generate undefined solutions to system (57) is called *domain of undefinable solutions* of the system.

The next result characterizes the domain of undefinable solutions to system (2) when $a_n b_n \alpha_n \beta_n \neq 0$, $n \in \mathbb{N}_0$.

Theorem 4 *Assume that* $a_n b_n \alpha_n \beta_n \neq 0$, $n \in \mathbb{N}_0$. *Then the domain of undefinable solutions to system* (2) *is the following set*:

$$\begin{aligned}
\mathcal{U} = \bigcup_{m\in\mathbb{N}_0} \Bigg\{ & (x_{-2}, x_{-1}, y_{-2}, y_{-1}) \in \mathbb{R}^4 : \\
& \frac{1}{x_{-1}x_{-2}} = g_0^{-1} \circ f_1^{-1} \circ \cdots \circ g_{2m-2}^{-1} \circ f_{2m-1}^{-1} \circ g_{2m}^{-1} \circ f_{2m+1}^{-1}(0) \\
& \textit{or } \frac{1}{x_{-1}x_{-2}} = g_0^{-1} \circ f_1^{-1} \circ \cdots \circ g_{2m-2}^{-1} \circ f_{2m-1}^{-1} \circ g_{2m}^{-1}(0) \\
& \textit{or } \frac{1}{y_{-1}y_{-2}} = f_0^{-1} \circ g_1^{-1} \circ \cdots \circ f_{2m-2}^{-1} \circ g_{2m-1}^{-1} \circ f_{2m}^{-1}(0) \\
& \textit{or } \frac{1}{y_{-1}y_{-2}} = f_0^{-1} \circ g_1^{-1} \circ \cdots \circ g_{2m-1}^{-1} \circ f_{2m}^{-1} \circ g_{2m+1}^{-1}(0) \Bigg\} \\
& \cup \Big\{ (x_{-2}, x_{-1}, y_{-2}, y_{-1}) \in \mathbb{R}^4 : \\
& x_{-2} = 0 \textit{ or } x_{-1} = 0 \textit{ or } y_{-2} = 0 \textit{ or } y_{-1} = 0 \Big\},
\end{aligned} \tag{58}$$

where

$$f_n(t) = a_n t + b_n, \qquad g_n(t) = \alpha_n t + \beta_n, \quad n \in \mathbb{N}_0.$$

Proof We have already proved that the set

$$\big\{ (x_{-2}, x_{-1}, y_{-2}, y_{-1}) \in \mathbb{R}^4 : x_{-2} = 0 \text{ or } x_{-1} = 0 \text{ or } y_{-2} = 0 \text{ or } y_{-1} = 0 \big\}$$

belongs to the domain of undefinable solutions to system (2).

If $x_{-j} \neq 0 \neq y_{-j}$, $j = \overline{1,2}$ (*i.e.*, $x_n \neq 0 \neq y_n$ for every $n \geq -2$), then such a solution $(x_n, y_n)_{n\geq-2}$ is not defined if and only if

$$a_n + b_n y_{n-1} y_{n-2} = 0 \quad \text{or} \quad \alpha_n + \beta_n x_{n-1} x_{n-2} = 0 \tag{59}$$

for some $n \in \mathbb{N}_0$, which is equivalent to

$$v_{n-1} = -b_n/a_n \quad \text{or} \quad u_{n-1} = -\beta_n/\alpha_n \tag{60}$$

for some $n \in \mathbb{N}_0$.

Note that

$$f_n^{-1}(0) = -b_n/a_n \quad \text{and} \quad g_n^{-1}(0) = -\beta_n/\alpha_n, \quad n \in \mathbb{N}_0. \tag{61}$$

We have

$$v_{2m-1} = (g_{2m-1} \circ f_{2m-2} \circ \cdots \circ f_2 \circ g_1 \circ f_0)(v_{-1}), \tag{62}$$

$$v_{2m} = (g_{2m} \circ f_{2m-1} \circ \cdots \circ g_2 \circ f_1 \circ g_0)(u_{-1}), \tag{63}$$

$$u_{2m-1} = (f_{2m-1} \circ g_{2m-2} \circ \cdots \circ g_2 \circ f_1 \circ g_0)(u_{-1}), \tag{64}$$

$$u_{2m} = (f_{2m} \circ g_{2m-1} \circ \cdots \circ f_2 \circ g_1 \circ f_0)(v_{-1}) \tag{65}$$

for $m \in \mathbb{N}_0$.

From (61) and (62) we have that

$$-\frac{b_{2m}}{a_{2m}} = v_{2m-1} = (g_{2m-1} \circ f_{2m-2} \circ \cdots \circ f_2 \circ g_1 \circ f_0)(v_{-1})$$

for some $m \in \mathbb{N}_0$ if and only if

$$\frac{1}{y_{-1}y_{-2}} = f_0^{-1} \circ g_1^{-1} \circ \cdots \circ f_{2m-2}^{-1} \circ g_{2m-1}^{-1} \circ f_{2m}^{-1}(0). \tag{66}$$

From (61) and (63) we have that

$$-\frac{b_{2m+1}}{a_{2m+1}} = v_{2m} = (g_{2m} \circ f_{2m-1} \circ \cdots \circ g_2 \circ f_1 \circ g_0)(u_{-1})$$

for some $m \in \mathbb{N}_0$ if and only if

$$\frac{1}{x_{-1}x_{-2}} = g_0^{-1} \circ f_1^{-1} \circ \cdots \circ g_{2m-2}^{-1} \circ f_{2m-1}^{-1} \circ g_{2m}^{-1} \circ f_{2m+1}^{-1}(0). \tag{67}$$

From (61) and (64) we have that

$$-\frac{\beta_{2m}}{\alpha_{2m}} = u_{2m-1} = (f_{2m-1} \circ g_{2m-2} \circ \cdots \circ g_2 \circ f_1 \circ g_0)(u_{-1})$$

for some $m \in \mathbb{N}_0$ if and only if

$$\frac{1}{x_{-1}x_{-2}} = g_0^{-1} \circ f_1^{-1} \circ \cdots \circ g_{2m-2}^{-1} \circ f_{2m-1}^{-1} \circ g_{2m}^{-1}(0). \tag{68}$$

From (61) and (65) we have that

$$-\frac{\beta_{2m+1}}{\alpha_{2m+1}} = u_{2m} = (f_{2m} \circ g_{2m-1} \circ \cdots \circ f_2 \circ g_1 \circ f_0)(v_{-1})$$

for some $m \in \mathbb{N}_0$ if and only if

$$\frac{1}{y_{-1}y_{-2}} = f_0^{-1} \circ g_1^{-1} \circ \cdots \circ g_{2m-1}^{-1} \circ f_{2m}^{-1} \circ g_{2m+1}^{-1}(0). \tag{69}$$

From (66)-(69) we see that the first union in (58) also belongs to the domain of undefinable solutions, finishing the proof of the theorem. □

Remark 1 Quantities

$$g_0^{-1}\circ f_1^{-1}\circ\cdots\circ g_{2m-2}^{-1}\circ f_{2m-1}^{-1}\circ g_{2m}^{-1}\circ f_{2m+1}^{-1}(0), \tag{70}$$

$$g_0^{-1}\circ f_1^{-1}\circ\cdots\circ g_{2m-2}^{-1}\circ f_{2m-1}^{-1}\circ g_{2m}^{-1}(0), \tag{71}$$

$$f_0^{-1}\circ g_1^{-1}\circ\cdots\circ f_{2m-2}^{-1}\circ g_{2m-1}^{-1}\circ f_{2m}^{-1}(0), \tag{72}$$

$$f_0^{-1}\circ g_1^{-1}\circ\cdots\circ g_{2m-1}^{-1}\circ f_{2m}^{-1}\circ g_{2m+1}^{-1}(0) \tag{73}$$

can be calculated for every $m\in\mathbb{N}_0$.

Indeed, note that

$$g_0^{-1}\circ f_1^{-1}\circ\cdots\circ g_{2m-2}^{-1}\circ f_{2m-1}^{-1}\circ g_{2m}^{-1}\circ f_{2m+1}^{-1}(0)=\left(\prod_{j=0}^{m}\bigl(g_{2j}^{-1}\circ f_{2j+1}^{-1}\bigr)\right)(t)\Big|_{t=0}, \tag{74}$$

$$g_0^{-1}\circ f_1^{-1}\circ\cdots\circ g_{2m-2}^{-1}\circ f_{2m-1}^{-1}\circ g_{2m}^{-1}(0)=\left(\prod_{j=0}^{m-1}\bigl(g_{2j}^{-1}\circ f_{2j+1}^{-1}\bigr)\right)(t)\Big|_{t=g_{2m}^{-1}(0)}, \tag{75}$$

$$f_0^{-1}\circ g_1^{-1}\circ\cdots\circ f_{2m-2}^{-1}\circ g_{2m-1}^{-1}\circ f_{2m}^{-1}(0)=\left(\prod_{j=0}^{m-1}\bigl(f_{2j}^{-1}\circ g_{2j+1}^{-1}\bigr)\right)(t)\Big|_{t=f_{2m}^{-1}(0)}, \tag{76}$$

$$f_0^{-1}\circ g_1^{-1}\circ\cdots\circ g_{2m-1}^{-1}\circ f_{2m}^{-1}\circ g_{2m+1}^{-1}(0)=\left(\prod_{j=0}^{m}\bigl(f_{2j}^{-1}\circ g_{2j+1}^{-1}\bigr)\right)(t)\Big|_{t=0}, \tag{77}$$

and also that

$$\bigl(g_{2j}^{-1}\circ f_{2j+1}^{-1}\bigr)(t)=\frac{t}{\alpha_{2j}a_{2j+1}}-\frac{b_{2j+1}}{\alpha_{2j}a_{2j+1}}-\frac{\beta_{2j}}{\alpha_{2j}},\quad j\in\mathbb{N}_0, \tag{78}$$

and

$$\bigl(f_{2j}^{-1}\circ g_{2j+1}^{-1}\bigr)(t)=\frac{t}{a_{2j}\alpha_{2j+1}}-\frac{\beta_{2j+1}}{a_{2j}\alpha_{2j+1}}-\frac{b_{2j}}{a_{2j}},\quad j\in\mathbb{N}_0. \tag{79}$$

On the other hand, if

$$h_j(t)=c_jt+d_j,\quad j\in\mathbb{N}_0,$$

it is easy to see that

$$(h_0\circ h_1\circ\cdots\circ h_n)(t)=\left(\prod_{j=0}^{n}c_j\right)t+\sum_{i=0}^{n}d_i\prod_{j=0}^{i-1}c_j,\quad n\in\mathbb{N}_0. \tag{80}$$

From (74)-(80) explicit formulas for the quantities in (70)-(73) are easily obtained.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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