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# Periodic solutions of a generalized system of piecewise linear difference equations

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**Abstract**
In this article we consider the global behavior of the following system of piecewise linear difference equations: $x_{n+1} = |x_n| - y_n - b$ and $y_{n+1} = x_n + |y_n|$ where $b$ is any positive real number and the initial condition $(x_0, y_0)$ is an element of $R^2$. By mathematical induction and direct computations we show that the solution to the system is eventually one of two particular prime period 3 solutions or the unique equilibrium solution.

**Keywords:** difference equations; periodic solutions; equilibrium solution

## 1 Introduction

Recently there has been a surge of interest in systems of both rational and piecewise linear difference equations due to their practical applications in evolutionary biology, neural networks, and ecology [1–3]. There has been particular interest in the global behavior of systems of piecewise linear difference equations [1, 4–7].

We continue the trend by generalizing one of our recent papers [8]. In this paper we consider the behavior of the generalized system of piecewise linear difference equations,

$$\begin{cases} x_{n+1} = |x_n| - y_n - b, \\ y_{n+1} = x_n + |y_n|, \end{cases} \qquad n = 0, 1, \ldots,$$

where the parameter $b \in (0, \infty)$ and the initial condition $(x_0, y_0) \in R^2$. This system originated from a larger project at the University of Rhode Island which involves the following family of systems:

$$\begin{cases} x_{n+1} = |x_n| + ay_n + b, \\ y_{n+1} = x_n + c|y_n| + d, \end{cases} \qquad n = 0, 1, \ldots,$$

where the parameters $a, b, c, d \in \{-1, 0, 1\}$ and the initial condition $(x_0, y_0) \in R^2$. Each of the 81 systems in this family is designated a number $N$, where $N = 27(a + 1) + 9(b + 1) + 3(c + 1) + (d + 1) + 1$. The system considered in this paper is a generalized form of system number 8. Related work has been given recently in [8, 9].

Interest in the area began in 1984 when Devaney published his famous paper introducing the Gingerbreadman map:

$$\begin{cases} x_{n+1} = |x_n| - y_n + 1, \\ y_{n+1} = x_n, \end{cases} \qquad n = 0, 1, \ldots,$$

with initial the condition $(-0.1, 0)$. See [2, 3]. The Gingerbreadman map was Devaney's response to the 1978 generalized Lozi equation. The Lozi equation,

$$\begin{cases} x_{n+1} = -a|x_n| + y_n + 1, \\ y_{n+1} = bx_n, \end{cases} \qquad n = 0, 1, \ldots,$$

with parameters $a, b \in R$ and the initial condition $(x_0, y_0) \in R^2$, had been used to examine an attractor that was observed by Lorenz in the Hénon map, a non-linear system of difference equations,

$$\begin{cases} x_{n+1} = -ax_n^2 + y_n + 1, \\ y_{n+1} = bx_n, \end{cases} \qquad n = 0, 1, \ldots,$$

with parameters $a, b \in R$ and the initial condition $(x_0, y_0) \in R^2$ that modeled weather patterns [10–12]. We believe that our family of piecewise linear difference equations are the prototypes for more elaborate piecewise difference equations that, in many cases as in the Lozi equation, exhibit complicated behavior.

## 2 Preliminaries

The following definitions [2] are used in this paper. A *difference equation of the first order* is an equation of the form

$$x_{n+1} = f(x_n), \quad n = 0, 1, \ldots, \tag{1}$$

where $f$ is a continuous function which maps some set $J$ into $J$. The set $J$ is usually an interval of real numbers, or a union of intervals, but it may even be a discrete set such as the set of integers.

A *solution* of (1) is a sequence $\{x_n\}_{n=0}^{\infty}$ which satisfies (1) for all $n \geq 0$. If we prescribe a set of *initial conditions*

$$x_0 \in J,$$

then

$$\begin{aligned} x_1 &= f(x_0), \\ x_2 &= f(x_1), \\ &\ldots \end{aligned}$$

and so the solution $\{x_n\}_{n=0}^{\infty}$ of (1) exists for all $n \geq 0$ and is uniquely determined by the initial conditions.

A solution of (1) which is constant for all $n \geq 0$ is called an *equilibrium solution* of (1).

A solution $\{x_n\}_{n=0}^{\infty}$ of (1) is called *periodic with period p* (or a *period p cycle*) if there exists an integer $p \geq 1$ such that

$$x_{n+p} = x_n \quad \text{for all } n \geq 0. \tag{2}$$

We say that the solution is *periodic with prime period p* if $p$ is the smallest positive integer for which (2) holds. In this case, a $p$-tuple

$$\begin{pmatrix} x_{n+1} \\ x_{n+2} \\ \vdots \\ x_{n+p} \end{pmatrix}$$

of any $p$ consecutive values of the solution is called a *p cycle* of (1). A solution $\{x_n\}_{n=0}^{\infty}$ of (1) is called *eventually periodic with period p* if there exists an integer $N \geq 0$ such that $\{x_n\}_{n=N}^{\infty}$ is periodic with period $p$.

We denote the absolute value of $x$ by $|x|$. So

$$|x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

## 3 Main result

In this paper we consider the behavior of the system of piecewise linear difference equations,

$$\begin{cases} x_{n+1} = |x_n| - y_n - b, \\ y_{n+1} = x_n + |y_n|, \end{cases} \qquad n = 0, 1, \ldots, \tag{3}$$

where the parameter $b \in (0, \infty)$ and the initial condition $(x_0, y_0) \in R^2$.

Through extensive study of specific cases of $b$ we found that system (3) has the unique equilibrium solution $(\bar{x}, \bar{y}) = (-\frac{2b}{5}, -\frac{b}{5})$ and two period 3 cycles

$$P_3^1 = \begin{pmatrix} 0, & -b \\ 0, & b \\ -2b, & b \end{pmatrix} \quad \text{and} \quad P_3^2 = \begin{pmatrix} 0, & -\frac{b}{3} \\ -\frac{2b}{3}, & \frac{b}{3} \\ -\frac{2b}{3}, & -\frac{b}{3} \end{pmatrix},$$

where

$$\begin{pmatrix} a_1, & b_1 \\ a_2, & b_2 \\ a_3, & b_3 \end{pmatrix}$$

represents the consecutive solutions $(a_1, b_1)$, $(a_2, b_2)$, and $(a_3, b_3)$ of the system.

The main result of this paper is as follows.

Set

$$\mathcal{L}_1 = \{(0, y_0) | y_0 \geq 0\},$$

$$\mathcal{L}_2 = \{(0, y_0) | y_0 \leq 0\},$$

$$Q_1 = \{(x, y) \in \mathbf{R} \times \mathbf{R} | x \geq 0 \text{ and } y \geq 0\},$$

$$Q_2 = \{(x, y) \in \mathbf{R} \times \mathbf{R} | x < 0 \text{ and } y \geq 0\},$$

$$Q_3 = \{(x,y) \in \mathbf{R} \times \mathbf{R} | x < 0 \text{ and } y < 0\},$$
$$Q_4 = \{(x,y) \in \mathbf{R} \times \mathbf{R} | x \geq 0 \text{ and } y < 0\}.$$

**Theorem 1** *Let $\{(x_n,y_n)\}_{n=0}^{\infty}$ be a solution to system* (3) *with $b \in (0,\infty)$ and the initial condition $(x_0,y_0) \in R^2$. Then the solution $\{(x_n,y_n)\}_{n=N}^{\infty}$ is the equilibrium solution $(-\frac{2b}{5},-\frac{b}{5})$ or eventually prime period* 3 *cycle $P_3^1$ or $P_3^2$.*

In each lemma that follows we examine the separate sections of $R^2$. The compilation of the sections addressed amount to all of $R^2$ and so the proof of Theorem 1 is a direct consequence of the following lemmas.

**Lemma 2** *Let $\{(x_n,y_n)\}_{n=0}^{\infty}$ be a solution to system* (3) *and suppose that there exists an integer $N \geq 0$ such that $y_N = -x_N - b \geq 0$. Then $(x_{N+1},y_{N+1}) = (0,-b)$, and so $\{(x_n,y_n)\}_{n=N+1}^{\infty}$ is the period* 3 *cycle $P_3^1$.*

*Proof* We have

$$x_{N+1} = |x_N| - y_N - b = -x_N - (-x_N - b) - b = 0 \quad \text{and}$$
$$y_{N+1} = x_N + |y_N| = x_N + (-x_N - b) = -b.$$

Hence, $(x_{N+1},y_{N+1}) = (0,-b) \in P_3^1$. □

**Lemma 3** *Recall that $\mathcal{L}_1 = \{(0,y_0)|y_0 \geq 0\}$ and let $\{(x_n,y_n)\}_{n=0}^{\infty}$ be a solution to system* (3)*. Suppose $(x_0,y_0) \in \mathcal{L}_1$. Then $\{(x_n,y_n)\}_{n=2}^{\infty}$ is the prime period* 3 *cycle $P_3^1$.*

*Proof* We have

$$x_1 = |x_0| - y_0 - b = 0 - y_0 - b = -y_0 - b < 0 \quad \text{and}$$
$$y_1 = x_0 + |y_0| = 0 + y_0 = y_0 \geq 0.$$

We see that $y_1 = -x_1 - b$. It follows by Lemma 2, $(x_2,y_2) = (0,-b) \in P_3^1$. □

**Lemma 4** *Recall that $\mathcal{L}_2 = \{(0,y_0)|y_0 \leq 0\}$ and let $\{(x_n,y_n)\}_{n=0}^{\infty}$ be a solution to system* (3)*. Suppose $(x_0,y_0) \in \mathcal{L}_2$. Then $\{(x_n,y_n)\}_{n=0}^{\infty}$ is eventually either the prime period* 3 *cycle $P_3^1$ or $P_3^2$.*

*Proof* We have

$$x_1 = |x_0| - y_0 - b = -y_0 - b \quad \text{and}$$
$$y_1 = x_0 + |y_0| = -y_0 \geq 0.$$

Case 1: Suppose $y_0 < -b$. Then $x_1 = -y_0 - b > 0$ and by direct computations $x_3 = 2y_0 + 2b < 0$ and $y_3 = -2y_0 - 3b$.

Case 1a: Suppose $y_0 \leq -\frac{3b}{2}$. Then $y_3 = -2y_0 - 3b \geq 0$. In addition, we see that $y_3 = -x_3 - b \geq 0$. We now apply Lemma 2 and conclude that $(x_4,y_4) = (0,-b) \in P_3^1$.

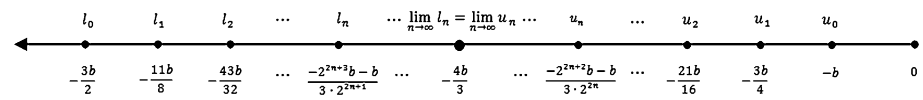


**Figure 1 Sequences $l_n$ and $u_n$ on the negative $x$-axis.**

Case 1b: It remains to consider the case $-\frac{3b}{2} < y_0 < -b$. Then $y_3 = -2y_0 - 3b < 0$.

For the sake of contradiction, assume that it is false that there exists an integer $N$ such that $\{(x_n, y_n)\}_{n=N}^{\infty}$ is either the prime period 3 cycle $P_3^1$ or $P_3^2$. It follows from Lemma 2 that when $y_n = -x_n - b$, then $y_n < 0$ for every integer $n \geq 0$. It also follows from Lemma 3 that when $x_n = 0$, then $y_n < 0$ for every integer $n \geq 0$.

For each $n \geq 0$ let

$$l_n = \frac{-2^{2n+3}b - b}{3 \times 2^{2n+1}}, \qquad u_n = \frac{-2^{2n+2}b + b}{3 \times 2^{2n}} \quad \text{and} \quad \delta_n = \frac{2^{2n+2}b - b}{3}.$$

See Figure 1.

Observe that

$$-\frac{3b}{2} = l_0 < l_1 < l_2 < \cdots < -\frac{4b}{3} \quad \text{and} \quad \lim_{n\to\infty} l_n = -\frac{4b}{3},$$
$$-b = u_0 > u_1 > u_2 > \cdots > -\frac{4b}{3} \quad \text{and} \quad \lim_{n\to\infty} u_n = -\frac{4b}{3}.$$

For each integer $m$ such that $m \geq 1$, let $\mathrm{P}(m)$ be the following statement: for $y_0 \in (l_{m-1}, u_{m-1})$,

$$x_{3m+1} = 0 \quad \text{and} \quad y_{3m+1} = 2^{2m} y_0 + \delta_m.$$

If $y_0 \in [u_m, u_{m-1})$, then $y_{3m+1} \geq 0$, which will be a contradiction by Lemma 3.

If $y_0 \in (l_{m-1}, u_m)$, then $y_{3m+1} < 0$, and so

$$x_{3m+2} = -2^{2m} y_0 - (\delta_m + b) < 0 \quad \text{and} \quad y_{3m+2} = -2^{2m} y_0 - \delta_m > 0,$$
$$x_{3m+3} = 2^{2m+1} y_0 + 2\delta_m < 0 \quad \text{and} \quad y_{3m+3} = -2^{2m+1} y_0 - (2\delta_m + b).$$

If $y_0 \in (l_{m-1}, l_m]$, then $y_{3m+3} \geq 0$, which will be a contradiction by Lemma 2.

If $y_0 \in (l_m, u_m)$, then $y_{3m+3} < 0$.

Note that $y_0$ is now in $(l_m, u_m)$.

**Claim 4.1** $\mathrm{P}(m)$ *is true for* $m \geq 1$.

The proof of the Claim 4.1 will be by induction on $m$. We shall first show that $\mathrm{P}(1)$ is true. Recall that $x_3 = 2y_0 + 2b < 0$ and $y_3 = -2y_0 - 3b < 0$ when $y_0 \in (l_0, u_0) = (-\frac{3b}{2}, -b)$. Then

$$x_{3(1)+1} = x_4 = |x_3| - y_3 - b = 0,$$
$$y_{3(1)+1} = y_4 = x_3 + |y_3| = 4y_0 + 5b = 2^{2(1)} y_0 + \delta_1.$$

If $y_0 \in [u_1, u_0) = [-\frac{5b}{4}, -b)$, then $y_{3(1)+1} = 4y_0 + 5b \geq 0$. By Lemma 3 the solution is eventually one of the two prime period 3 solutions and we have a contradiction.

If $y_0 \in (l_0, u_1) = (-\frac{3b}{2}, -\frac{5b}{4})$, then $y_{3(1)+1} = 4y_0 + 5b < 0$, and so

$$\begin{aligned}
x_{3(1)+2} &= x_5 = |x_4| - y_4 - b = -4y_0 - 6b = -2^{2(1)}y_0 - (\delta_1 + b) < 0, \\
y_{3(1)+2} &= y_5 = x_4 + |y_4| = -4y_0 - 5b = -2^{2(1)}y_0 - \delta_1 > 0, \\
x_{3(1)+3} &= x_6 = |x_5| - y_5 - b = 8y_0 + 10b = 2^{2(1)+1}y_0 + 2\delta_1 < 0, \\
y_{3(1)+3} &= y_6 = x_5 + |y_5| = -8y_0 - 11b = -2^{2(1)+1}y_0 - (2\delta_1 + b).
\end{aligned}$$

If $y_0 \in (l_{1-1}, l_1] = (l_0, l_1] = (-\frac{3b}{2}, -\frac{11b}{8}]$, then $y_{3(1)+3} = -8y_0 - 11b \geq 0$. By Lemma 2 the solution is eventually one of the two prime period 3 solutions and we have a contradiction.

If $y_0 \in (l_1, u_1) = (-\frac{11b}{8}, -\frac{5b}{4})$, then $y_{3(1)+3} = -8y_0 - 11b < 0$. Hence P(1) is true.

Next, assume that P($N$) is true for some integer $N > 1$. We shall show that P($N + 1$) is true. Since P($N$) is true, $x_{3N+3} = 2^{2N+1}y_0 + 2\delta_N < 0$ and $y_{3N+3} = -2^{2N+1}y_0 - (2\delta_N + b) < 0$ when

$$y_0 \in (l_N, u_N) = \left(\frac{-2^{2N+3}b - b}{3 \times 2^{2N+1}}, \frac{-2^{2N+2}b + b}{3 \times 2^{2N}}\right).$$

Then

$$\begin{aligned}
x_{3(N+1)+1} &= x_{3N+4} = |x_{3N+3}| - y_{3N+3} - b = 0, \\
y_{3(N+1)+1} &= y_{3N+4} = x_{3N+3} + |y_{3N+3}| = 2^{2(N+1)}y_0 + 4\delta_N + b \\
&= 2^{2(N+1)}y_0 + \delta_{N+1}.
\end{aligned}$$

Note that

$$\delta_{N+1} = \frac{2^{2N+4}b - b}{3} = 4\left(\frac{2^{2N+2}b - b}{3}\right) + \frac{3b}{3} = 4\delta_N + b.$$

If $y_0 \in [u_{N+1}, u_{(N+1)-1}) = [u_{N+1}, u_N) = [\frac{-2^{2N+4}b+b}{3\times 2^{2N+2}}, \frac{-2^{2N+2}b+b}{3\times 2^{2N}})$, then

$$y_{3(N+1)+1} = 2^{2(N+1)}y_0 + \delta_{N+1} = 2^{2(N+1)}y_0 + \left(\frac{2^{2N+4}b - b}{3}\right) \geq 0.$$

By Lemma 3 the solution is eventually a prime period 3 solution and we have a contradiction.

If $y_0 \in (l_{(N+1)-1}, u_{(N+1)}) = (l_N, u_{N+1}) = (\frac{-2^{2N+3}b-b}{3\times 2^{2N+1}}, \frac{-2^{2N+4}b+b}{3\times 2^{2N+2}})$, then

$$y_{3(N+1)+1} = 2^{2(N+1)}y_0 + \delta_{N+1} = 2^{2(N+1)}y_0 + \left(\frac{2^{2N+4}b - b}{3}\right) < 0,$$

so that

$$\begin{aligned}
x_{3(N+1)+2} &= |x_{3N+4}| - y_{3N+4} - b = -2^{2(N+1)}y_0 - (\delta_{N+1} + b) \\
&= -2^{2N+2}y_0 + \left(\frac{-2^{2N+4}b - 2b}{3}\right) < 0,
\end{aligned}$$

$$y_{3(N+1)+2} = x_{3N+4} + |y_{3N+4}| = -2^{2(N+1)}y_0 - \delta_{N+1} > 0,$$

$$x_{3(N+1)+3} = |x_{3N+5}| - y_{3N+5} - b = 2^{2(N+1)+1}y_0 + 2\delta_{N+1} < 0,$$

$$y_{3(N+1)+3} = x_{3N+5} + |y_{3N+5}| = -2^{2(N+1)+1}y_0 - (2\delta_{N+1} + b).$$

If $y_0 \in (l_{(N+1)-1}, l_{N+1}] = (l_N, l_{N+1}] = (\frac{-2^{2N+3}b-b}{3\times 2^{2N+1}}, \frac{-2^{2N+5}b-b}{3\times 2^{2N+3}}]$, then

$$y_{3(N+1)+3} = -2^{2(N+1)+1}y_0 - (2\delta_{N+1} + b) = -2^{2N+3}y_0 + \left(\frac{-2^{2N+5}b - b}{3}\right) \geq 0.$$

By Lemma 2 the solution is eventually one of the two prime period 3 solutions and we have a contradiction.

If $y_0 \in (l_{N+1}, u_{N+1}) = (\frac{-2^{2N+5}b-b}{3\times 2^{2N+3}}, \frac{-2^{2N+4}b+b}{3\times 2^{2N+2}})$, then

$$y_{3(N+1)+3} = -2^{2(N+1)+1}y_0 - (2\delta_{N+1} + b) = -2^{2N+3}y_0 + \left(\frac{-2^{2N+5}b - b}{3}\right) < 0.$$

Hence, $\mathrm{P}(N+1)$ is true. By mathematical induction, we can conclude that $\mathrm{P}(m)$ is true for all integers $m \geq 1$. We can infer that any solution with initial condition in $\{(0,y) | -\frac{3b}{2} < y < -b\} - \{(0, -\frac{4b}{3})\}$ is eventually the prime period 3 cycle $P_3^1$.

Note that

$$\lim_{n\to\infty} l_n = -\frac{4b}{3} = \lim_{n\to\infty} u_n.$$

We also note that if $(x_0, y_0) = (0, -\frac{4b}{3})$, then by direct computations

$$(x_3, y_3) = \left(-\frac{2b}{3}, -\frac{b}{3}\right) \in P_3^2.$$

We can conclude that in Case 1, every solution is eventually the prime period 3 cycle $P_3^1$ or $P_3^2$.

Case 2: Suppose $-b \leq y_0 \leq 0$. Then $x_1 = -y_0 - b \leq 0$ and so

$$x_2 = |x_1| - y_1 - b = y_0 + b + y_0 - b = 2y_0 < 0,$$

$$y_2 = x_1 + |y_1| = -y_0 - b - y_0 = -2y_0 - b.$$

Case 2a: Suppose $-b \leq y_0 \leq -\frac{b}{2}$. Then $y_2 = -2y_0 - b \geq 0$. In addition, we see that $y_2 = -x_2 - b \geq 0$. We now apply Lemma 2 and conclude that $(x_3, y_3) = (0, -b) \in P_3^1$.

Case 2b: Suppose $-\frac{b}{2} < y_0 \leq 0$. Then $y_2 = -2y_0 - b < 0$, and so

$$x_3 = |x_2| - y_2 - b = -2y_0 + 2y_0 + b - b = 0,$$

$$y_3 = x_2 + |y_2| = 2y_0 + 2y_0 + b = 4y_0 + b.$$

Case 2bi: Suppose $-\frac{b}{4} \leq y_0 \leq 0$. Then $y_3 = 4y_0 + b \geq 0$. We now apply Lemma 3 and conclude that $(x_5, y_5) = (0, -b) \in P_3^1$.

Case 2bii: It remains to consider the case $-\frac{b}{2} < y_0 < -\frac{b}{4}$.

For the sake of contradiction, assume that it is false that there exists an integer $N$ such that $\{(x_n,y_n)\}_{n=N}^{\infty}$ is either the prime period 3 cycle $P_3^1$ or $P_3^2$. It follows from Lemma 2 that when $y_n = -x_n - b$, then $y_n < 0$ for every integer $n \geq 0$. It also follows from Lemma 3 that when $x_n = 0$, then $y_n < 0$ for every integer $n \geq 0$.

For each $n \geq 1$ let

$$\mathfrak{l}_n = \frac{-2^{2n-1}b - b}{3 \times 2^{2n-1}}, \qquad \mathfrak{u}_n = \frac{-2^{2n}b + b}{3 \times 2^{2n}}, \qquad \alpha_n = \frac{2^{2n}b - b}{3}.$$

Observe that

$$-\frac{b}{2} = \mathfrak{l}_1 < \mathfrak{l}_2 < \mathfrak{l}_3 < \cdots < -\frac{b}{3} \quad \text{and} \quad \lim_{n\to\infty} \mathfrak{l}_n = -\frac{b}{3},$$
$$-\frac{b}{4} = \mathfrak{u}_1 > \mathfrak{u}_2 > \mathfrak{u}_3 > \cdots > -\frac{b}{3} \quad \text{and} \quad \lim_{n\to\infty} \mathfrak{u}_n = -\frac{b}{3}.$$

For each integer $m$ such that $m \geq 1$, let $\mathrm{Q}(m)$ be the following statement: for $y_0 \in (\mathfrak{l}_m, \mathfrak{u}_m)$,

$$x_{3m+1} = -2^{2m}y_0 - (\alpha_m + b) < 0 \quad \text{and} \quad y_{3m+1} = -2^{2m}y_0 - \alpha_m > 0,$$
$$x_{3m+2} = 2^{2m+1}y_0 + 2\alpha_m < 0 \quad \text{and} \quad y_{3m+2} = -2^{2m+1}y_0 - (2\alpha_m + b).$$

If $y_0 \in (\mathfrak{l}_m, \mathfrak{l}_{m+1}]$, then $y_{3m+2} \geq 0$, which will be a contradiction by Lemma 2.

If $y_0 \in (\mathfrak{l}_{m+1}, \mathfrak{u}_m)$, then $y_{3m+2} < 0$, and so

$$x_{3m+3} = 0 \quad \text{and} \quad y_{3m+3} = 2^{2m+2}y_0 + (4\alpha_m + b).$$

If $y_0 \in [\mathfrak{u}_{m+1}, \mathfrak{u}_m)$, then $y_{3m+3} \geq 0$, which will be a contradiction by Lemma 3.

If $y_0 \in (\mathfrak{l}_{m+1}, \mathfrak{u}_{m+1})$, then $y_{m+3} < 0$.

Note that $y_0$ is now in $(\mathfrak{l}_{m+1}, \mathfrak{u}_{m+1})$.

The proof is similar to Case 1b, so it will be omitted. Note that

$$\lim_{n\to\infty} \mathfrak{l}_n = -\frac{b}{3} = \lim_{n\to\infty} \mathfrak{u}_n$$

and that $(0, -\frac{b}{3}) \in P_3^2$. So the solution is eventually one of the two prime period 3 solutions. □

**Lemma 5** *Let* $\{(x_n,y_n)\}_{n=0}^{\infty}$ *be a solution to system* (3) *where the initial condition* $(x_0,y_0) \in Q_1 = \{(x,y) \in \mathbf{R} \times \mathbf{R} | x \geq 0$ *and* $y \geq 0\}$. *Then the solution is eventually the prime period* 3 *cycle* $P_3^1$ *or* $P_3^2$.

*Proof* Let $(x_0,y_0) \in Q_1$. Then

$$x_1 = |x_0| - y_0 - b = x_0 - y_0 - b,$$
$$y_1 = x_0 + |y_0| = x_0 + y_0 \geq 0.$$

Case 1: Suppose $x_1 = x_0 - y_0 - b \geq 0$. Then, by direct computations, $x_4 = 0$ and so $(x_4,y_4) \in \mathcal{L}_1 \cup \mathcal{L}_2$. By Lemmas 3 and 4 the solution $\{(x_n,y_n)\}_{n=0}^{\infty}$ is eventually the prime period 3 cycle $P_3^1$ or $P_3^2$.

Case 2: Suppose $x_1 = x_0 - y_0 - b < 0$. Then by direct computations $x_3 = 0$ and so $(x_3, y_3) \in \mathcal{L}_1 \cup \mathcal{L}_2$. By Lemmas 3 and 4 the solution $\{(x_n, y_n)\}_{n=0}^{\infty}$ is eventually the prime period 3 cycle $P_3^1$ or $P_3^2$. □

**Lemma 6** *Let* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *be a solution to system* (3) *where the initial condition* $(x_0, y_0) \in Q_2 = \{(x,y) \in \mathbf{R} \times \mathbf{R} | x < 0$ *and* $y \geq 0\}$. *Then the solution is eventually the prime period* 3 *cycle* $P_3^1$ *or* $P_3^2$.

*Proof* Suppose $(x_0, y_0) \in Q_2$. Then

$$x_1 = |x_0| - y_0 - b = -x_0 - y_0 - b,$$

$$y_1 = x_0 + |y_0| = x_0 + y_0.$$

If $x_1 = -x_0 - y_0 - b < 0$ and $y_1 = x_0 + y_0 \geq 0$, then by Lemma 2, $(x_2, y_2) = (0, -b) \in P_3^1$.

If $x_1 = -x_0 - y_0 - b < 0$ and $y_1 = x_0 + y_0 < 0$, then $x_2 = |x_1| - y_1 - b = 0$. So $(x_2, y_2) \in \mathcal{L}_1 \cup \mathcal{L}_2$. By Lemmas 3 and 4 the solution is eventually the prime period 3 cycle $P_3^1$ or $P_3^2$.

If $x_1 = -x_0 - y_0 - b \geq 0$ and $y_1 = x_0 + y_0 < 0$, then by direct computations $x_5 = 0$. Thus, $(x_5, y_5) \in \mathcal{L}_1 \cup \mathcal{L}_2$. By Lemmas 3 and 4 the solution is eventually the prime period 3 cycle $P_3^1$ or $P_3^2$. Please note that we cannot have the case $x_1 \geq 0$ and $y_1 \geq 0$. Hence from all the possible cases one infers that the solution is eventually a prime period 3 cycle. □

**Lemma 7** *Let* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *be a solution to system* (3) *where the initial condition* $(x_0, y_0) \in Q_4 = \{(x,y) \in \mathbf{R} \times \mathbf{R} | x \geq 0$ *and* $y < 0\}$. *Then the solution is eventually the prime period* 3 *cycle* $P_3^1$ *or* $P_3^2$.

*Proof* Suppose $(x_0, y_0) \in Q_4$. Then

$$x_1 = |x_0| - y_0 - b = x_0 - y_0 - b,$$

$$y_1 = x_0 + |y_0| = x_0 - y_0 > 0.$$

If $x_1 = x_0 - y_0 - b \geq 0$, then $(x_1, y_1) \in Q_1$. By Lemma 5 the solution is eventually the prime period 3 cycle $P_3^1$ or $P_3^2$.

If $x_1 = x_0 - y_0 - b < 0$, then $(x_1, y_1) \in Q_2$. By Lemma 6 the solution is eventually the prime period 3 cycle $P_3^1$ or $P_3^2$. □

**Lemma 8** *Let* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *be a solution to system* (3) *where the initial condition* $(x_0, y_0) \in Q_3 = \{(x,y) \in \mathbf{R} \times \mathbf{R} | x < 0$ *and* $y < 0\}$. *Then the solution* $\{(x_n, y_n)\}_{n=N}^{\infty}$ *is the equilibrium solution or eventually prime period* 3 *cycle* $P_3^1$ *or* $P_3^2$.

*Proof* If $(x_0, y_0) = (-\frac{2b}{5}, -\frac{b}{5})$, then the solution $\{(x_n, y_n)\}_{n=0}^{\infty}$ is the equilibrium solution. So suppose $(x_0, y_0) \in Q_3 \backslash \{(-\frac{2b}{5}, -\frac{b}{5})\}$. It suffices to show that there exists an integer $N \geq 0$ such that $\{(x_n, y_n)\}_{n=N}^{\infty}$ is either the period 3 cycle $P_3^1$ or the period 3 cycle $P_3^2$.

For the sake of contradiction, assume that it is false that the solution $\{(x_n, y_n)\}_{n=0}^{\infty}$ is eventually prime period 3 cycle, which means that there exists an integer $N \geq 0$ such that $\{(x_n, y_n)\}_{n=N}^{\infty}$ is either the period 3 cycle $P_3^1$ or the period 3 cycle $P_3^2$. It follows from the previous lemmas that $x_n < 0$ and $y_n < 0$ for every integer $n \geq 0$.

Case 1: Suppose further that $x_0 < -\frac{b}{2}$ and $y_0 < 0$. Then by direct computations $y_2 = -2x_0 - b > 0$, which is a contradiction because $(x_2, y_2) \notin Q_3$.

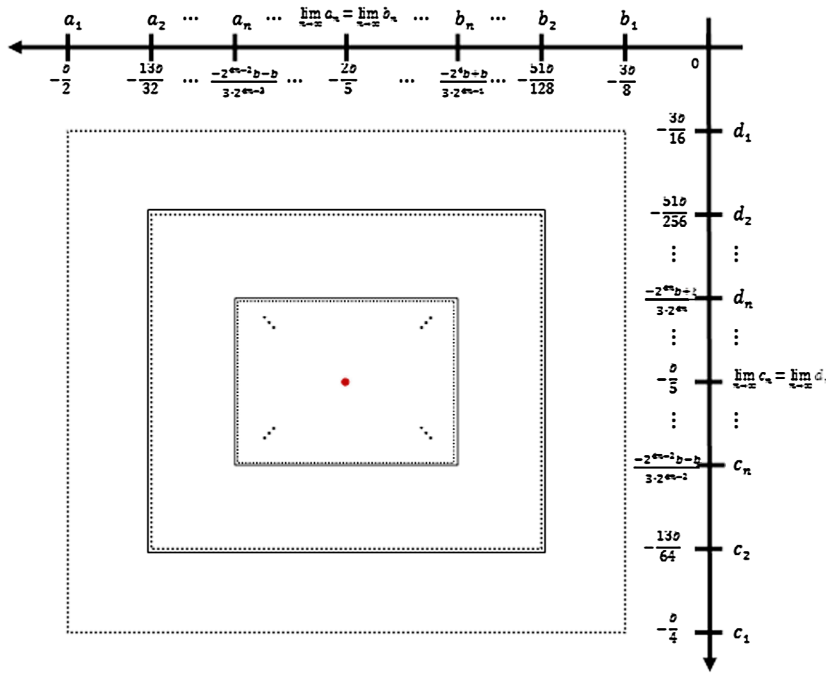


**Figure 2 Sequences $a_n$, $b_n$, $c_n$, and $d_n$ in the third quadrant.**

Case 2: Suppose $x_0 \geq -\frac{b}{2}$ and $y_0 < -\frac{b}{4}$. Then by direct computations $y_4 = -4y_0 - b > 0$, which is a contradiction because $(x_4, y_4) \notin Q_3$.

Case 3: Suppose $x_0 \geq -\frac{3b}{8}$ and $y_0 > -\frac{b}{4}$. Then by direct computations $y_6 = 8x_0 + 3b > 0$, which is a contradiction because $(x_6, y_6) \notin Q_3$.

Case 4: Suppose $x_0 < -\frac{3b}{8}$ and $y_0 \geq -\frac{3b}{16}$. Then by direct computations $y_8 = 16y_0 + 3b > 0$, which is a contradiction because $(x_8, y_8) \notin Q_3$.

It remains to consider the case $-\frac{b}{2} < x_0 < -\frac{3b}{8}$ and $-\frac{b}{4} < y_0 < -\frac{3b}{16}$. For each integer $n \geq 1$, let

$$a_n = \frac{-2^{4n-2}b - b}{5 \times 2^{4n-3}}, \qquad b_n = \frac{-2^{4n}b + b}{5 \times 2^{4n-1}}, \qquad c_n = \frac{-2^{4n-2}b - b}{5 \times 2^{4n-2}},$$
$$d_n = \frac{-2^{4n}b + b}{5 \times 2^{4n}} \quad \text{and} \quad \delta_n = \frac{2^{4n}b - b}{5}.$$

See Figure 2.

$$\begin{aligned}
&-\frac{b}{2} = a_1 < a_2 < \cdots < -\frac{2b}{5} \quad \text{and} \quad \lim_{n\to\infty} a_n = -\frac{2b}{5}, \\
&-\frac{3b}{8} = b_1 > b_2 > \cdots > -\frac{2b}{5} \quad \text{and} \quad \lim_{n\to\infty} b_n = -\frac{2b}{5}, \\
&-\frac{b}{4} = c_1 < c_2 < \cdots < -\frac{b}{5} \quad \text{and} \quad \lim_{n\to\infty} c_n = -\frac{b}{5}, \\
&-\frac{3b}{16} = d_1 > d_2 > \cdots > -\frac{b}{5} \quad \text{and} \quad \lim_{n\to\infty} d_n = -\frac{b}{5}.
\end{aligned}$$

There exists a unique integer $n$ such that $(x_0, y_0) \in (a_n, b_n) \times (c_n, d_n)$.

Recall that by assumption $x_n < 0$ and $y_n < 0$ for every integer $n \geq 1$. Suppose that $x_0 \in (a_1, b_1) = (-\frac{b}{2}, -\frac{3b}{8})$ and $y_0 \in (c_1, d_1) = (-\frac{b}{4}, -\frac{3b}{16})$.

For each integer $n$ with $n \geq 1$, let $\mathcal{R}(n)$ be the following statement: for $x_0 \in (a_n, b_n)$ and $y_0 \in (c_n, d_n)$,

$$\begin{aligned}
&x_{8n+1} = -2^{4n}x_0 - 2^{4n}y_0 - (3\delta_n + b), \qquad y_{8n+1} = 2^{4n}x_0 - 2^{4n}y_0 + \delta_n,\\
&x_{8n+2} = 2^{4n+1}y_0 + 2\delta_n, \qquad y_{8n+2} = -2^{4n+1}x_0 - (4\delta_n + b),\\
&x_{8n+3} = 2^{4n+1}x_0 - 2^{4n+1}y_0 + 2\delta_n, \qquad y_{8n+3} = 2^{4n+1}x_0 + 2^{4n+1}y_0 + (6\delta_n + b),\\
&x_{8n+4} = -2^{4n+2}x_0 - (8\delta_n + 2b), \qquad y_{8n+4} = -2^{4n+2}y_0 - (4\delta_n + b),\\
&x_{8n+5} = 2^{4n+2}x_0 + 2^{4n+2}y_0 + (12\delta_n + 2b), \qquad y_{8n+5} = -2^{4n+2}x_0 + 2^{4n+2}y_0 - (4\delta_n + b),\\
&x_{8n+6} = -2^{4n+3}y_0 - (8\delta_n + 2b), \qquad y_{8n+6} = 2^{4n+3}x_0 + (16\delta_n + 3b),\\
&x_{8n+7} = -2^{4n+3}x_0 + 2^{4n+3}y_0 - (8\delta_n + 2b),\\
&y_{8n+7} = -2^{4n+3}x_0 - 2^{4n+3}y_0 - (24\delta_n + 5b),\\
&x_{8n+8} = 2^{4n+4}x_0 + (32\delta_n + 6b), \qquad y_{8n+8} = 2^{4n+4}y_0 + (16\delta_n + 3b).
\end{aligned}$$

Now, $x_0 \in (a_{n+1}, b_{n+1})$ and $y_0 \in (c_{n+1}, d_{n+1})$, otherwise we have a contradiction.

**Claim 8.1** *$\mathcal{R}(n)$ is true for $n \geq 1$.*

The proof of the Claim 8.1 will be by induction on $n$. We shall first show that $\mathcal{R}(1)$ is true. We have $x_0 \in (a_1, b_1) = (-\frac{b}{2}, -\frac{3b}{8})$ and $y_0 \in (c_1, d_1) = (-\frac{b}{4}, -\frac{3b}{16})$.

Suppose further that $x_0 \in (a_1, a_2] = (-\frac{b}{2}, -\frac{13b}{32}]$ and $y_0 \in (c_1, d_1) = (-\frac{b}{4}, -\frac{3b}{16})$.

Hence

$$\begin{aligned}
&x_{8(1)+1} = x_9 = -16x_0 - 16y_0 - 10b = -2^{4(1)}x_0 - 2^{4(1)}y_0 - (3\delta_1 + b),\\
&y_{8(1)+1} = y_9 = 16x_0 - 16y_0 + 3b = 2^{4(1)}x_0 - 2^{4(1)}y_0 + \delta_1,\\
&x_{8(1)+2} = x_{10} = 32y_0 + 6b = 2^{4(1)+1}y_0 + 2\delta_1,\\
&y_{8(1)+2} = y_{10} = -32x_0 - 13b = -2^{4(1)+1}x_0 - (4\delta_1 + b) \geq 0,
\end{aligned}$$

which is a contradiction.

Suppose further that $x_0 \in (a_2, b_1) = (-\frac{13b}{32}, -\frac{3b}{8})$ and $y_0 \in (c_1, c_2] = (-\frac{b}{4}, -\frac{13b}{64}]$.

Hence

$$\begin{aligned}
&x_{8(1)+3} = x_{11} = 32x_0 - 32y_0 + 6b = 2^{4(1)+1}x_0 - 2^{4(1)+1}y_0 + 2\delta_1,\\
&y_{8(1)+3} = y_{11} = 32x_0 + 32y_0 + 19b = 2^{4(1)+1}x_0 + 2^{4(1)+1}y_0 + (6\delta_1 + b),\\
&x_{8(1)+4} = x_{12} = -64x_0 - 26b = -2^{4(1)+2}x_0 - (8\delta_1 + 2b),\\
&y_{8(1)+4} = y_{12} = -64y_0 - 13b = -2^{4(1)+2}y_0 - (4\delta_1 + b) \geq 0,
\end{aligned}$$

which is a contradiction.

Suppose further that $x_0 \in [b_2, b_1) = [-\frac{51b}{128}, -\frac{3b}{8})$ and $y_0 \in (c_2, d_1) = (-\frac{13b}{64}, -\frac{3b}{16})$.

Hence

$$\begin{aligned}
x_{8(1)+5} &= x_{13} = 64x_0 + 64y_0 + 38b = 2^{4(1)+2}x_0 + 2^{4(1)+2}y_0 + (12\delta_1 + 2b),\\
y_{8(1)+5} &= y_{13} = -64x_0 + 64y_0 - 13b = -2^{4(1)+2}x_0 + 2^{4(1)+2}y_0 - (4\delta_1 + b),\\
x_{8(1)+6} &= x_{14} = -128y_0 - 26b = -2^{4(1)+3}y_0 - (8\delta_1 + 2b),\\
y_{8(1)+6} &= y_{14} = 128x_0 + 51b = 2^{4(1)+3}x_0 + (16\delta_1 + 3b) \geq 0,
\end{aligned}$$

which is a contradiction.

Suppose further that $x_0 \in (a_2, b_2) = (-\frac{13b}{32}, -\frac{51b}{128})$ and $y_0 \in [d_2, d_1) = [-\frac{51b}{256}, -\frac{3b}{16})$.

Hence

$$\begin{aligned}
x_{8(1)+7} &= x_{15} = -128x_0 + 128y_0 - 26b = -2^{4(1)+3}x_0 + 2^{4(1)+3b}y_0 - (8\delta_1 + 2b),\\
y_{8(1)+7} &= y_{15} = -128x_0 - 128y_0 - 77b = -2^{4(1)+3}x_0 - 2^{4(1)+3b}y_0 - (24\delta_1 + 5b),\\
x_{8(1)+8} &= x_{16} = 256x_0 + 102b = 2^{4(1)+4}x_0 + (32\delta_1 + 6b),\\
y_{8(1)+8} &= y_{16} = 256y_0 + 51b = 2^{4(1)+4}y_0 + (16\delta_1 + 3b) \geq 0,
\end{aligned}$$

which is a contradiction.

So we have $x_0 \in (a_2, b_2) = (-\frac{13b}{32}, -\frac{51b}{128})$ and $y_0 \in (c_2, d_2) = (-\frac{13b}{64}, -\frac{51b}{256})$. Hence $\mathcal{R}(1)$ is true.

Suppose $\mathcal{R}(N)$ is true. We shall show that $\mathcal{R}(N+1)$ is true. Since $\mathcal{R}(N)$ is true, we know

$$x_{8N+8} = 2^{4N+4}x_0 + (32\delta_N + 6b), \qquad y_{8N+8} = 2^{4n+4}y_0 + (16\delta_N + 3b).$$

Recall that

$$\begin{aligned}
&x_0 \in (a_{N+1}, b_{N+1}) = \left(\frac{-2^{4N+2}b - b}{5 \times 2^{4N+1}}, \frac{-2^{4N+4}b + b}{5 \times 2^{4N+3}}\right) \quad \text{and}\\
&y_0 \in (c_{N+1}, d_{N+1}) = \left(\frac{-2^{4N+2}b - b}{5 \times 2^{4N+2}}, \frac{-2^{4N+4}b + b}{5 \times 2^{4N+4}}\right).
\end{aligned}$$

Suppose $x_0 \in (a_{(N+1)}, a_{(N+1)+1}] = (a_{N+1}, a_{N+2}] = (\frac{-2^{4N+2}b-b}{5\times 2^{4N+1}}, \frac{-2^{4N+6}b-b}{5\times 2^{4N+5}}]$, and $y_0 \in (c_{N+1}, d_{N+1}) = (\frac{-2^{4N+2}b-b}{5\times 2^{4N+2}}, \frac{-2^{4N+4}b+b}{5\times 2^{4N+4}})$.

Hence

$$\begin{aligned}
x_{8(N+1)+1} &= x_{8N+9} = -2^{4(N+1)}x_0 - 2^{4(N+1)}y_0 - (48\delta_N + 10b),\\
&= -2^{4(N+1)}x_0 - 2^{4(N+1)}y_0 - (3\delta_{N+1} + b),\\
y_{8(N+1)+1} &= y_{8N+9} = 2^{4(N+1)}x_0 - 2^{4(N+1)}y_0 + 16\delta_N + 3b,\\
&= 2^{4(N+1)}x_0 - 2^{4(N+1)}y_0 + \delta_{N+1}.
\end{aligned}$$

Note that

$$16\delta_N + 3b = 2^4\left(\frac{2^{2N}b - b}{5}\right) + \frac{15b}{5} = \frac{2^{4N+4}b - b}{5} = \delta_{N+1}.$$

Hence

$$x_{8(N+1)+2} = x_{8N+10} = 2^{4(N+1)+1}y_0 + 2\delta_{N+1},$$
$$y_{8(N+1)+2} = y_{8N+10} = -2^{4(N+1)+1}x_0 - (4\delta_{N+1} + b).$$

We see that

$$y_{8(N+1)+2} = -2^{4(N+1)+1}x_0 - (4\delta_{N+1} + b) = -2^{4N+5}x_0 + \left(\frac{-2^{4N+6}b - b}{5}\right) \geq 0,$$

which is a contradiction.

Next suppose $x_0 \in (a_{(N+1)+1}, b_{N+1}) = (\frac{-2^{4N+6}b-b}{5\times 2^{4N+5}}, \frac{-2^{4N+4}b+b}{5\times 2^{4N+3}})$ and $y_0 \in (c_{N+1}, c_{(N+1)+1}] = (\frac{-2^{4N+2}b-b}{5\times 2^{4N+2}}, \frac{-2^{4N+6}b-b}{5\times 2^{4N+6}}]$.

Hence

$$x_{8(N+1)+3} = x_{8N+11} = 2^{4(N+1)+1}x_0 - 2^{4(N+1)+1}y_0 + 2\delta_{N+1},$$
$$y_{8(N+1)+3} = y_{8N+11} = 2^{4(N+1)+1}x_0 + 2^{4(N+1)+1}y_0 + (6\delta_{N+1} + b),$$
$$x_{8(N+1)+4} = x_{8N+12} = -2^{4(N+1)+2}x_0 - (8\delta_{N+1} + 2b),$$
$$y_{8(N+1)+4} = y_{8N+12} = -2^{4(N+1)+2}y_0 - (4\delta_{N+1} + b).$$

We see that

$$y_{8(N+1)+4} = -2^{4(N+1)+2}y_0 - (4\delta_{N+1} + b) = -2^{4N+6}y_0 - \left(\frac{2^{4N+6}b + b}{5}\right) \geq 0,$$

which is a contradiction.

Next suppose $x_0 \in [b_{(N+1)+1}, b_{N+1}) = [\frac{-2^{4N+8}b+b}{5\times 2^{4N+7}}, \frac{-2^{4N+4}b+b}{5\times 2^{4N+3}})$ and $y_0 \in (c_{(N+1)+1}, d_{N+1}) = (\frac{-2^{4N+6}b-b}{5\times 2^{4N+6}}, \frac{-2^{4N+4}b+b}{5\times 2^{4N+4}})$.

Hence

$$x_{8(N+1)+5} = x_{8N+13} = 2^{4(N+1)+2}x_0 + 2^{4(N+1)+2}y_0 + 12\delta_{N+1} + 2b,$$
$$y_{8(N+1)+5} = y_{8N+13} = -2^{4(N+1)+2}x_0 + 2^{4(N+1)+2}y_0 - (4\delta_{N+1} + b),$$
$$x_{8(N+1)+6} = x_{8N+14} = -2^{4(N+1)+3}y_0 - (8\delta_{N+1} + 2b),$$
$$y_{8(N+1)+6} = y_{8N+14} = 2^{4(N+1)+3}x_0 + (16\delta_{N+1} + 3b).$$

We see that

$$\begin{aligned} y_{8(N+1)+6} &= 2^{4(N+1)+3}x_0 + (16\delta_{N+1} + 3b) \\ &= 2^{4N+7}x_0 + \left(\frac{2^{4N+8}b - b}{5}\right) \geq 0, \end{aligned}$$

which is a contradiction.

Next suppose $x_0 \in (a_{(N+1)+1}, b_{(N+1)+1}) = (\frac{-2^{4N+6}b-b}{5\times 2^{4N+5}}, \frac{-2^{4N+8}b+b}{5\times 2^{4N+7}})$ and $y_0 \in [d_{(N+1)+1}, d_{N+1}) = [\frac{-2^{4N+8}b+b}{5\times 2^{4N+8}}, \frac{-2^{4N+4}b+b}{5\times 2^{4N+4}})$.

Hence

$$\begin{aligned}
x_{8(N+1)+7} &= x_{8N+15} = -2^{4(N+1)+3}x_0 + 2^{4(N+1)+3}y_0 - (8\delta_{N+1} + 2b),\\
y_{8(N+1)+7} &= y_{8N+15} = -2^{4(N+1)+3}x_0 - 2^{4(N+1)+3}y_0 - (24\delta_{N+1} + 5b),\\
x_{8(N+1)+8} &= x_{8N+16} = 2^{4(N+1)+4}x_0 + (32\delta_{N+1} + 6b),\\
y_{8(N+1)+8} &= y_{8N+16} = 2^{4(N+1)+4}y_0 + (16\delta_{N+1} + 3b).
\end{aligned}$$

We see that

$$y_{8(N+1)+8} = 2^{4(N+1)+4}y_0 + (16\delta_{N+1} + 3b) = 2^{4N+8}y_0 + \left(\frac{2^{4N+8}b - b}{5}\right) \geq 0,$$

which is a contradiction and so $\mathcal{R}(N+1)$ is true.

Thus the proof is complete. That is we showed by mathematical induction that $\mathcal{R}(n)$ is true for every integer $n \geq 1$. Therefore for any initial condition $(x_0, y_0) \in Q_3 \backslash \{(-\frac{2b}{5}, -\frac{b}{5})\}$ there exists a natural number $N$ so that $(x_N, y_N) \notin Q_3 \backslash \{(-\frac{2b}{5}, -\frac{b}{5})\}$, which allows us to apply previous lemmas to conclude that the solution is eventually the prime period 3 cycle $P_3^1$ or $P_3^2$.

Note that

$$\lim_{n\to\infty} a_n = -\frac{2b}{5} = \lim_{n\to\infty} b_n \quad \text{and} \quad \lim_{n\to\infty} c_n = -\frac{b}{5} = \lim_{n\to\infty} d_n.$$

We also note that $(-\frac{2b}{5}, -\frac{b}{5})$ is the equilibrium solution. □

## 4 Discussion and conclusion

The system of piecewise linear difference equations examined in this paper was created as a prototype to understand the global behavior of systems like the Lozi equation. Although there has been some progress in examining the behavior of the Lozi equation [4], it still remains an enigma. We believe that this paper contributes broadly to the overall understanding of systems whose global behavior still remains unknown.

We utilized mathematical induction, proof by contradiction, and direct computations to show that every solution of system (3) is eventually either one of the two prime period 3 solutions or the equilibrium solution. We believe that investigating these types of systems of piecewise linear difference equations will give us the germ of generality that is required to understand systems with a more complicated behavior.

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
All authors carried out the proof. WT coordinated the work of the team and orchestrated the proof's design. All authors read and approved the final manuscript.

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**References**

1. Chua, LO, Yang, L: Cellular neural networks: applications. IEEE Trans. Circuits Syst. **38**, 1273-1290 (1991)
2. Grove, EA, Ladas, G: Periodicities in Nonlinear Difference Equations. Chapman & Hall/CRC, New York (2005)
3. Kulenovic, MRS, Merino, O: Discrete Dynamical Systems and Difference Equations with Mathematica. Chapman & Hall/CRC, New York (2002)
4. Botella-Soler, V, Castelo, JM, Oteo, JA, Ros, J: Bifurcations in the Lozi map. J. Phys. A, Math. Theor. **44**, 305101 (2011)
5. Devaney, RL: A piecewise linear model of the zones of instability of an area-preserving map. Physica D **10**, 387-393 (1984)
6. Devaney, RL: The Gingerbreadman. Algorithms **3**, 15-16 (1992)
7. Feuer, J: Two classes of piecewise-linear difference equations with a periodic parameter and eventual periodicity. J. Differ. Equ. Appl. **14**, 309-319 (2008)
8. Grove, EA, Lapierre, E, Tikjha, W: On the global behavior of $x_{n+1} = |x_n| - y_n - 1$ and $y_{n+1} = x_n + |y_n|$. CUBO **14**, 125-166 (2012)
9. Lapierre, EG, Lenbury, Y, Tikjha, W: On the global character of the system of piecewise linear difference equations $x_{n+1} = |x_n| - y_n - 1$ and $y_{n+1} = x_n - |y_n|$. Adv. Differ. Equ. **2010**, Article ID 573281 (2010)
10. Henon, M: A two-dimensional mapping with a strange attractor. Commun. Math. Phys. **50**, 69-77 (1976)
11. Lozi, R: Un attracteur étrange du type attracteur de Hénon. J. Phys. Colloq. **39**(C5), C5-9-C5-10 (1978)
12. Peitgen, HO, Saupe, D (eds.): The Science of Fractal Images. Springer, New York (1988)