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Existence and uniqueness of solution for a boundary value problem of fractional order involving two Caputo's fractional derivatives

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Abstract
In this paper we prove the existence and uniqueness of solution for a boundary value problem of fractional order involving two Caputo's fractional derivatives. Our investigation is based on Hölder's inequality together with Banach contraction principle and Schaefer's fixed point theorem.

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Keywords: boundary value problem; Hölder's inequality; Schaefer's fixed point theorem; Banach contraction principle

1 Introduction

Boundary value problems for fractional differential equations arise from the study of models of viscoelasticity, electrochemistry, control, porous media, electromagnetic, *etc.* (see [1, 2] and [3]). Therefore, they have received much attention.

The existence theory for initial value problems involving fractional derivatives has received considerable attention during recent decades, we mention, for example, [4–7] and [8]. However, quite recently, the theory of boundary value problems for fractional differential equations has received attention from many researchers. The attention drawn to the theory of the existence, multiplicity, and uniqueness of solutions to boundary value problems for fractional order differential equations is evident from the increased number of recent publications; see, for example, [9, 10] and [11], and the references therein.

Motivated by the above work, we investigate the existence and uniqueness of solution for a boundary value problem of fractional differential equation of the form

$$\begin{cases} {}^{C}D^{\alpha}u(t) = f(t, u(t), {}^{C}D^{\beta}u(t)), \quad t \in J := [0,1], \\ u(0) = \lambda_1 u(\eta), \qquad u'(0) = 0, \qquad u''(0) = 0, \qquad \ldots, \\ u^{(m-2)}(0) = 0, \qquad u(1) = \lambda_2 u(\eta), \end{cases} \tag{1.1}$$

where $\alpha \in (m-1, m]$, $m \in \mathbb{N}$, $m \geq 2$, $\beta > 0$, $\alpha - \beta \geq 1$, $0 < \eta < 1$ with $(\lambda_2 - \lambda_1)\eta^{m-1} \neq (1 - \lambda_1)$, $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function satisfying some assumptions that will be specified later and ${}^{C}D^{\alpha}$, ${}^{C}D^{\beta}$ are the Caputo derivatives of orders α and β with the lower limit zero, respectively.

2 Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper. Let $C(J,\mathbb{X})$ be the space of all continuous functions defined on J. Define the space $\mathbb{X} = \{u(t) \mid u(t) \in C(J), {}^{\mathrm{C}}D^{\beta}u(t) \in C(J)\}$ endowed with the norm $\|u\|_{\mathbb{X}} = \max_{t\in J} \|u(t)\| + \max_{t\in J} \|{}^{\mathrm{C}}D^{\beta}u(t)\|$. It is clear that $(\mathbb{X}, \|\cdot\|)$ is a Banach space [12].

For measurable functions $m : J \to \mathbb{R}$, define the norm

$$\|m\|_{L^p(J)} = \begin{cases} (\int_J |m(t)|^p\,dt)^{\frac{1}{p}}, & 1 \le p < \infty, \\ \inf_{\mu(\bar{J})=0}\{\sup_{t\in J-\bar{J}} |m(t)|\}, & p = \infty, \end{cases}$$

where $\mu(\bar{J})$ is the Lebesgue measure on $\bar{J}$. Let $L^p(J,\mathbb{R})$ be the Banach space of all Lebesgue measurable functions $m : J \to \mathbb{R}$ with $\|m\|_{L^p(J)} < \infty$.

We need some basic definitions and properties [13, 14] of fractional calculus which are used in this paper.

Definition 2.1 The Riemann-Liouville fractional integral of order q with the lower limit zero for a function $h : [0,\infty) \to \mathbb{R}$ is defined by

$$I^q h(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{h(s)}{(t-s)^{1-q}}\,ds, \quad t > 0, q > 0$$

provided the right-hand side is point-wise defined on $[0,\infty)$, where $\Gamma(\cdot)$ is the gamma function.

Definition 2.2 The Riemann-Liouville derivative of order q with the lower limit zero for a function $h : [0,\infty) \to \mathbb{R}$ can be written as

$${}^{\mathrm{L}}D^q h(t) = \frac{1}{\Gamma(n-q)} \frac{d^n}{dt^n} \int_0^t \frac{h(s)}{(t-s)^{q+1-n}}\,ds, \quad t > 0, n-1 < q < n.$$

Definition 2.3 The Caputo derivative of order $q > 0$ for a function $h : [0,\infty) \to \mathbb{R}$ can be written as

$${}^{\mathrm{C}}D^q h(t) = {}^{\mathrm{L}}D^q \left[h(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} h^{(k)}(0) \right], \quad t > 0, n-1 < q < n.$$

Lemma 2.1

(i) *If $h \in C^n([0,\infty),\mathbb{R})$, then the Caputo derivative of order $q > 0$ for a function $h : [0,\infty) \to \mathbb{R}$ can be written as*

$${}^{\mathrm{C}}D^q h(t) = \frac{1}{\Gamma(n-q)} \int_0^t \frac{h^{(n)}(s)}{(t-s)^{q+1-n}}\,ds, \quad t > 0, n-1 < q < n.$$

(ii) *If $x \in L(0,1)$, $\rho > \sigma > 0$, then*

$${}^{\mathrm{C}}D^{\sigma} I^{\rho} x(t) = I^{\rho-\sigma} x(t), \qquad I^{\rho} I^{\sigma} x(t) = I^{\rho+\sigma} x(t).$$

(iii) *If $\rho > 0$, $k > 0$, then*

$${}^{\mathrm{C}}D^{\rho} t^{k-1} = \frac{\Gamma(k)}{\Gamma(k-\rho)} t^{k-\rho-1}.$$

(iv) *If* $\rho > 0$, C *is a constant, then*

$$^{\mathrm{C}}D^{\rho} C = 0.$$

It is useful to mention that Definition 2.3 (generalization of the classical Caputo derivative), where the integrable function h can be discontinuous, is more general than that of the classical Caputo derivative described in (i) in the above lemma (see [15] and [16]).

Lemma 2.2 [17] *Let* $\alpha > 0$, *then the differential equation*

$$^{\mathrm{C}}D^{\alpha} h(t) = 0$$

has solutions $h(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{m-1} t^{m-1}$, $c_i \in \mathbb{R}$, $i = 0, 1, \ldots, m-1$, $m = [\alpha] + 1$.

Lemma 2.3 [17] *Let* $\alpha > 0$, *then*

$$I^{\alpha}\,{}^{\mathrm{C}}D^{\alpha} h(t) = h(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{m-1} t^{m-1}$$

for some $c_i \in \mathbb{R}$, $i = 0, 1, \ldots, m-1$, $m = -[-\alpha]$.

Now, let us recall the definition of a solution of the fractional boundary value problem (1.1).

Definition 2.4 A function $u \in C(J, \mathbb{X})$ with its α-derivative existing on J is said to be a solution of the fractional boundary value problem (1.1) if u satisfies the equation ${}^{\mathrm{C}}D^{\alpha} u(t) = f(t, u(t), {}^{\mathrm{C}}D^{\beta} u(t))$ a.e. on J and the conditions $u(0) = \lambda_1 u(\eta), u'(0) = 0, u''(0) = 0, \ldots, u^{(m-2)}(0) = 0, u(1) = \lambda_2 u(\eta)$.

To study the nonlinear problem (1.1), we first consider the associated linear problem

$$\begin{cases} {}^{\mathrm{C}}D^{\alpha} u(t) = h(t), \quad t \in J, \\ u(0) = \lambda_1 u(\eta), \qquad u'(0) = 0, \qquad u''(0) = 0, \qquad \ldots, \\ u^{(m-2)}(0) = 0, \qquad u(1) = \lambda_2 u(\eta), \end{cases} \tag{2.1}$$

where $h \in C(J, \mathbb{R})$.

Lemma 2.4 *A unique solution of equation* (2.1) *satisfies the following integral equation*:

$$\begin{aligned} u(t) = {} & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)\,ds + \frac{\lambda_1 \eta^{m-1} + (1-\lambda_1) t^{m-1}}{(\lambda_1 - 1) + (\lambda_2 - \lambda_1)\eta^{m-1}} \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)\,ds \\ & - \frac{\lambda_1 + (\lambda_2 - \lambda_1) t^{m-1}}{(\lambda_1 - 1) + (\lambda_2 - \lambda_1)\eta^{m-1}} \int_0^{\eta} \frac{(\eta - s)^{\alpha-1}}{\Gamma(\alpha)} h(s)\,ds. \end{aligned} \tag{2.2}$$

Proof By Lemma 2.3, the general solution of (2.1) can be written as

$$u(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)\,ds - c_0 - c_1 t - c_2 t^2 + \cdots - c_{m-1} t^{m-1}, \tag{2.3}$$

where $c_0, c_1, \ldots, c_{m-1} \in \mathbb{R}$ are arbitrary constants. In view of Lemma 2.1, we obtain

$$u'(t) = \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} h(s)\,ds - c_1 - 2c_2 t + \cdots - (m-1)c_{m-1}t^{m-2},$$
$$u''(t) = \int_0^t \frac{(t-s)^{\alpha-3}}{\Gamma(\alpha-2)} h(s)\,ds - 2c_2 + \cdots - (m-1)(m-2)c_{m-1}t^{m-3}, \qquad \ldots.$$

From $u'(0) = 0, u''(0) = 0, \ldots, u^{(m-2)}(0) = 0$, it follows that $c_1 = c_2 = \cdots = c_{m-2} = 0$. Using $u(0) = \lambda_1 u(\eta)$ and $u(1) = \lambda_2 u(\eta)$, we get

$$(\lambda_1 - 1)c_0 + \lambda_1 \eta^{m-1} c_{m-1} = \lambda_1 \int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)\,ds$$

and

$$(\lambda_2 - 1)c_0 + \bigl(\lambda_2 \eta^{m-1} - 1\bigr)c_{m-1} = \lambda_2 \int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)\,ds - \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)\,ds,$$

respectively. Therefore, we get

$$\begin{aligned} c_0 &= \frac{\lambda_1}{(\lambda_1 - 1) + (\lambda_2 - \lambda_1)\eta^{m-1}} \int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)\,ds \\ &\quad - \frac{\lambda_1 \eta^{m-1}}{(\lambda_1 - 1) + (\lambda_2 - \lambda_1)\eta^{m-1}} \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)\,ds, \\ c_{m-1} &= \frac{\lambda_2 - \lambda_1}{(\lambda_1 - 1) + (\lambda_2 - \lambda_1)\eta^{m-1}} \int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)\,ds \\ &\quad + \frac{\lambda_1 - 1}{(\lambda_1 - 1) + (\lambda_2 - \lambda_1)\eta^{m-1}} \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)\,ds. \end{aligned}$$

Substituting the values of c_0 and c_{m-1} in (2.3), we obtain the result. This completes the proof. □

As a consequence of Lemma 2.4, we have the following result which is useful in what follows.

Lemma 2.5 *Let $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous function. A function $u \in \mathbb{X}$ is a solution of the integral equation*

$$\begin{aligned} u(t) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f\bigl(s, u(s), {}^{\mathrm{C}}D^\beta u(s)\bigr)\,ds \\ &\quad + \frac{\lambda_1 \eta^{m-1} + (1-\lambda_1)t^{m-1}}{(\lambda_1 - 1) + (\lambda_2 - \lambda_1)\eta^{m-1}} \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} f\bigl(s, u(s), {}^{\mathrm{C}}D^\beta u(s)\bigr)\,ds \\ &\quad - \frac{\lambda_1 + (\lambda_2 - \lambda_1)t^{m-1}}{(\lambda_1 - 1) + (\lambda_2 - \lambda_1)\eta^{m-1}} \int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} f\bigl(s, u(s), {}^{\mathrm{C}}D^\beta u(s)\bigr)\,ds \end{aligned}$$

if and only if u is a solution of the fractional boundary value problem (1.1).

Lemma 2.6 (Hölder's inequality) *Assume that $q, p \geq 1$ and $\frac{1}{q} + \frac{1}{p} = 1$. If $l \in L^q(J, \mathbb{R})$ and $m \in L^p(J, \mathbb{R})$, then for $1 \leq p \leq \infty$, $lm \in L^1(J, \mathbb{R})$ and $\|lm\|_{L^1(J)} \leq \|l\|_{L^q(J)} \cdot \|m\|_{L^p(J)}$.*

Lemma 2.7 (Bochner's theorem) *A measurable function $g : J \to \mathbb{R}$ is Bochner integrable if $|g|$ is Lebesgue integrable.*

Lemma 2.8 (Schaefer's fixed point theorem) *Let $F : C(J,\mathbb{X}) \to C(J,\mathbb{X})$ be a completely continuous operator. If the set $E(F) = \{u \in C(J,\mathbb{X}) : u = \lambda Fu$ for some $\lambda \in (0,1)\}$ is bounded, then F has at least a fixed point.*

3 Main results

Before stating and proving the main results, we introduce the following hypotheses.

(H1) $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is Lebesgue measurable with respect to t on J.

(H2) There exists a constant $\alpha_1 \in [0,\alpha)$ and a real-valued function $m(t) \in L^{\frac{1}{\alpha_1}}(J,\mathbb{R}_+)$ such that

$$\bigl|f(t,u_2,v_2) - f(t,u_1,v_1)\bigr| \le m(t)\bigl(|u_2 - u_1| + |v_2 - v_1|\bigr)$$

for each $t \in J$ and all $u_i, v_i \in \mathbb{X}$, $i = 1,2$.

(H3) There exists a constant $\alpha_2 \in [0,\alpha)$ and a real-valued function $h(t) \in L^{\frac{1}{\alpha_2}}(J,\mathbb{R}_+)$ such that

$$\bigl|f(t,u,v)\bigr| \le h(t)$$

for each $t \in J$ and all $u, v \in \mathbb{X}$.

For brevity, let $M = \|m\|_{L^{\frac{1}{\alpha_1}}(J,\mathbb{R}_+)}$ and $H = \|h\|_{L^{\frac{1}{\alpha_2}}(J,\mathbb{R}_+)}$.

Our first result is based on the Banach contraction principle.

Theorem 3.1 *Assume that* (H1)-(H3) *hold. If*

$$\Omega = M\left[\frac{(1+\Lambda_1)}{\Gamma(\alpha)(\frac{\alpha-\alpha_1}{1-\alpha_1})^{1-\alpha_1}} + \left(\frac{1}{\Gamma(\alpha-\beta)(\frac{\alpha-\beta-\alpha_2}{1-\alpha_2})^{1-\alpha_2}} + \frac{\Lambda_2}{\Gamma(\alpha)(\frac{\alpha-\alpha_2}{1-\alpha_2})^{1-\alpha_2}}\right)\right] < 1, \tag{3.1}$$

where

$$\Lambda_1 = \frac{2|\lambda_1| + |1-\lambda_1| + |\lambda_2-\lambda_1|}{|(\lambda_1-1) + (\lambda_2-\lambda_1)\eta^{m-1}|},$$
$$\Lambda_2 = \frac{(|1-\lambda_1| + |\lambda_2-\lambda_1|)\Gamma(m)}{|(\lambda_1-1) + (\lambda_2-\lambda_1)\eta^{m-1}|\Gamma(m-\beta)},$$

then the boundary value problem (1.1) *has a unique solution.*

Proof For each $t \in J$, we have

$$\begin{aligned}
\int_0^t \bigl|(t-s)^{\alpha-1} f\bigl(s,u(s),{}^{\mathrm{C}}D^\beta u(s)\bigr)\bigr|\,ds &\le \left(\int_0^t (t-s)^{\frac{\alpha-1}{1-\alpha_2}}\,ds\right)^{1-\alpha_2}\left(\int_0^t \bigl(h(s)\bigr)^{\frac{1}{\alpha_2}}\,ds\right)^{\alpha_2}\\
&\le \left(\int_0^t (t-s)^{\frac{\alpha-1}{1-\alpha_2}}\,ds\right)^{1-\alpha_2}\left(\int_0^1 \bigl(h(s)\bigr)^{\frac{1}{\alpha_2}}\,ds\right)^{\alpha_2}\\
&\le \frac{H}{(\frac{\alpha-\alpha_2}{1-\alpha_2})^{1-\alpha_2}}.
\end{aligned}$$

Thus $|(t-s)^{\alpha-1}f(s,u(s),{}^{\mathrm{C}}D^{\beta}u(s))|$ is Lebesgue integrable with respect to $s\in[0,t]$ for all $t\in J$ and $u\in C(J,\mathbb{X})$. Then $(t-s)^{\alpha-1}f(s,u(s),{}^{\mathrm{C}}D^{\beta}u(s))$ is Bochner integrable with respect to $s\in[0,t]$ for all $t\in J$ due to Lemma 2.7.

In the same manner we can show that $(1-s)^{\alpha-1}f(s,u(s),{}^{\mathrm{C}}D^{\beta}u(s))$ and $(\eta-s)^{\alpha-1}f(s,u(s),{}^{\mathrm{C}}D^{\beta}u(s))$ are also Bochner integrable with respect to $s\in[0,t]$ for all $t\in J$ due to Lemma 2.7.

Hence, the boundary value problem (1.1) is equivalent to the following integral equation:

$$\begin{aligned} u(t) ={}& \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f\bigl(s,u(s),{}^{\mathrm{C}}D^{\beta}u(s)\bigr)\,ds \\ &+ \frac{\lambda_1\eta^{m-1}+(1-\lambda_1)t^{m-1}}{(\lambda_1-1)+(\lambda_2-\lambda_1)\eta^{m-1}} \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} f\bigl(s,u(s),{}^{\mathrm{C}}D^{\beta}u(s)\bigr)\,ds \\ &- \frac{\lambda_1+(\lambda_2-\lambda_1)t^{m-1}}{(\lambda_1-1)+(\lambda_2-\lambda_1)\eta^{m-1}} \int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} f\bigl(s,u(s),{}^{\mathrm{C}}D^{\beta}u(s)\bigr)\,ds, \quad t\in J. \end{aligned}$$

Let

$$r \ge H\biggl[\frac{1+\Lambda_1+\Lambda_2}{\Gamma(\alpha)(\frac{\alpha-\alpha_2}{1-\alpha_2})^{1-\alpha_2}} + \frac{1}{\Gamma(\alpha-\beta)(\frac{\alpha-\beta-\alpha_2}{1-\alpha_2})^{1-\alpha_2}}\biggr].$$

Now we define the operator F on $B_r := \{u\in C(J,\mathbb{X}) : \|u\|_{\mathbb{X}} \le r\}$ as follows:

$$\begin{aligned} (Fu)(t) ={}& \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f\bigl(s,u(s),{}^{\mathrm{C}}D^{\beta}u(s)\bigr)\,ds \\ &+ \frac{\lambda_1\eta^{m-1}+(1-\lambda_1)t^{m-1}}{(\lambda_1-1)+(\lambda_2-\lambda_1)\eta^{m-1}} \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} f\bigl(s,u(s),{}^{\mathrm{C}}D^{\beta}u(s)\bigr)\,ds \\ &- \frac{\lambda_1+(\lambda_2-\lambda_1)t^{m-1}}{(\lambda_1-1)+(\lambda_2-\lambda_1)\eta^{m-1}} \int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} f\bigl(s,u(s),{}^{\mathrm{C}}D^{\beta}u(s)\bigr)\,ds \end{aligned} \tag{3.2}$$

for each $t\in J$. Therefore, the existence of a solution of the fractional boundary value problem (1.1) is equivalent to the fact that the operator F has a fixed point in B_r. We shall use the Banach contraction principle to prove that F has a fixed point. The proof is divided into two steps.

Step 1. $Fu\in B_r$ for every $u\in B_r$.

For every $u\in B_r$ and $\delta>0$, by (H3) and Hölder's inequality, we get

$$\begin{aligned} &\bigl|(Fu)(t+\delta)-(Fu)(t)\bigr| \\ &\quad\le \biggl|\int_t^{t+\delta} \frac{(t+\delta-s)^{\alpha-1}}{\Gamma(\alpha)} f\bigl(s,u(s),{}^{\mathrm{C}}D^{\beta}u(s)\bigr)\,ds\biggr| \\ &\qquad+ \biggl|\int_0^t \frac{((t+\delta-s)^{\alpha-1}-(t-s)^{\alpha-1})}{\Gamma(\alpha)} f\bigl(s,u(s),{}^{\mathrm{C}}D^{\beta}u(s)\bigr)\,ds\biggr| \\ &\qquad+ \biggl|\frac{(1-\lambda_1)((t+\delta)^{m-1}-t^{m-1})}{(\lambda_1-1)+(\lambda_2-\lambda_1)\eta^{m-1}} \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} f\bigl(s,u(s),{}^{\mathrm{C}}D^{\beta}u(s)\bigr)\,ds\biggr| \\ &\qquad+ \biggl|\frac{(\lambda_1-\lambda_2)((t+\delta)^{m-1}-t^{m-1})}{(\lambda_1-1)+(\lambda_2-\lambda_1)\eta^{m-1}} \int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} f\bigl(s,u(s),{}^{\mathrm{C}}D^{\beta}u(s)\bigr)\,ds\biggr| \\ &\quad\le \int_t^{t+\delta} \frac{(t+\delta-s)^{\alpha-1}}{\Gamma(\alpha)} \bigl|f\bigl(s,u(s),{}^{\mathrm{C}}D^{\beta}u(s)\bigr)\bigr|\,ds \end{aligned}$$

$$
\begin{aligned}
&+\int_0^t \frac{((t+\delta-s)^{\alpha-1}-(t-s)^{\alpha-1})}{\Gamma(\alpha)}\big|f\big(s,u(s),{}^{\mathrm{C}}D^{\beta}u(s)\big)\big|\,ds \\
&+\frac{|1-\lambda_1|((t+\delta)^{m-1}-t^{m-1})}{|(\lambda_1-1)+(\lambda_2-\lambda_1)\eta^{m-1}|}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\big|f\big(s,u(s),{}^{\mathrm{C}}D^{\beta}u(s)\big)\big|\,ds \\
&+\frac{|\lambda_1-\lambda_2|((t+\delta)^{m-1}-t^{m-1})}{|(\lambda_1-1)+(\lambda_2-\lambda_1)\eta^{m-1}|}\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}\big|f\big(s,u(s),{}^{\mathrm{C}}D^{\beta}u(s)\big)\big|\,ds \\
\le{}&\int_t^{t+\delta}\frac{(t+\delta-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)\,ds+\int_0^t \frac{((t+\delta-s)^{\alpha-1}-(t-s)^{\alpha-1})}{\Gamma(\alpha)}h(s)\,ds \\
&+\frac{|1-\lambda_1|((t+\delta)^{m-1}-t^{m-1})}{|(\lambda_1-1)+(\lambda_2-\lambda_1)\eta^{m-1}|}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)\,ds \\
&+\frac{|\lambda_1-\lambda_2|((t+\delta)^{m-1}-t^{m-1})}{|(\lambda_1-1)+(\lambda_2-\lambda_1)\eta^{m-1}|}\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)\,ds \\
\le{}&\frac{1}{\Gamma(\alpha)}\left(\int_t^{t+\delta}(t+\delta-s)^{\frac{\alpha-1}{1-\alpha_2}}\,ds\right)^{1-\alpha_2}\left(\int_t^{t+\delta}\big(h(s)\big)^{\frac{1}{\alpha_2}}\,ds\right)^{\alpha_2} \\
&+\frac{1}{\Gamma(\alpha)}\left(\int_0^t\big((t+\delta-s)^{\alpha-1}-(t-s)^{\alpha-1}\big)^{\frac{1}{1-\alpha_2}}\,ds\right)^{1-\alpha_2}\left(\int_0^t\big(h(s)\big)^{\frac{1}{\alpha_2}}\,ds\right)^{\alpha_2} \\
&+\frac{|1-\lambda_1|((t+\delta)^{m-1}-t^{m-1})}{|(\lambda_1-1)+(\lambda_2-\lambda_1)\eta^{m-1}|}\frac{1}{\Gamma(\alpha)}\left(\int_0^1(1-s)^{\frac{\alpha-1}{1-\alpha_2}}\,ds\right)^{1-\alpha_2}\left(\int_0^1\big(h(s)\big)^{\frac{1}{\alpha_2}}\,ds\right)^{\alpha_2} \\
&+\frac{|\lambda_1-\lambda_2|((t+\delta)^{m-1}-t^{m-1})}{|(\lambda_1-1)+(\lambda_2-\lambda_1)\eta^{m-1}|}\frac{1}{\Gamma(\alpha)}\left(\int_0^\eta(\eta-s)^{\frac{\alpha-1}{1-\alpha_2}}\,ds\right)^{1-\alpha_2}\left(\int_0^\eta\big(h(s)\big)^{\frac{1}{\alpha_2}}\,ds\right)^{\alpha_2} \\
\le{}&\frac{1}{\Gamma(\alpha)}\left(\int_t^{t+\delta}(t+\delta-s)^{\frac{\alpha-1}{1-\alpha_2}}\,ds\right)^{1-\alpha_2}\left(\int_t^{t+\delta}\big(h(s)\big)^{\frac{1}{\alpha_2}}\,ds\right)^{\alpha_2} \\
&+\frac{1}{\Gamma(\alpha)}\left(\int_0^t\big((t+\delta-s)^{\frac{\alpha-1}{1-\alpha_2}}-(t-s)^{\frac{\alpha-1}{1-\alpha_2}}\big)\,ds\right)^{1-\alpha_2}\left(\int_0^t\big(h(s)\big)^{\frac{1}{\alpha_2}}\,ds\right)^{\alpha_2} \\
&+\frac{|1-\lambda_1|((t+\delta)^{m-1}-t^{m-1})}{|(\lambda_1-1)+(\lambda_2-\lambda_1)\eta^{m-1}|}\frac{1}{\Gamma(\alpha)}\left(\int_0^1(1-s)^{\frac{\alpha-1}{1-\alpha_2}}\,ds\right)^{1-\alpha_2}\left(\int_0^1\big(h(s)\big)^{\frac{1}{\alpha_2}}\,ds\right)^{\alpha_2} \\
&+\frac{|\lambda_1-\lambda_2|((t+\delta)^{m-1}-t^{m-1})}{|(\lambda_1-1)+(\lambda_2-\lambda_1)\eta^{m-1}|}\frac{1}{\Gamma(\alpha)}\left(\int_0^\eta(\eta-s)^{\frac{\alpha-1}{1-\alpha_2}}\,ds\right)^{1-\alpha_2}\left(\int_0^\eta\big(h(s)\big)^{\frac{1}{\alpha_2}}\,ds\right)^{\alpha_2} \\
\le{}&\frac{\|h\|_{L^{\frac{1}{\alpha_2}}(J,\mathbb{R}_+)}}{\Gamma(\alpha)(\frac{\alpha-\alpha_2}{1-\alpha_2})^{1-\alpha_2}}\Bigg[\delta^{\alpha-\alpha_2}+\big((t+\delta)^{\frac{\alpha-\alpha_2}{1-\alpha_2}}-\delta^{\frac{\alpha-\alpha_2}{1-\alpha_2}}-t^{\frac{\alpha-\alpha_2}{1-\alpha_2}}\big)^{1-\alpha_2} \\
&+\frac{|1-\lambda_1|((t+\delta)^{m-1}-t^{m-1})}{|(\lambda_1-1)+(\lambda_2-\lambda_1)\eta^{m-1}|}+\frac{|\lambda_1-\lambda_2|((t+\delta)^{m-1}-t^{m-1})}{|(\lambda_1-1)+(\lambda_2-\lambda_1)\eta^{m-1}|}\eta^{\alpha-\alpha_2}\Bigg] \\
\le{}&\frac{H}{\Gamma(\alpha)(\frac{\alpha-\alpha_2}{1-\alpha_2})^{1-\alpha_2}}\Bigg[\delta^{\alpha-\alpha_2}+\big((t+\delta)^{\frac{\alpha-\alpha_2}{1-\alpha_2}}-\delta^{\frac{\alpha-\alpha_2}{1-\alpha_2}}-t^{\frac{\alpha-\alpha_2}{1-\alpha_2}}\big)^{1-\alpha_2} \\
&+\frac{(|1-\lambda_1|+|\lambda_1-\lambda_2|)}{|(\lambda_1-1)+(\lambda_2-\lambda_1)\eta^{m-1}|}\big((t+\delta)^{m-1}-t^{m-1}\big)\Bigg],
\end{aligned}
$$

and in view of Lemma 2.1, we have

$$
\begin{aligned}
&\big|{}^{\mathrm{C}}D^{\beta}(Fu)(t+\delta)-{}^{\mathrm{C}}D^{\beta}(Fu)(t)\big| \\
&\quad\le\int_t^{t+\delta}\frac{(t+\delta-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\big|f\big(s,u(s),{}^{\mathrm{C}}D^{\beta}u(s)\big)\big|\,ds
\end{aligned}
$$

$$
\begin{aligned}
&+ \int_0^t \frac{((t+\delta-s)^{\alpha-\beta-1}-(t-s)^{\alpha-\beta-1})}{\Gamma(\alpha-\beta)} \bigl|f\bigl(s,u(s),{}^{\mathrm{C}}D^{\beta}u(s)\bigr)\bigr|\,ds \\
&+ \biggl[\frac{|1-\lambda_1|}{|(\lambda_1-1)+(\lambda_2-\lambda_1)\eta^{m-1}|}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} \bigl|f\bigl(s,u(s),{}^{\mathrm{C}}D^{\beta}u(s)\bigr)\bigr|\,ds \\
&+ \frac{|\lambda_1-\lambda_2|}{|(\lambda_1-1)+(\lambda_2-\lambda_1)\eta^{m-1}|}\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} \bigl|f\bigl(s,u(s),{}^{\mathrm{C}}D^{\beta}u(s)\bigr)\bigr|\,ds\biggr] \\
&\times \frac{\Gamma(m)}{\Gamma(m-\beta-1)}\bigl((t+\delta)^{m-\beta-1}-(t)^{m-\beta-1}\bigr) \\
\le{}& \int_t^{t+\delta} \frac{(t+\delta-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}h(s)\,ds + \int_0^t \frac{((t+\delta-s)^{\alpha-\beta-1}-(t-s)^{\alpha-\beta-1})}{\Gamma(\alpha-\beta)}h(s)\,ds \\
&+ \biggl[\frac{|1-\lambda_1|}{|(\lambda_1-1)+(\lambda_2-\lambda_1)\eta^{m-1}|}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)\,ds \\
&+ \frac{|\lambda_1-\lambda_2|}{|(\lambda_1-1)+(\lambda_2-\lambda_1)\eta^{m-1}|}\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)\,ds\biggr] \\
&\times \frac{\Gamma(m)}{\Gamma(m-\beta-1)}\bigl((t+\delta)^{m-\beta-1}-(t)^{m-\beta-1}\bigr) \\
\le{}& \frac{1}{\Gamma(\alpha-\beta)}\biggl(\int_t^{t+\delta}(t+\delta-s)^{\frac{\alpha-\beta-1}{1-\alpha_2}}\,ds\biggr)^{1-\alpha_2}\biggl(\int_t^{t+\delta}\bigl(h(s)\bigr)^{\frac{1}{\alpha_2}}\,ds\biggr)^{\alpha_2} \\
&+ \frac{1}{\Gamma(\alpha-\beta)}\biggl(\int_0^t \bigl((t+\delta-s)^{\alpha-\beta-1}-(t-s)^{\alpha-\beta-1}\bigr)^{\frac{1}{1-\alpha_2}}\,ds\biggr)^{1-\alpha_2}\biggl(\int_0^t\bigl(h(s)\bigr)^{\frac{1}{\alpha_2}}\,ds\biggr)^{\alpha_2} \\
&+ \biggl[\frac{|1-\lambda_1|}{|(\lambda_1-1)+(\lambda_2-\lambda_1)\eta^{m-1}|}\biggl(\int_0^1(1-s)^{\frac{\alpha-1}{1-\alpha_2}}\,ds\biggr)^{1-\alpha_2}\biggl(\int_0^1\bigl(h(s)\bigr)^{\frac{1}{\alpha_2}}\,ds\biggr)^{\alpha_2} \\
&+ \frac{|\lambda_1-\lambda_2|}{|(\lambda_1-1)+(\lambda_2-\lambda_1)\eta^{m-1}|}\biggl(\int_0^\eta(\eta-s)^{\frac{\alpha-1}{1-\alpha_2}}\,ds\biggr)^{1-\alpha_2}\biggl(\int_0^\eta\bigl(h(s)\bigr)^{\frac{1}{\alpha_2}}\,ds\biggr)^{\alpha_2}\biggr] \\
&\times \frac{\Gamma(m)((t+\delta)^{m-\beta-1}-(t)^{m-\beta-1})}{\Gamma(m-\beta-1)\Gamma(\alpha)} \\
\le{}& \frac{1}{\Gamma(\alpha-\beta)}\biggl(\int_t^{t+\delta}(t+\delta-s)^{\frac{\alpha-\beta-1}{1-\alpha_2}}\,ds\biggr)^{1-\alpha_2}\biggl(\int_t^{t+\delta}\bigl(h(s)\bigr)^{\frac{1}{\alpha_2}}\,ds\biggr)^{\alpha_2} \\
&+ \frac{1}{\Gamma(\alpha-\beta)}\biggl(\int_0^t \bigl((t+\delta-s)^{\frac{\alpha-\beta-1}{1-\alpha_2}}-(t-s)^{\frac{\alpha-\beta-1}{1-\alpha_2}}\bigr)\,ds\biggr)^{1-\alpha_2}\biggl(\int_0^t\bigl(h(s)\bigr)^{\frac{1}{\alpha_2}}\,ds\biggr)^{\alpha_2} \\
&+ \biggl[\frac{|1-\lambda_1|}{|(\lambda_1-1)+(\lambda_2-\lambda_1)\eta^{m-1}|}\biggl(\int_0^1(1-s)^{\frac{\alpha-1}{1-\alpha_2}}\,ds\biggr)^{1-\alpha_2}\biggl(\int_0^1\bigl(h(s)\bigr)^{\frac{1}{\alpha_2}}\,ds\biggr)^{\alpha_2} \\
&+ \frac{|\lambda_1-\lambda_2|}{|(\lambda_1-1)+(\lambda_2-\lambda_1)\eta^{m-1}|}\biggl(\int_0^\eta(\eta-s)^{\frac{\alpha-1}{1-\alpha_2}}\,ds\biggr)^{1-\alpha_2}\biggl(\int_0^\eta\bigl(h(s)\bigr)^{\frac{1}{\alpha_2}}\,ds\biggr)^{\alpha_2}\biggr] \\
&\times \frac{\Gamma(m)((t+\delta)^{m-\beta-1}-(t)^{m-\beta-1})}{\Gamma(m-\beta-1)\Gamma(\alpha)} \\
\le{}& H\biggl[\frac{\delta^{\alpha-\beta-\alpha_2}}{\Gamma(\alpha-\beta)(\frac{\alpha-\beta-\alpha_2}{1-\alpha_2})^{1-\alpha_2}} + \frac{((t+\delta)^{\frac{\alpha-\beta-\alpha_2}{1-\alpha_2}}-\delta^{\frac{\alpha-\beta-\alpha_2}{1-\alpha_2}}-t^{\frac{\alpha-\beta-\alpha_2}{1-\alpha_2}})^{1-\alpha_2}}{\Gamma(\alpha-\beta)(\frac{\alpha-\beta-\alpha_2}{1-\alpha_2})^{1-\alpha_2}} \\
&+ \frac{|1-\lambda_1|+|\lambda_1-\lambda_2|}{|(\lambda_1-1)+(\lambda_2-\lambda_1)\eta^{m-1}|}\frac{\Gamma(m)((t+\delta)^{m-\beta-1}-(t)^{m-\beta-1})}{\Gamma(m-\beta-1)\Gamma(\alpha)(\frac{\alpha-\alpha_2}{1-\alpha_2})^{1-\alpha_2}}\biggr].
\end{aligned}
$$

Hence, we get

$$
\begin{aligned}
&\bigl\|(Fu)(t+\delta)-(Fu)(t)\bigr\|_{\mathbb{X}}\\
&\quad\leq \frac{H}{\Gamma(\alpha)(\frac{\alpha-\alpha_2}{1-\alpha_2})^{1-\alpha_2}}\biggl[\delta^{\alpha-\alpha_2}+\bigl((t+\delta)^{\frac{\alpha-\alpha_2}{1-\alpha_2}}-\delta^{\frac{\alpha-\alpha_2}{1-\alpha_2}}-t^{\frac{\alpha-\alpha_2}{1-\alpha_2}}\bigr)^{1-\alpha_2}\\
&\qquad+\frac{(|1-\lambda_1|+|\lambda_1-\lambda_2|)}{|(\lambda_1-1)+(\lambda_2-\lambda_1)\eta^{m-1}|}\bigl((t+\delta)^{m-1}-t^{m-1}\bigr)\biggr]\\
&\qquad+H\biggl[\frac{\delta^{\alpha-\beta-\alpha_2}+((t+\delta)^{\frac{\alpha-\beta-\alpha_2}{1-\alpha_2}}-\delta^{\frac{\alpha-\beta-\alpha_2}{1-\alpha_2}}-t^{\frac{\alpha-\beta-\alpha_2}{1-\alpha_2}})^{1-\alpha_2}}{\Gamma(\alpha-\beta)(\frac{\alpha-\beta-\alpha_2}{1-\alpha_2})^{1-\alpha_2}}\\
&\qquad+\frac{|1-\lambda_1|+|\lambda_1-\lambda_2|}{(\lambda_1-1)+(\lambda_2-\lambda_1)\eta^{m-1}}\frac{\Gamma(m)((t+\delta)^{m-\beta-1}-(t)^{m-\beta-1})}{\Gamma(m-\beta-1)\Gamma(\alpha)(\frac{\alpha-\alpha_2}{1-\alpha_2})^{1-\alpha_2}}\biggr].
\end{aligned}
$$

It is obvious that the right-hand side of the above inequality tends to zero as $\delta\to 0$. Therefore, F is continuous on J. Moreover, for $u\in B_r$ and all $t\in J$, we get

$$
\begin{aligned}
&\bigl|(Fu)(t)\bigr|\\
&\quad\leq\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\bigl|f\bigl(s,u(s),{}^{\mathrm{C}}D_t^{\beta}u(s)\bigr)\bigr|\,ds\\
&\qquad+\biggl|\frac{\lambda_1\eta^{m-1}+(1-\lambda_1)t^{m-1}}{(\lambda_1-1)+(\lambda_2-\lambda_1)\eta^{m-1}}\biggr|\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\bigl|f\bigl(s,u(s),{}^{\mathrm{C}}D_t^{\beta}u(s)\bigr)\bigr|\,ds\\
&\qquad+\biggl|\frac{\lambda_1+(\lambda_2-\lambda_1)t^{m-1}}{(\lambda_1-1)+(\lambda_2-\lambda_1)\eta^{m-1}}\biggr|\int_0^{\eta}\frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}\bigl|f\bigl(s,u(s),{}^{\mathrm{C}}D_t^{\beta}u(s)\bigr)\bigr|\,ds\\
&\quad\leq\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)\,ds+\frac{|\lambda_1\eta^{m-1}|+|(1-\lambda_1)t^{m-1}|}{|(\lambda_1-1)+(\lambda_2-\lambda_1)\eta^{m-1}|}\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)\,ds\\
&\qquad+\frac{|\lambda_1|+|(\lambda_2-\lambda_1)t^{m-1}|}{|(\lambda_1-1)+(\lambda_2-\lambda_1)\eta^{m-1}|}\int_0^{\eta}\frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)\,ds\\
&\quad\leq\frac{1}{\Gamma(\alpha)}\biggl(\int_0^t(t-s)^{\frac{\alpha-1}{1-\alpha_2}}\,ds\biggr)^{1-\alpha_2}\biggl(\int_0^t\bigl(h(s)\bigr)^{\frac{1}{\alpha_2}}\,ds\biggr)^{\alpha_2}\\
&\qquad+\frac{|\lambda_1||\eta^{m-1}|+|1-\lambda_1||t^{m-1}|}{|(\lambda_1-1)+(\lambda_2-\lambda_1)\eta^{m-1}|}\frac{1}{\Gamma(\alpha)}\biggl(\int_0^1(1-s)^{\frac{\alpha-1}{1-\alpha_2}}\,ds\biggr)^{1-\alpha_2}\biggl(\int_0^1\bigl(h(s)\bigr)^{\frac{1}{\alpha_2}}\,ds\biggr)^{\alpha_2}\\
&\qquad+\frac{|\lambda_1|+|\lambda_2-\lambda_1||t^{m-1}|}{|(\lambda_1-1)+(\lambda_2-\lambda_1)\eta^{m-1}|}\frac{1}{\Gamma(\alpha)}\biggl(\int_0^{\eta}(\eta-s)^{\frac{\alpha-1}{1-\alpha_2}}\,ds\biggr)^{1-\alpha_2}\biggl(\int_0^{\eta}\bigl(h(s)\bigr)^{\frac{1}{\alpha_2}}\,ds\biggr)^{\alpha_2}\\
&\quad\leq\frac{1}{\Gamma(\alpha)}\biggl(\int_0^t(t-s)^{\frac{\alpha-1}{1-\alpha_2}}\,ds\biggr)^{1-\alpha_2}\biggl(\int_0^1\bigl(h(s)\bigr)^{\frac{1}{\alpha_2}}\,ds\biggr)^{\alpha_2}\\
&\qquad+\frac{|\lambda_1|+|1-\lambda_1|}{|(\lambda_1-1)+(\lambda_2-\lambda_1)\eta^{m-1}|}\frac{1}{\Gamma(\alpha)}\biggl(\int_0^1(1-s)^{\frac{\alpha-1}{1-\alpha_2}}\,ds\biggr)^{1-\alpha_2}\biggl(\int_0^1\bigl(h(s)\bigr)^{\frac{1}{\alpha_2}}\,ds\biggr)^{\alpha_2}\\
&\qquad+\frac{|\lambda_1|+|\lambda_2-\lambda_1|}{|(\lambda_1-1)+(\lambda_2-\lambda_1)\eta^{m-1}|}\frac{1}{\Gamma(\alpha)}\biggl(\int_0^{\eta}(\eta-s)^{\frac{\alpha-1}{1-\alpha_2}}\,ds\biggr)^{1-\alpha_2}\biggl(\int_0^1\bigl(h(s)\bigr)^{\frac{1}{\alpha_2}}\,ds\biggr)^{\alpha_2}\\
&\quad\leq\frac{\|h\|_{L^{\frac{1}{\alpha_2}}(J,\mathbb{R}_+)}}{\Gamma(\alpha)(\frac{\alpha-\alpha_2}{1-\alpha_2})^{1-\alpha_2}}\biggl(t^{\alpha-\alpha_2}+\frac{|\lambda_1|+|1-\lambda_1|}{|(\lambda_1-1)+(\lambda_2-\lambda_1)\eta^{m-1}|}+\frac{(|\lambda_1|+|\lambda_2-\lambda_1|)\eta^{\alpha-\alpha_2}}{|(\lambda_1-1)+(\lambda_2-\lambda_1)\eta^{m-1}|}\biggr)\\
&\quad\leq\frac{H}{\Gamma(\alpha)(\frac{\alpha-\alpha_2}{1-\alpha_2})^{1-\alpha_2}}\biggl(1+\frac{2|\lambda_1|+|1-\lambda_1|+|\lambda_2-\lambda_1|}{|(\lambda_1-1)+(\lambda_2-\lambda_1)\eta^{m-1}|}\biggr)=\frac{H(1+\Lambda_1)}{\Gamma(\alpha)(\frac{\alpha-\alpha_2}{1-\alpha_2})^{1-\alpha_2}}.
\end{aligned}
$$

In view of Lemma 2.1, we have

$$
\begin{aligned}
&\left|{}^{\mathrm{C}}D^{\beta}(Fu)(t)\right| \\
&\quad\le \int_0^t \frac{(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\left|f\left(s,u(s),{}^{\mathrm{C}}D^{\beta}u(s)\right)\right|\,ds \\
&\qquad+\left[\frac{|1-\lambda_1|}{|(\lambda_1-1)+(\lambda_2-\lambda_1)\eta^{m-1}|}\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\left|f\left(s,u(s),{}^{\mathrm{C}}D^{\beta}u(s)\right)\right|\,ds\right. \\
&\qquad+\left.\frac{|\lambda_2-\lambda_1|}{|(\lambda_1-1)+(\lambda_2-\lambda_1)\eta^{m-1}|}\int_0^{\eta}\frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}\left|f\left(s,u(s),{}^{\mathrm{C}}D^{\beta}u(s)\right)\right|\,ds\right]{}^{\mathrm{C}}D^{\beta}t^{m-1} \\
&\quad\le \int_0^t\frac{(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}h(s)\,ds+\left[\frac{|1-\lambda_1|}{|(\lambda_1-1)+(\lambda_2-\lambda_1)\eta^{m-1}|}\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)\,ds\right. \\
&\qquad+\left.\frac{|\lambda_2-\lambda_1|}{|(\lambda_1-1)+(\lambda_2-\lambda_1)\eta^{m-1}|}\int_0^{\eta}\frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)\,ds\right]\frac{\Gamma(m)t^{m-\beta-1}}{\Gamma(m-\beta)} \\
&\quad\le \frac{1}{\Gamma(\alpha-\beta)}\left(\int_0^t(t-s)^{\frac{\alpha-\beta-1}{1-\alpha_2}}\,ds\right)^{1-\alpha_2}\left(\int_0^1\left(h(s)\right)^{\frac{1}{\alpha_2}}\,ds\right)^{\alpha_2} \\
&\qquad+\left[\frac{|1-\lambda_1|}{|(\lambda_1-1)+(\lambda_2-\lambda_1)\eta^{m-1}|}\frac{1}{\Gamma(\alpha)}\left(\int_0^1(1-s)^{\frac{\alpha-1}{1-\alpha_2}}\,ds\right)^{1-\alpha_2}\left(\int_0^1\left(h(s)\right)^{\frac{1}{\alpha_2}}\,ds\right)^{\alpha_2}\right. \\
&\qquad+\left.\frac{|\lambda_2-\lambda_1|}{|(\lambda_1-1)+(\lambda_2-\lambda_1)\eta^{m-1}|}\frac{1}{\Gamma(\alpha)}\left(\int_0^{\eta}(\eta-s)^{\frac{\alpha-1}{1-\alpha_2}}\,ds\right)^{1-\alpha_2}\left(\int_0^{\eta}\left(h(s)\right)^{\frac{1}{\alpha_2}}\,ds\right)^{\alpha_2}\right] \\
&\qquad\times\frac{\Gamma(m)t^{m-\beta-1}}{\Gamma(m-\beta)} \\
&\quad\le H\left[\frac{1}{\Gamma(\alpha-\beta)(\frac{\alpha-\beta-\alpha_2}{1-\alpha_2})^{1-\alpha_2}}+\frac{(|1-\lambda_1|+|\lambda_2-\lambda_1|)\Gamma(m)}{|(\lambda_1-1)+(\lambda_2-\lambda_1)\eta^{m-1}|\Gamma(m-\beta)\Gamma(\alpha)(\frac{\alpha-\alpha_2}{1-\alpha_2})^{1-\alpha_2}}\right] \\
&\quad= H\left(\frac{1}{\Gamma(\alpha-\beta)(\frac{\alpha-\beta-\alpha_2}{1-\alpha_2})^{1-\alpha_2}}+\frac{\Lambda_2}{\Gamma(\alpha)(\frac{\alpha-\alpha_2}{1-\alpha_2})^{1-\alpha_2}}\right).
\end{aligned}
$$

Therefore,

$$
\|Fu\|_{\mathbb{X}}\le H\left(\frac{1}{\Gamma(\alpha-\beta)(\frac{\alpha-\beta-\alpha_2}{1-\alpha_2})^{1-\alpha_2}}+\frac{1+\Lambda_1+\Lambda_2}{\Gamma(\alpha)(\frac{\alpha-\alpha_2}{1-\alpha_2})^{1-\alpha_2}}\right)\le r.
$$

Notice that $(Fu)(t)$ and ${}^{\mathrm{C}}D^{\beta}(Fu)(t)$ are continuous on J. Thus, we can conclude that for all $u\in B_r$, $Fu\in B_r$, *i.e.*, $F:B_r\to B_r$.

Step 2. F is a contraction mapping on B_r.

For $u,v\in B_r$ and any $t\in J$, using (H2) and Hölder's inequality, we get

$$
\begin{aligned}
&\left|(Fu)(t)-(Fv)(t)\right| \\
&\quad\le \int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left|f\left(s,u(s),{}^{\mathrm{C}}D^{\beta}u(s)\right)-f\left(s,v(s),{}^{\mathrm{C}}D^{\beta}v(s)\right)\right|\,ds \\
&\qquad+\left|\frac{\lambda_1\eta^{m-1}+(1-\lambda_1)t^{m-1}}{(\lambda_1-1)+(\lambda_2-\lambda_1)\eta^{m-1}}\right| \\
&\qquad\times\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\left|f\left(s,u(s),{}^{\mathrm{C}}D^{\beta}u(s)\right)-f\left(s,v(s),{}^{\mathrm{C}}D^{\beta}v(s)\right)\right|\,ds
\end{aligned}
$$

$$
\begin{aligned}
&\quad + \left| \frac{\lambda_1 + (\lambda_2 - \lambda_1)t^{m-1}}{(\lambda_1 - 1) + (\lambda_2 - \lambda_1)\eta^{m-1}} \right| \\
&\quad \times \int_0^\eta \frac{(\eta - s)^{\alpha-1}}{\Gamma(\alpha)} \left| f\left(s, u(s), {}^{\mathrm{C}}D^\beta u(s)\right) - f\left(s, v(s), {}^{\mathrm{C}}D^\beta v(s)\right) \right| ds \\
&\le \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} m(s) \left(\left| u(s) - v(s) \right| + \left| {}^{\mathrm{C}}D^\beta u(s) - {}^{\mathrm{C}}D^\beta v(s) \right| \right) ds \\
&\quad + \frac{|\lambda_1 \eta^{m-1}| + |(1 - \lambda_1)t^{m-1}|}{|(\lambda_1 - 1) + (\lambda_2 - \lambda_1)\eta^{m-1}|} \\
&\quad \times \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} m(s) \left(\left| u(s) - v(s) \right| + \left| {}^{\mathrm{C}}D^\beta u(s) - {}^{\mathrm{C}}D^\beta v(s) \right| \right) ds \\
&\quad + \frac{|\lambda_1 + |(\lambda_2 - \lambda_1)t^{m-1}|}{|(\lambda_1 - 1) + (\lambda_2 - \lambda_1)\eta^{m-1}|} \\
&\quad \times \int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} m(s) \left(\left| u(s) - v(s) \right| + \left| {}^{\mathrm{C}}D^\beta u(s) - {}^{\mathrm{C}}D^\beta v(s) \right| \right) ds \\
&\le \left[\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} m(s)\, ds + \frac{|\lambda_1| + |1 - \lambda_1|}{|(\lambda_1 - 1) + (\lambda_2 - \lambda_1)\eta^{m-1}|} \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} m(s)\, ds \right. \\
&\quad \left. + \frac{|\lambda_1| + |\lambda_2 - \lambda_1|}{|(\lambda_1 - 1) + (\lambda_2 - \lambda_1)\eta^{m-1}|} \int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} m(s)\, ds \right] \|u - v\|_{\mathbb{X}} \\
&\le \left[\frac{1}{\Gamma(\alpha)} \left(\int_0^t (t-s)^{\frac{\alpha-1}{1-\alpha_1}}\, ds \right)^{1-\alpha_1} \left(\int_0^1 \left(m(s)\right)^{\frac{1}{\alpha_1}}\, ds \right)^{\alpha_1} \right. \\
&\quad + \frac{|\lambda_1| + |1 - \lambda_1|}{|(\lambda_1 - 1) + (\lambda_2 - \lambda_1)\eta^{m-1}|} \frac{1}{\Gamma(\alpha)} \left(\int_0^1 (1-s)^{\frac{\alpha-1}{1-\alpha_1}}\, ds \right)^{1-\alpha_1} \left(\int_0^1 \left(m(s)\right)^{\frac{1}{\alpha_1}}\, ds \right)^{\alpha_1} \\
&\quad \left. + \frac{|\lambda_1| + |\lambda_2 - \lambda_1|}{|(\lambda_1 - 1) + (\lambda_2 - \lambda_1)\eta^{m-1}|} \frac{1}{\Gamma(\alpha)} \left(\int_0^\eta (\eta-s)^{\frac{\alpha-1}{1-\alpha_1}}\, ds \right)^{1-\alpha_1} \left(\int_0^\eta \left(m(s)\right)^{\frac{1}{\alpha_1}}\, ds \right)^{\alpha_1} \right] \\
&\quad \times \|u - v\|_{\mathbb{X}} \\
&\le \frac{\|m\|_{L^{\frac{1}{\alpha_1}}(J,\mathbb{R}_+)}}{\Gamma(\alpha)(\frac{\alpha-\alpha_1}{1-\alpha_1})^{1-\alpha_1}} \left[t^{\alpha-\alpha_1} + \frac{|\lambda_1| + |1 - \lambda_1|}{|(\lambda_1 - 1) + (\lambda_2 - \lambda_1)\eta^{m-1}|} + \frac{(|\lambda_1| + |\lambda_2 - \lambda_1|)\eta^{\alpha-\alpha_1}}{|(\lambda_1 - 1) + (\lambda_2 - \lambda_1)\eta^{m-1}|} \right] \\
&\quad \times \|u - v\|_{\mathbb{X}} \\
&\le \frac{M}{\Gamma(\alpha)(\frac{\alpha-\alpha_1}{1-\alpha_1})^{1-\alpha_1}} \left[1 + \frac{2|\lambda_1| + |1 - \lambda_1| + |\lambda_2 - \lambda_1|}{|(\lambda_1 - 1) + (\lambda_2 - \lambda_1)\eta^{m-1}|} \right] \|u - v\|_{\mathbb{X}} \\
&= \frac{M(1 + \Lambda_1)}{\Gamma(\alpha)(\frac{\alpha-\alpha_1}{1-\alpha_1})^{1-\alpha_1}} \|u - v\|_{\mathbb{X}}.
\end{aligned}
$$

Similarly, we can get

$$
\begin{aligned}
&\left| {}^{\mathrm{C}}D^\beta (Fu)(t) - {}^{\mathrm{C}}D^\beta (Fv)(t) \right| \\
&\le \int_0^t \frac{(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} \left| f\left(s, u(s), {}^{\mathrm{C}}D^\beta u(s)\right) - f\left(s, v(s), {}^{\mathrm{C}}D^\beta v(s)\right) \right| ds \\
&\quad + \left[\frac{|1 - \lambda_1|}{|(\lambda_1 - 1) + (\lambda_2 - \lambda_1)\eta^{m-1}|} \right. \\
&\quad \times \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} \left| f\left(s, u(s), {}^{\mathrm{C}}D^\beta u(s)\right) - f\left(s, v(s), {}^{\mathrm{C}}D^\beta v(s)\right) \right| ds
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{|\lambda_2 - \lambda_1|}{|(\lambda_1 - 1) + (\lambda_2 - \lambda_1)\eta^{m-1}|} \\
&\times \int_0^\eta \frac{(\eta - s)^{\alpha-1}}{\Gamma(\alpha)} \bigl|f\bigl(s, u(s), {}^{\mathrm{C}}D^\beta u(s)\bigr) - f\bigl(s, v(s), {}^{\mathrm{C}}D^\beta v(s)\bigr)\bigr|\, ds \Biggr] \frac{\Gamma(m) t^{m-\beta-1}}{\Gamma(m-\beta)} \\
\le{}& \int_0^t \frac{(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} m(s)\bigl(\bigl|u(s) - v(s)\bigr| + \bigl|{}^{\mathrm{C}}D^\beta u(s) - {}^{\mathrm{C}}D^\beta v(s)\bigr|\bigr)\, ds \\
&+ \Biggl[\frac{|1 - \lambda_1|}{|(\lambda_1 - 1) + (\lambda_2 - \lambda_1)\eta^{m-1}|} \\
&\times \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} m(s)\bigl(\bigl|u(s) - v(s)\bigr| + \bigl|{}^{\mathrm{C}}D^\beta u(s) - {}^{\mathrm{C}}D^\beta v(s)\bigr|\bigr)\, ds \\
&+ \frac{|\lambda_2 - \lambda_1|}{|(\lambda_1 - 1) + (\lambda_2 - \lambda_1)\eta^{m-1}|} \\
&\times \int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} m(s)\bigl(\bigl|u(s) - v(s)\bigr| + \bigl|{}^{\mathrm{C}}D^\beta u(s) - {}^{\mathrm{C}}D^\beta v(s)\bigr|\bigr)\, ds \Biggr] \frac{\Gamma(m) t^{m-\beta-1}}{\Gamma(m-\beta)} \\
\le{}& \frac{1}{\Gamma(\alpha-\beta)} \biggl(\int_0^t (t-s)^{\frac{\alpha-\beta-1}{1-\alpha_1}}\, ds \biggr)^{1-\alpha_1} \biggl(\int_0^t \bigl(m(s)\bigr)^{\frac{1}{\alpha_1}}\, ds \biggr)^{\alpha_1} \|u - v\|_{\mathbb{X}} \\
&+ \Biggl[\frac{|1 - \lambda_1|}{|(\lambda_1 - 1) + (\lambda_2 - \lambda_1)\eta^{m-1}|} \frac{1}{\Gamma(\alpha)} \biggl(\int_0^1 (1-s)^{\frac{\alpha-1}{1-\alpha_1}}\, ds \biggr)^{1-\alpha_1} \biggl(\int_0^1 \bigl(m(s)\bigr)^{\frac{1}{\alpha_1}}\, ds \biggr)^{\alpha_1} \\
&+ \frac{|\lambda_2 - \lambda_1|}{|(\lambda_1 - 1) + (\lambda_2 - \lambda_1)\eta^{m-1}|} \frac{1}{\Gamma(\alpha)} \biggl(\int_0^\eta (\eta-s)^{\frac{\alpha-1}{1-\alpha_1}}\, ds \biggr)^{1-\alpha_1} \biggl(\int_0^\eta \bigl(m(s)\bigr)^{\frac{1}{\alpha_1}}\, ds \biggr)^{\alpha_1} \Biggr] \\
&\times \frac{\Gamma(m)}{\Gamma(m-\beta)} \|u - v\|_{\mathbb{X}} \\
\le{}& \Biggl[\frac{1}{\Gamma(\alpha-\beta)(\frac{\alpha-\beta-\alpha_1}{1-\alpha_1})^{1-\alpha_1}} + \frac{(|1-\lambda_1| + |\lambda_2 - \lambda_1|)\Gamma(m)}{|(\lambda_1 - 1) + (\lambda_2 - \lambda_1)\eta^{m-1}|\Gamma(m-\beta)\Gamma(\alpha)(\frac{\alpha-\alpha_1}{1-\alpha_1})^{1-\alpha_1}} \Biggr] \\
&\times M \|u - v\|_{\mathbb{X}} \\
={}& \biggl(\frac{1}{\Gamma(\alpha-\beta)(\frac{\alpha-\beta-\alpha_2}{1-\alpha_2})^{1-\alpha_2}} + \frac{\Lambda_2}{\Gamma(\alpha)(\frac{\alpha-\alpha_2}{1-\alpha_2})^{1-\alpha_2}} \biggr) M \|u - v\|_{\mathbb{X}}.
\end{aligned}
$$

So we obtain $\|Fu - Fv\|_{\mathbb{X}} \le \Omega \|u - v\|_{\mathbb{X}}$. Thus, F is a contraction due to condition (3.1). By the Banach contraction principle, we can deduce that F has a unique fixed point which is just the unique solution of the fractional boundary value problem (1.1). □

Our second result is based on the well-known Schaefer's fixed point theorem. We make the following assumptions:

(H4) $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous.

(H5) There exists a constant $L > 0$ such that

$$
\bigl\|f(t, u, v)\bigr\| \le L\bigl(1 + |u| + |v|\bigr)
$$

for each $t \in J$ and all $u, v \in \mathbb{X}$, with

$$
L \biggl(\frac{1}{\Gamma(\alpha - \beta + 1)} + \frac{1 + \Lambda_1 + \Lambda_2}{\Gamma(\alpha + 1)} \biggr) \neq 1.
$$

Theorem 3.2 *Assume that* (H4) *and* (H5) *hold and there exists a constant* $M^* > 0$ *such that*

$$M^* \geq \frac{L(\frac{1}{\Gamma(\alpha-\beta+1)} + \frac{1+\Lambda_1+\Lambda_2}{\Gamma(\alpha+1)})}{1 - L(\frac{1}{\Gamma(\alpha-\beta+1)} + \frac{1+\Lambda_1+\Lambda_2}{\Gamma(\alpha+1)})}.$$

Then the fractional boundary value problem (1.1) *has at least one solution on* J.

Proof Transform the fractional boundary value problem (1.1) into a fixed point problem. Consider the operator $F : C(J,\mathbb{X}) \to C(J,\mathbb{X})$ defined as (3.2). It is obvious that F is well defined due to (H4).

For the sake of convenience, we subdivide the proof into several steps.

Step 1. F is continuous.

Let $\{u_n\}$ be a sequence such that $u_n \to u$ in $C(J,\mathbb{X})$. Then, for each $t \in J$, we have

$$\begin{aligned}
&\bigl|(Fu_n)(t) - (Fu)(t)\bigr| \\
&\quad\leq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \bigl|f\bigl(s,u_n(s),{}^{\mathrm{C}}D^\beta u_n(s)\bigr) - f\bigl(s,u(s),{}^{\mathrm{C}}D^\beta u(s)\bigr)\bigr|\,ds \\
&\qquad + \left|\frac{\lambda_1\eta^{m-1} + (1-\lambda_1)t^{m-1}}{(\lambda_1-1) + (\lambda_2-\lambda_1)\eta^{m-1}}\right| \\
&\qquad \times \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} \bigl|f\bigl(s,u_n(s),{}^{\mathrm{C}}D^\beta u_n(s)\bigr) - f\bigl(s,u(s),{}^{\mathrm{C}}D^\beta u(s)\bigr)\bigr|\,ds \\
&\qquad + \left|\frac{\lambda_1 + (\lambda_2-\lambda_1)t^{m-1}}{(\lambda_1-1) + (\lambda_2-\lambda_1)\eta^{m-1}}\right| \\
&\qquad \times \int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} \bigl|f\bigl(s,u_n(s),{}^{\mathrm{C}}D^\beta u_n(s)\bigr) - f\bigl(s,u(s),{}^{\mathrm{C}}D^\beta u(s)\bigr)\bigr|\,ds \\
&\quad\leq \biggl[\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\,ds + \frac{|\lambda_1| + |1-\lambda_1|}{|(\lambda_1-1) + (\lambda_2-\lambda_1)\eta^{m-1}|} \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\,ds \\
&\qquad + \frac{|\lambda_1| + |\lambda_2-\lambda_1|}{|(\lambda_1-1) + (\lambda_2-\lambda_1)\eta^{m-1}|} \int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}\,ds\biggr] \\
&\qquad \times \sup_{t\in J}\bigl|f\bigl(s,u_n(t),{}^{\mathrm{C}}D^\beta u_n(t)\bigr) - f\bigl(s,u(t),{}^{\mathrm{C}}D^\beta u(t)\bigr)\bigr| \\
&\quad\leq \left(\frac{1+\Lambda_1}{\Gamma(\alpha+1)}\right) \sup_{t\in J}\bigl|f\bigl(s,u_n(t),{}^{\mathrm{C}}D^\beta u_n(t)\bigr) - f\bigl(s,u(t),{}^{\mathrm{C}}D^\beta u(t)\bigr)\bigr|.
\end{aligned}$$

Also, we can get

$$\begin{aligned}
&\bigl|{}^{\mathrm{C}}D^\beta (Fu_n)(t) - {}^{\mathrm{C}}D^\beta (Fu)(t)\bigr| \\
&\quad\leq \int_0^t \frac{(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} \bigl|f\bigl(s,u_n(s),{}^{\mathrm{C}}D^\beta u_n(s)\bigr) - f\bigl(s,u(s),{}^{\mathrm{C}}D^\beta u(s)\bigr)\bigr|\,ds \\
&\qquad + \biggl[\frac{|1-\lambda_1|}{|(\lambda_1-1) + (\lambda_2-\lambda_1)\eta^{m-1}|} \\
&\qquad \times \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} \bigl|f\bigl(s,u_n(s),{}^{\mathrm{C}}D^\beta u_n(s)\bigr) - f\bigl(s,u(s),{}^{\mathrm{C}}D^\beta u(s)\bigr)\bigr|\,ds \\
&\qquad + \frac{|\lambda_2-\lambda_1|}{|(\lambda_1-1) + (\lambda_2-\lambda_1)\eta^{m-1}|}
\end{aligned}$$

$$\begin{aligned}
&\times \int_0^\eta \frac{(\eta - s)^{\alpha-1}}{\Gamma(\alpha)} \bigl|f\bigl(s, u_n(s), {}^{\mathrm{C}}D^\beta u_n(s)\bigr) - f\bigl(s, u(s), {}^{\mathrm{C}}D^\beta u(s)\bigr)\bigr| \, ds\Biggr] \frac{\Gamma(m) t^{m-\beta-1}}{\Gamma(m-\beta)} \\
&\leq \Biggl[\int_0^t \frac{(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\, ds + \frac{\Gamma(m)}{\Gamma(m-\beta)}\biggl(\frac{|1-\lambda_1|}{|(\lambda_1 - 1) + (\lambda_2 - \lambda_1)\eta^{m-1}|} \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\, ds \\
&\quad + \frac{|\lambda_2 - \lambda_1|}{|(\lambda_1 - 1) + (\lambda_2 - \lambda_1)\eta^{m-1}|} \int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}\, ds\biggr)\Biggr] \\
&\quad \times \sup_{t\in J}\bigl|f\bigl(s, u_n(t), {}^{\mathrm{C}}D^\beta u_n(t)\bigr) - f\bigl(s, u(t), {}^{\mathrm{C}}D^\beta u(t)\bigr)\bigr| \\
&\leq \biggl(\frac{1}{\Gamma(\alpha-\beta+1)} + \frac{\Lambda_2}{\Gamma(\alpha+1)}\biggr) \sup_{t\in J}\bigl|f\bigl(s, u_n(t), {}^{\mathrm{C}}D^\beta u_n(t)\bigr) - f\bigl(s, u(t), {}^{\mathrm{C}}D^\beta u(t)\bigr)\bigr|.
\end{aligned}$$

Thus, we get

$$\begin{aligned}
&\bigl\|(Fu_n)(t) - (Fu)(t)\bigr\|_{\mathbb{X}} \\
&\quad \leq \biggl(\frac{1}{\Gamma(\alpha-\beta+1)} + \frac{1+\Lambda_1+\Lambda_2}{\Gamma(\alpha+1)}\biggr) \sup_{t\in J}\bigl|f\bigl(s, u_n(t), {}^{\mathrm{C}}D^\beta u_n(t)\bigr) - f\bigl(s, u(t), {}^{\mathrm{C}}D^\beta u(t)\bigr)\bigr|.
\end{aligned}$$

Since f is continuous, then $\|(Fu_n)(t) - (Fu)(t)\|_{\mathbb{X}} \to 0$ as $n \to \infty$.

Step 2. F maps bounded sets into bounded sets in $C(J, \mathbb{X})$.

Indeed, it is enough to show that for any $r' > 0$, there exists $l > 0$ such that for each $u \in B_{r'} = \{u \in C(J, \mathbb{X}) : \|u\|_{\mathbb{X}} < r'\}$, we have $\|Fu\|_{\mathbb{X}} \leq l$.

Then, for each $t \in J$ and (H5), we have

$$\begin{aligned}
\bigl|(Fu)(t)\bigr| &\leq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \bigl|f\bigl(s, u(s), {}^{\mathrm{C}}D^\beta u(s)\bigr)\bigr|\, ds \\
&\quad + \biggl|\frac{\lambda_1\eta^{m-1} + (1-\lambda_1)t^{m-1}}{(\lambda_1 - 1) + (\lambda_2 - \lambda_1)\eta^{m-1}}\biggr| \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} \bigl|f\bigl(s, u(s), {}^{\mathrm{C}}D^\beta u(s)\bigr)\bigr|\, ds \\
&\quad + \biggl|\frac{\lambda_1 + (\lambda_2-\lambda_1)t^{m-1}}{(\lambda_1 - 1) + (\lambda_2 - \lambda_1)\eta^{m-1}}\biggr| \int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} \bigl|f\bigl(s, u(s), {}^{\mathrm{C}}D^\beta u(s)\bigr)\bigr|\, ds \\
&\leq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} L\bigl(1 + \bigl|u(s)\bigr| + \bigl|{}^{\mathrm{C}}D^\beta u(s)\bigr|\bigr)\, ds \\
&\quad + \frac{|\lambda_1| + |1-\lambda_1|}{|(\lambda_1 - 1) + (\lambda_2 - \lambda_1)\eta^{m-1}|} \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} L\bigl(1 + \bigl|u(s)\bigr| + \bigl|{}^{\mathrm{C}}D^\beta u(s)\bigr|\bigr)\, ds \\
&\quad + \frac{|\lambda_1| + |\lambda_2-\lambda_1|}{|(\lambda_1 - 1) + (\lambda_2 - \lambda_1)\eta^{m-1}|} \int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} L\bigl(1 + \bigl|u(s)\bigr| + \bigl|{}^{\mathrm{C}}D^\beta u(s)\bigr|\bigr)\, ds \\
&\leq \frac{L(1+\Lambda_1)}{\Gamma(\alpha+1)}\bigl(1 + \|u\|_{\mathbb{X}}\bigr).
\end{aligned}$$

Also, we can get

$$\begin{aligned}
&\bigl|\bigl({}^{\mathrm{C}}D^\beta Fu\bigr)(t)\bigr| \\
&\quad \leq \int_0^t \frac{(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} \bigl|f\bigl(s, u(s), {}^{\mathrm{C}}D^\beta u(s)\bigr)\bigr|\, ds \\
&\qquad + \Biggl[\frac{|1-\lambda_1|}{|(\lambda_1 - 1) + (\lambda_2 - \lambda_1)\eta^{m-1}|} \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} \bigl|f\bigl(s, u(s), {}^{\mathrm{C}}D^\beta u(s)\bigr)\bigr|\, ds
\end{aligned}$$

$$
\begin{aligned}
&\quad + \frac{|\lambda_2-\lambda_1|}{|(\lambda_1-1)+(\lambda_2-\lambda_1)\eta^{m-1}|}\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}\bigl|f\bigl(s,u(s),{}^{\mathrm{C}}D^\beta u(s)\bigr)\bigr|\,ds\Biggr]\frac{\Gamma(m)t^{m-\beta-1}}{\Gamma(m-\beta)} \\
&\le \int_0^t \frac{(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}L\bigl(1+\bigl|u(s)\bigr|+\bigl|{}^{\mathrm{C}}D^\beta u(s)\bigr|\bigr)\,ds \\
&\quad + \Biggl[\frac{|\lambda_1|+|1-\lambda_1|}{|(\lambda_1-1)+(\lambda_2-\lambda_1)\eta^{m-1}|}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}L\bigl(1+\bigl|u(s)\bigr|+\bigl|{}^{\mathrm{C}}D^\beta u(s)\bigr|\bigr)\,ds \\
&\quad + \frac{|\lambda_1|+|\lambda_2-\lambda_1|}{|(\lambda_1-1)+(\lambda_2-\lambda_1)\eta^{m-1}|}\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}L\bigl(1+\bigl|u(s)\bigr|+\bigl|{}^{\mathrm{C}}D^\beta u(s)\bigr|\bigr)\,ds\Biggr]\frac{\Gamma(m)}{\Gamma(m-\beta)} \\
&\le L\biggl(\frac{1}{\Gamma(\alpha-\beta+1)}+\frac{\Lambda_2}{\Gamma(\alpha+1)}\biggr)\bigl(1+\|u\|_{\mathbb{X}}\bigr).
\end{aligned}
$$

Hence, we get

$$
\bigl\|(Fu)(t)\bigr\|_{\mathbb{X}} \le L\biggl(\frac{1}{\Gamma(\alpha-\beta+1)}+\frac{1+\Lambda_1+\Lambda_2}{\Gamma(\alpha+1)}\biggr)\bigl(1+\|u\|_{\mathbb{X}}\bigr) := l.
$$

Step 3. F maps bounded sets into equicontinuous sets of $C(J,\mathbb{X})$.

For $u \in B_{r'}$ and $t_1, t_2 \in J$ such that $t_1 < t_2$. Then, using (H5), we have

$$
\begin{aligned}
&\bigl|(Fu)(t_2)-(Fu)(t_1)\bigr| \\
&\le \biggl|\int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}f\bigl(s,u(s),{}^{\mathrm{C}}D^\beta u(s)\bigr)\,ds\biggr| \\
&\quad + \biggl|\int_0^{t_1} \frac{((t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1})}{\Gamma(\alpha)}f\bigl(s,u(s),{}^{\mathrm{C}}D^\beta u(s)\bigr)\,ds\biggr| \\
&\quad + \biggl|\frac{(1-\lambda_1)(t_2^{m-1}-t_1^{m-1})}{(\lambda_1-1)+(\lambda_2-\lambda_1)\eta^{m-1}}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}f\bigl(s,u(s),{}^{\mathrm{C}}D^\beta u(s)\bigr)\,ds\biggr| \\
&\quad + \biggl|\frac{(\lambda_1-\lambda_2)(t_2^{m-1}-t_1^{m-1})}{(\lambda_1-1)+(\lambda_2-\lambda_1)\eta^{m-1}}\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}f\bigl(s,u(s),{}^{\mathrm{C}}D^\beta u(s)\bigr)\,ds\biggr| \\
&\le \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}\bigl|f\bigl(s,u(s),{}^{\mathrm{C}}D^\beta u(s)\bigr)\bigr|\,ds \\
&\quad + \int_0^{t_1} \frac{((t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1})}{\Gamma(\alpha)}\bigl|f\bigl(s,u(s),{}^{\mathrm{C}}D^\beta u(s)\bigr)\bigr|\,ds \\
&\quad + \frac{|(1-\lambda_1)(t_2^{m-1}-t_1^{m-1})|}{|(\lambda_1-1)+(\lambda_2-\lambda_1)\eta^{m-1}|}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\bigl|f\bigl(s,u(s),{}^{\mathrm{C}}D^\beta u(s)\bigr)\bigr|\,ds \\
&\quad + \frac{|(\lambda_1-\lambda_2)(t_2^{m-1}-t_1^{m-1})|}{|(\lambda_1-1)+(\lambda_2-\lambda_1)\eta^{m-1}|}\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}\bigl|f\bigl(s,u(s),{}^{\mathrm{C}}D^\beta u(s)\bigr)\bigr|\,ds \\
&\le \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}L\bigl(1+\bigl|u(s)\bigr|+\bigl|{}^{\mathrm{C}}D^\beta u(s)\bigr|\bigr)\,ds \\
&\quad + \int_0^{t_1} \frac{((t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1})}{\Gamma(\alpha)}L\bigl(1+\bigl|u(s)\bigr|+\bigl|{}^{\mathrm{C}}D^\beta u(s)\bigr|\bigr)\,ds \\
&\quad + \frac{|1-\lambda_1|(t_2^{m-1}-t_1^{m-1})}{|(\lambda_1-1)+(\lambda_2-\lambda_1)\eta^{m-1}|}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}L\bigl(1+\bigl|u(s)\bigr|+\bigl|{}^{\mathrm{C}}D^\beta u(s)\bigr|\bigr)\,ds \\
&\quad + \frac{|\lambda_1-\lambda_2|(t_2^{m-1}-t_1^{m-1})}{|(\lambda_1-1)+(\lambda_2-\lambda_1)\eta^{m-1}|}\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}L\bigl(1+\bigl|u(s)\bigr|+\bigl|{}^{\mathrm{C}}D^\beta u(s)\bigr|\bigr)\,ds
\end{aligned}
$$

$$
\begin{aligned}
&\leq \left[\int_{t_1}^{t_2} (t_2-s)^{\alpha-1}\,ds + \int_0^{t_1} \bigl((t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\bigr)\,ds \right. \\
&\quad + \frac{|1-\lambda_1|(t_2^{m-1}-t_1^{m-1})}{|(\lambda_1-1)+(\lambda_2-\lambda_1)\eta^{m-1}|}\int_0^1 (1-s)^{\alpha-1}\,ds \\
&\quad \left. + \frac{|\lambda_1-\lambda_2|(t_2^{m-1}-t_1^{m-1})}{|(\lambda_1-1)+(\lambda_2-\lambda_1)\eta^{m-1}|}\int_0^\eta (\eta-s)^{\alpha-1}\,ds\right]\frac{L}{\Gamma(\alpha)}\bigl(1+\|u\|_{\mathbb{X}}\bigr) \\
&= \left[(t_2-t_1)^\alpha + \bigl(t_1^\alpha - t_2^\alpha\bigr) - (t_2-t_1)^\alpha + \frac{|1-\lambda_1|(t_2^{m-1}-t_1^{m-1})}{|(\lambda_1-1)+(\lambda_2-\lambda_1)\eta^{m-1}|}\right. \\
&\quad \left. + \frac{|\lambda_1-\lambda_2|(t_2^{m-1}-t_1^{m-1})}{|(\lambda_1-1)+(\lambda_2-\lambda_1)\eta^{m-1}|}\eta^\alpha\right]\frac{L}{\Gamma(\alpha+1)}\bigl(1+\|u\|_{\mathbb{X}}\bigr) \\
&\leq \left[\bigl(t_1^\alpha - t_2^\alpha\bigr) + \frac{(|1-\lambda_1|+|\lambda_1-\lambda_2|)(t_2^{m-1}-t_1^{m-1})}{|(\lambda_1-1)+(\lambda_2-\lambda_1)\eta^{m-1}|}\right]\frac{L}{\Gamma(\alpha+1)}\bigl(1+\|u\|_{\mathbb{X}}\bigr).
\end{aligned}
$$

Also, we can get

$$
\begin{aligned}
&\bigl|{}^{\mathrm{C}}D^\beta (Fu)(t_2) - {}^{\mathrm{C}}D^\beta (Fu)(t_1)\bigr| \\
&\quad \leq \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\bigl|f\bigl(s,u(s),{}^{\mathrm{C}}D^\beta u(s)\bigr)\bigr|\,ds \\
&\qquad + \int_0^{t_1} \frac{((t_2-s)^{\alpha-\beta-1}-(t_1-s)^{\alpha-\beta-1})}{\Gamma(\alpha-\beta)}\bigl|f\bigl(s,u(s),{}^{\mathrm{C}}D^\beta u(s)\bigr)\bigr|\,ds \\
&\qquad + \left[\frac{|1-\lambda_1|}{|(\lambda_1-1)+(\lambda_2-\lambda_1)\eta^{m-1}|}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\bigl|f\bigl(s,u(s),{}^{\mathrm{C}}D^\beta u(s)\bigr)\bigr|\,ds\right. \\
&\qquad \left. + \frac{|\lambda_1-\lambda_2|}{|(\lambda_1-1)+(\lambda_2-\lambda_1)\eta^{m-1}|}\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}\bigl|f\bigl(s,u(s),{}^{\mathrm{C}}D^\beta u(s)\bigr)\bigr|\,ds\right] \\
&\qquad \times \frac{\Gamma(m)(t_2^{m-\beta-1}-t_1^{m-\beta-1})}{\Gamma(m-\beta)} \\
&\quad \leq \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}L\bigl(1+\bigl|u(s)\bigr|+\bigl|{}^{\mathrm{C}}D^\beta u(s)\bigr|\bigr)\,ds \\
&\qquad + \int_0^{t_1} \frac{((t_2-s)^{\alpha-\beta-1}-(t_1-s)^{\alpha-\beta-1})}{\Gamma(\alpha-\beta)}L\bigl(1+\bigl|u(s)\bigr|+\bigl|{}^{\mathrm{C}}D^\beta u(s)\bigr|\bigr)\,ds \\
&\qquad + \left[\frac{|1-\lambda_1|}{|(\lambda_1-1)+(\lambda_2-\lambda_1)\eta^{m-1}|}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}L\bigl(1+\bigl|u(s)\bigr|+\bigl|{}^{\mathrm{C}}D^\beta u(s)\bigr|\bigr)\,ds\right. \\
&\qquad \left. + \frac{|\lambda_1-\lambda_2|}{|(\lambda_1-1)+(\lambda_2-\lambda_1)\eta^{m-1}|}\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}L\bigl(1+\bigl|u(s)\bigr|+\bigl|{}^{\mathrm{C}}D^\beta u(s)\bigr|\bigr)\,ds\right] \\
&\qquad \times \frac{\Gamma(m)(t_2^{m-\beta-1}-t_1^{m-\beta-1})}{\Gamma(m-\beta)} \\
&\quad \leq \left[\frac{(t_1^{\alpha-\beta}-t_2^{\alpha-\beta})}{\Gamma(\alpha-\beta+1)} + \frac{(|1-\lambda_1|+|\lambda_1-\lambda_2|)\Gamma(m)(t_2^{m-\beta-1}-t_1^{m-\beta-1})}{|(\lambda_1-1)+(\lambda_2-\lambda_1)\eta^{m-1}|\Gamma(m-\beta)\Gamma(\alpha+1)}\right]L\bigl(1+\|u\|_{\mathbb{X}}\bigr).
\end{aligned}
$$

Hence, we get

$$
\begin{aligned}
&\bigl\|(Fu)(t_2)-(Fu)(t_1)\bigr\|_{\mathbb{X}} \\
&\quad \leq \left[\frac{(t_1^\alpha-t_2^\alpha)}{\Gamma(\alpha+1)} + \frac{\Lambda_2\Gamma(m-\beta)(t_2^{m-1}-t_1^{m-1})}{\Gamma(\alpha+1)\Gamma(m)} + \frac{(t_1^{\alpha-\beta}-t_2^{\alpha-\beta})}{\Gamma(\alpha-\beta+1)} + \frac{\Lambda_2(t_2^{m-\beta-1}-t_1^{m-\beta-1})}{\Gamma(\alpha+1)}\right] \\
&\qquad \times L\bigl(1+\|u\|_{\mathbb{X}}\bigr).
\end{aligned}
$$

Now, using the fact that the functions $t_1^{\alpha} - t_2^{\alpha}$, $t_2^{m-1} - t_1^{m-1}$, $t_1^{\alpha-\beta} - t_2^{\alpha-\beta}$, and $t_2^{m-\beta-1} - t_1^{m-\beta-1}$ are uniformly continuous on J, we conclude that the right-hand side of the above inequality tends to zero as $t_2 \to t_1$, therefore F is equicontinuous. As a consequence of Steps 1-3 together with the Arzela-Ascoli theorem, we can conclude that F is continuous and completely continuous.

Step 4. *A priori* bounds.

Now it remains to show that the set $E(F) = \{u \in C(J,\mathbb{X}) : u = \lambda Fu \text{ for some } \lambda \in (0,1)\}$ is bounded. Let $u \in E(F)$, then $u = \lambda Fu$ for some $\lambda \in (0,1)$. Thus, for each $t \in J$, we have

$$\begin{aligned} u(t) = \lambda \biggl[&\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f\bigl(s, u(s), {}^{\mathrm{C}}D^{\beta}u(s)\bigr)\,ds \\ &+ \frac{\lambda_1 \eta^{m-1} + (1-\lambda_1)t^{m-1}}{(\lambda_1 - 1) + (\lambda_2 - \lambda_1)\eta^{m-1}} \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} f\bigl(s, u(s), {}^{\mathrm{C}}D^{\beta}u(s)\bigr)\,ds \\ &- \frac{\lambda_1 + (\lambda_2 - \lambda_1)t^{m-1}}{(\lambda_1 - 1) + (\lambda_2 - \lambda_1)\eta^{m-1}} \int_0^{\eta} \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} f\bigl(s, u(s), {}^{\mathrm{C}}D^{\beta}u(s)\bigr)\,ds \biggr]. \end{aligned}$$

For each $t \in J$, we have

$$\bigl|u(t)\bigr| \le \frac{L(1+\Lambda_1)}{\Gamma(\alpha+1)}\bigl(1 + \|u\|_{\mathbb{X}}\bigr)$$

and

$$\bigl|{}^{\mathrm{C}}D^{\beta}u(t)\bigr| \le L\biggl(\frac{1}{\Gamma(\alpha-\beta+1)} + \frac{\Lambda_2}{\Gamma(\alpha+1)}\biggr)\bigl(1 + \|u\|_{\mathbb{X}}\bigr).$$

Therefore,

$$\|u\|_{\mathbb{X}} \le L\biggl(\frac{1}{\Gamma(\alpha-\beta+1)} + \frac{1+\Lambda_1+\Lambda_2}{\Gamma(\alpha+1)}\biggr)\bigl(1 + \|u\|_{\mathbb{X}}\bigr).$$

Thus, for every $t \in J$, we have

$$\|u\|_{\mathbb{X}} \le \frac{L(\frac{1}{\Gamma(\alpha-\beta+1)} + \frac{1+\Lambda_1+\Lambda_2}{\Gamma(\alpha+1)})}{1 - L(\frac{1}{\Gamma(\alpha-\beta+1)} + \frac{1+\Lambda_1+\Lambda_2}{\Gamma(\alpha+1)})} \le M^*.$$

This shows that the set $E(F)$ is bounded.

As a consequence of Schaefer's fixed point theorem, we deduce that F has a fixed point which is a solution of the fractional boundary value problem (1.1). The proof is complete. □

4 Example

In this section, we give two examples to illustrate the usefulness of our main results.

Example 4.1 Let us consider the following fractional boundary value problem:

$$\begin{cases} {}^{\mathrm{C}}D^{\frac{5}{2}}x(t) = \frac{e^{-at}(x(t)+{}^{\mathrm{C}}D^{\frac{1}{2}}x(t))}{(1+9e^t)(1+x(t)+{}^{\mathrm{C}}D^{\frac{1}{2}}x(t))}, & t \in J_1 := [0,1], \\ x(0) = \frac{1}{3}x(\frac{1}{5}), \qquad x'(0) = 0, \qquad x(1) = \frac{1}{4}x(\frac{1}{5}), \end{cases} \tag{4.1}$$

where $a > 0$ is a constant.

Here, $m = 3$, $\alpha = \frac{5}{2}$, $\beta = \frac{1}{2}$, $\lambda_1 = \frac{1}{3}$, $\lambda_2 = \frac{1}{4}$, and $\eta = \frac{1}{5}$.

Set

$$f(t,x,y) = \frac{e^{-at}(x+y)}{(1+9e^t)(1+x+y)}, \quad (t,x,y) \in J_1 \times [0,\infty) \times [0,\infty).$$

Let x_1, x_2, y_1 and $y_2 \in [0,\infty)$ and $t \in J_1$. Then we have

$$\begin{aligned}
&\bigl|f(t,x_2,y_2) - f(t,x_1,y_1)\bigr| \\
&\quad = \frac{e^{-at}}{1+9e^t}\left|\frac{x_2+y_2}{1+x_2+y_2} - \frac{x_1+y_1}{1+x_1+y_1}\right| \\
&\quad = \frac{e^{-at}(|x_2-x_1| + |y_2-y_1|)}{(1+9e^t)(1+x_2+y_2)(1+x_1+y_1)} \\
&\quad \le \frac{e^{-at}}{1+9e^t}\bigl(|x_2-x_1| + |y_2-y_1|\bigr) \\
&\quad \le \frac{e^{-at}}{10}\bigl(|x_2-x_1| + |y_2-y_1|\bigr).
\end{aligned}$$

Obviously, for all $x, y \in [0,\infty)$ and each $t \in J_1$,

$$\bigl|f(t,x,y)\bigr| = \frac{e^{-at}}{1+9e^t}\left|\frac{x+y}{1+x+y}\right| \le \frac{e^{-at}}{1+9e^t} \le \frac{e^{-at}}{10}.$$

For $t \in J_1$ and $b \in (0,\alpha)$, let $m(t) = h(t) = \frac{e^{-at}}{10} \in L^{\frac{1}{b}}(J_1,\mathbb{R})$, $M = \|\frac{e^{-at}}{10}\|_{L^{\frac{1}{b}}(J_1,\mathbb{R})}$.

Choosing some $a > 0$ large enough and suitable $b \in (0,\frac{5}{2})$ ($b = \frac{1}{2}$ say), one can arrive at the following inequality:

$$\Omega = M\left[\frac{(1+\Lambda_1)}{\Gamma(\frac{5}{2})(\frac{\frac{5}{2}-\frac{1}{2}}{1-\frac{1}{2}})^{1-\frac{1}{2}}} + \left(\frac{1}{\Gamma(2)(\frac{2-\frac{1}{2}}{1-\frac{1}{2}})^{1-\frac{1}{2}}} + \frac{\Lambda_2}{\Gamma(\frac{5}{2})(\frac{\frac{5}{2}-\frac{1}{2}}{1-\frac{1}{2}})^{1-\frac{1}{2}}}\right)\right] < 1,$$

where $\Lambda_1 \approx 2.0731707$ and $\Lambda_2 \approx 1.65128707$. Thus all the assumptions in Theorem 3.1 are satisfied, our results can be applied to problem (4.1).

Example 4.2 Let us consider the following fractional boundary value problem:

$$\begin{cases} {}^C D^{\frac{5}{2}} x(t) = \frac{1}{5+e^{t-1}}(1 + x(t) + {}^C D^{\frac{1}{2}} x(t)), & t \in J_1, \\ x(0) = \frac{1}{3}x(\frac{1}{5}), \qquad x'(0) = 0, \qquad x(1) = \frac{1}{4}x(\frac{1}{5}). \end{cases} \tag{4.2}$$

Set

$$f(t,x,y) = \frac{1}{5+e^{t-1}}(1+x+y), \quad (t,x,y) \in J_1 \times [0,\infty) \times [0,\infty).$$

Let $x, y \in [0,\infty)$ and $t \in J_1$. Then we have

$$\bigl|f(t,x,y)\bigr| = \frac{1}{5+e^{t-1}}|1+x+y| \le \frac{1}{5+e^{t-1}}\bigl(1+|x|+|y|\bigr) \le \frac{1}{6}\bigl(1+|x|+|y|\bigr).$$

According to (H5), $L = \frac{1}{6}$. Then we have

$$L\left(\frac{1}{\Gamma(\alpha-\beta+1)}+\frac{1+\Lambda_1+\Lambda_2}{\Gamma(\alpha+1)}\right) = \frac{1}{6}\times\left(\frac{1}{\Gamma(3)}+\frac{1+2.0731707+1.65128707}{\Gamma(\frac{7}{2})}\right) \approx 0.3202658 \neq 1.$$

Finally, we can get

$$\frac{L(\frac{1}{\Gamma(\alpha-\beta+1)}+\frac{1+\Lambda_1+\Lambda_2}{\Gamma(\alpha+1)})}{1-L(\frac{1}{\Gamma(\alpha-\beta+1)}+\frac{1+\Lambda_1+\Lambda_2}{\Gamma(\alpha+1)})} = \frac{0.3202658}{1-0.3202658} \approx 0.47116328 > 0.$$

This gives that $M^* > 0$. Thus all the assumptions in Theorem 3.2 are satisfied, our results can be applied to problem (4.2).

Competing interests
The author declares that they have no competing interests.

Author's contributions
The author read and approved the final revised manuscript.

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