Advances in Difference Equations
a SpringerOpen Journal

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A generalization of Morley's congruence

Jianxin Liu[1], Hao Pan[2] and Yong Zhang[3*]

*Correspondence:
yongzhang1982@163.com
[3]Department of Mathematics and
Physics, Nanjing Institute of
Technology, Nanjing, 211167,
People's Republic of China
Full list of author information is
available at the end of the article

Abstract

We establish an explicit formula for q-analog of Morley's congruence.

MSC: Primary 11B68; secondary 11B65; 11B83

Keywords: Morley's congruence; q-analog; binomial sums

1 Introduction

For arbitrary positive integer n, let

$$[n]_q = \frac{1-q^n}{1-q} = 1 + q + q^2 + \cdots + q^{n-1},$$

which is the q-analog of an integer n since $\lim_{q \to 1}(1 - q^n)/(1 - q) = n$. Also, for $n, m \in \mathbb{Z}$, define the q-binomial coefficients by

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \frac{[n]_q[n-1]_q \cdots [n-m+1]_q}{[m]_q[m-1]_q \cdots [1]_q}$$

when $m \geq 0$, and if $m < 0$ we set $\begin{bmatrix} n \\ m \end{bmatrix}_q = 0$. It is easy to check that

$$\begin{bmatrix} n+1 \\ m \end{bmatrix}_q = q^m \begin{bmatrix} n \\ m \end{bmatrix}_q + \begin{bmatrix} n \\ m-1 \end{bmatrix}_q.$$

Some combinatorial and arithmetical properties of the binomial sums

$$\sum_{k=0}^{n} \binom{n}{k}^a \quad \text{and} \quad \sum_{k=0}^{n} (-1)^k \binom{n}{k}^a$$

have been investigated by several authors (*e.g.*, Calkin [1], Cusick [2], McIntosh [3], Perlstadt [4]). Indeed, we know (*cf.* [5], equations (3.81) and (6.6)) that

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^2 = (-1)^n \binom{2n}{n} \tag{1.1}$$

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and

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}^3 = (-1)^n\binom{2n}{n}\binom{3n}{n}. \tag{1.2}$$

However, by using asymptotic methods, de Bruijn [6] has showed that no closed form exists for the sum $\sum_{k=0}^{n}(-1)^k\binom{n}{k}^a$ when $a \geq 4$. Wilf proved (in a personal communication with Calkin; see [1]) that the sum $\sum_{k=0}^{n}\binom{n}{k}^a$ has no closed form provided that $3 \leq a \leq 9$.

As a q-analog of (1.1), we have

$$\sum_{k=0}^{2n}(-1)^k q^{(n-k)^2}\begin{bmatrix} 2n \\ k \end{bmatrix}_q^2 = (-1)^n\begin{bmatrix} 2n \\ n \end{bmatrix}_{q^2}. \tag{1.3}$$

Indeed, from the well-known q-binomial theorem (*cf.* Corollary 10.2.2 of [7])

$$\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{\binom{k}{2}} x^k = (x;q)_n,$$

where

$$(x;q)_n = \begin{cases} (1-x)(1-xq)\cdots(1-xq^{n-1}), & \text{if } n \geq 1, \\ 1, & \text{if } n = 0, \end{cases}$$

it follows that

$$(x^2;q^2)_{2n} = (x;q)_{2n}(-x;q)_{2n} = \left(\sum_{k=0}^{2n}\begin{bmatrix} 2n \\ k \end{bmatrix}_q (-1)^k q^{\binom{k}{2}} x^k\right)\left(\sum_{k=0}^{2n}\begin{bmatrix} 2n \\ k \end{bmatrix}_q q^{\binom{k}{2}} x^k\right)$$

$$= \sum_{m=0}^{4n} x^m \sum_{k=0}^{2n}\begin{bmatrix} 2n \\ k \end{bmatrix}_q\begin{bmatrix} 2n \\ m-k \end{bmatrix}_q (-1)^k q^{\binom{k}{2}+\binom{m-k}{2}},$$

whence (1.3) is derived by comparing the coefficients of x^{2n} in the equation above.

As early as 1895, with the help of De Moivre's theorem, Morley [8] proved that

$$(-1)^{\frac{p-1}{2}}\binom{p-1}{(p-1)/2} \equiv 4^{p-1} \pmod{p^3}. \tag{1.4}$$

In [9], Pan gave a q-analog of Morley's congruence and showed that

$$(-1)^{\frac{n-1}{2}} q^{\frac{n^2-1}{4}}\begin{bmatrix} n-1 \\ (n-1)/2 \end{bmatrix}_{q^2} \equiv (-q;q)_{n-1}^2 - \frac{n^2-1}{24}(1-q)^2[n]_q^2 \pmod{\Phi_n(q)^3}, \tag{1.5}$$

where

$$\Phi_n(q) = \prod_{\substack{1 \leq j \leq n \\ (j,n)=1}} (q - e^{2\pi ij/n})$$

is the nth cyclotomic polynomial. In this section, we shall establish a generalization of Morley's congruence (1.4) proved by Cai and Granville [10], Theorem 6:

$$\sum_{k=0}^{p-1}(-1)^{(a-1)k}\binom{p-1}{k}^a \equiv 2^{a(p-1)} \pmod{p^3} \tag{1.6}$$

for any prime $p \geq 5$ and positive integer a. We also shall obtain a generalization of (1.5) in view of (1.3).

Theorem 1.1 *Let n be a positive odd integer. Then*

$$\sum_{k=0}^{n-1}(-1)^{(a-1)k}q^{a\binom{k+1}{2}}\begin{bmatrix}n-1\\k\end{bmatrix}_q^a$$

$$\equiv (-q;q)_{n-1}^a + \frac{a(a-1)(n^2-1)}{24}(1-q)^2[n]_q^2 \pmod{\Phi_n(q)^3}. \tag{1.7}$$

Furthermore, we have

$$q^{a(n^2-1)/4}\sum_{k=0}^{n-1}(-1)^k q^{a((n-1)/2-k)^2}\begin{bmatrix}n-1\\k\end{bmatrix}_q^{2a}$$

$$\equiv (-q;q)_{n-1}^{2a} + \frac{a(a-2)(n^2-1)}{24}(1-q)^2[n]_q^2 \pmod{\Phi_n(q)^3}. \tag{1.8}$$

Remark Clearly (1.6) is the special case of (1.7) in the limiting case $q-> 1$ for $n = p$.

2 Some lemmas

In this section, the following lemmas will be used in the proof of Theorem 1.1.

Lemma 2.1

$$q^{kn} \equiv 1 - k(1-q)[n]_q + \frac{k(k-1)}{2}(1-q)^2[n]_q^2 \pmod{[n]_q^3}. \tag{2.1}$$

Proof

$$q^{kn} = \sum_{j=0}^{k}(-1)^j\binom{k}{j}(1-q)^j[n]_q^j \equiv 1 - k(1-q)[n]_q + \frac{k(k-1)}{2}(1-q)^2[n]_q^2 \pmod{[n]_q^3}. \qquad \square$$

Lemma 2.2 *Let n be a positive odd integer. Then*

$$\sum_{j=1}^{(n-1)/2}\frac{1}{[2j]_q^2} = \sum_{j=1}^{(n-1)/2}\frac{q^{2j}}{[2j]_q^2} + (1-q)\sum_{j=1}^{(n-1)/2}\frac{1}{[2j]_q}$$

$$\equiv -\frac{n^2-1}{24}(1-q)^2 - Q_n(2,q)(1-q) \pmod{\Phi_n(q)}, \tag{2.2}$$

where the q-Fermat quotient is defined by

$$Q_n(m,q) = \frac{(q^m;q^m)_{n-1}/(q;q)_{n-1} - 1}{[n]_q}.$$

Lemma 2.3 *Let n be a positive odd integer. Then*

$$2 \sum_{j=1}^{(n-1)/2} \frac{1}{[2j]_q} + 2Q_n(2,q) - Q_n(2,q)^2 [n]_q$$

$$\equiv \left(Q_n(2,q)(1-q) + \frac{n^2-1}{8}(1-q)^2 \right) [n]_q \pmod{\Phi_n(q)^2}. \tag{2.3}$$

When n is an odd prime, the above two lemmas have been proved in [9], equation (2.7) and [9], Theorem 1.1, respectively. Of course, clearly the same discussions are also valid for general odd n.

3 Proofs of Theorem 1.1

In this section, we shall prove (1.7) and (1.8).

Proof By the properties of the q-binomial coefficients, we know that

$$(-1)^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q = \prod_{j=1}^{k} \frac{[j]_q - [n]_q}{q^j [j]_q}$$

$$\equiv q^{-\binom{k+1}{2}} \left(1 - \sum_{j=1}^{k} \frac{[n]_q}{[j]_q} + \sum_{1 \le i < j \le k} \frac{[n]_q^2}{[i]_q [j]_q} \right) \pmod{\Phi_n(q)^3}.$$

Thus

$$(-1)^{ak} q^{a\binom{k+1}{2}} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q^a$$

$$\equiv 1 - a \sum_{j=1}^{k} \frac{[n]_q}{[j]_q} + a \sum_{1 \le i < j \le k} \frac{[n]_q^2}{[i]_q [j]_q} + \binom{a}{2} \left(\sum_{j=1}^{k} \frac{[n]_q}{[j]_q} \right)^2 \pmod{\Phi_n(q)^3}.$$

Noting that

$$\left(\sum_{j=1}^{k} \frac{1}{[j]_q} \right)^2 = 2 \sum_{1 \le i < j \le k} \frac{1}{[i]_q [j]_q} + \sum_{j=1}^{k} \frac{1}{[j]_q^2},$$

we have

$$\sum_{k=1}^{n-1} (-1)^{(a-1)k} q^{a\binom{k+1}{2}} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q^a$$

$$\equiv \sum_{k=1}^{n-1} (-1)^k \left(1 - a \sum_{j=1}^{k} \frac{[n]_q}{[j]_q} + a \sum_{1 \le i < j \le k} \frac{[n]_q^2}{[i]_q [j]_q} + \binom{a}{2} \left(\sum_{j=1}^{k} \frac{[n]_q}{[j]_q} \right)^2 \right)$$

$$= \left(-a \sum_{j=1}^{n-1} \frac{[n]_q}{[j]_q} + a^2 \sum_{1 \le i < j \le n-1} \frac{[n]_q^2}{[i]_q [j]_q} + \binom{a}{2} \sum_{j=1}^{n-1} \frac{[n]_q^2}{[j]_q^2} \right) \sum_{k=j}^{n-1} (-1)^k$$

$$= -a \sum_{\substack{j=1 \\ 2|j}}^{n-1} \frac{[n]_q}{[j]_q} + a^2 \sum_{\substack{1 \le i < j \le n-1 \\ 2|j}} \frac{[n]_q^2}{[i]_q [j]_q} + \binom{a}{2} \sum_{\substack{j=1 \\ 2|j}}^{n-1} \frac{[n]_q^2}{[j]_q^2} \pmod{\Phi_n(q)^3}. \tag{3.1}$$

Thus letting $a = 1$ in (3.1), we get

$$\sum_{\substack{1 \le i < j \le n-1 \\ 2|j}} \frac{[n]_q^2}{[i]_q [j]_q} \equiv (-q; q)_{n-1} - 1 + \sum_{\substack{j=1 \\ 2|j}}^{n-1} \frac{[n]_q}{[j]_q} \pmod{\Phi_n(q)^3}, \tag{3.2}$$

whence by (2.3) and (3.2)

$$\sum_{\substack{1 \le i < j \le n-1 \\ 2|j}} \frac{1}{[i]_q [j]_q} \equiv \frac{Q_n(2,q)^2}{2} + \frac{Q_n(2,q)}{2}(1-q) + \frac{n^2-1}{16}(1-q)^2 \pmod{\Phi_n(q)}.$$

Recalling (2.2) and (2.3), then we obtain

$$\sum_{k=0}^{n-1} (-1)^{(a-1)k} q^{a\binom{k+1}{2}} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q^a$$

$$\equiv 1 + a[n]_q Q_n(2,q) + \binom{a}{2} [n]_q^2 Q_n(2,q)^2 + \binom{a}{2} \frac{n^2-1}{12}(1-q)^2 [n]_q^2$$

$$\equiv \sum_{j=0}^{a} \binom{a}{j} [n]_q^j Q_n(2,q)^j + \binom{a}{2} \frac{n^2-1}{12}(1-q)^2 [n]_q^2$$

$$= (-q; q)_{n-1}^a + \frac{a(a-1)(n^2-1)}{24}(1-q)^2 [n]_q^2 \pmod{\Phi_n(q)^3}.$$

Let us turn to (1.8). Similarly

$$\sum_{k=0}^{n-1} (-1)^k q^{2a\binom{k+1}{2}} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q^{2a} - q^{a(n^2-1)/4} \sum_{k=0}^{n-1} (-1)^k q^{a((n-1)/2-k)^2} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q^{2a}$$

$$= 1 - q^{a\binom{n}{2}} + \sum_{k=1}^{n-1} (-1)^k q^{2a\binom{k+1}{2}} \left(1 - q^{a(\binom{n}{2}-nk)}\right) \begin{bmatrix} n-1 \\ k \end{bmatrix}_q^{2a}$$

$$\equiv 1 - q^{a\binom{n}{2}} + \sum_{k=1}^{n-1} (-1)^k \left(1 - q^{a(\binom{n}{2}-nk)}\right) \left(1 - 2a \sum_{j=1}^{k} \frac{[n]_q}{[j]_q}\right) \pmod{\Phi_n(q)^3}.$$

Recalling (2.1), then we have

$$1 - q^{a(\binom{n}{2}-nk)} \equiv a\left(\frac{n-1}{2} - k\right)(1 - q^n) + \binom{a((n-1)/2 - k)}{2}(1 - q^n)^2 \pmod{\Phi_n(q)^3},$$

therefore

$$\sum_{k=1}^{n-1} (-1)^k \left(1 - q^{a(\binom{n}{2}-nk)}\right) \left(1 - 2a \sum_{j=1}^{k} \frac{[n]_q}{[j]_q}\right)$$

$$\equiv \frac{q^{a\binom{n}{2}} - q^{-a\binom{n}{2}}}{1 + q^{an}} - 2a^2 \sum_{j=1}^{n-1} \frac{[n]_q}{[j]_q} \sum_{k=j}^{n-1} (-1)^k \left(\frac{n-1}{2} - k\right)(1 - q^n)$$

$$= \frac{q^{a\binom{n}{2}} - q^{-a\binom{n}{2}}}{1 + q^{an}} + a^2(1 - q^n)[n]_q \left(\sum_{\substack{j=1 \\ 2|j}}^{n-1} \frac{j}{[j]_q} + \sum_{\substack{j=1 \\ 2 \nmid j}}^{n-1} \frac{n-j}{[j]_q}\right) \pmod{\Phi_n(q)^3}.$$

Since

$$q^{an} = \left(1 - \left(1 - q^n\right)\right)^a \equiv 1 - a\left(1 - q^n\right) + \binom{a}{2}\left(1 - q^n\right)^2 \pmod{\Phi_n(q)^3}$$

and

$$\frac{1}{2 - a(1 - q^n)} = \frac{1}{2}\frac{1}{1 - a(1 - q^n)/2} \equiv \frac{1}{2}\left(1 + a\left(1 - q^n\right)/2\right) \pmod{\Phi_n(q)^2},$$

we have

$$\frac{q^{a\binom{n}{2}} - q^{-a\binom{n}{2}}}{1 + q^{an}} \equiv -\frac{a(n-1)(1 - q^n) + a(n-1)/2 \cdot (1 - q^n)^2}{2 - a(1 - q^n)}$$

$$\equiv -\frac{a(n-1)}{2}\left(\left(1 - q^n\right) + \frac{a+1}{2}\left(1 - q^n\right)^2\right) \pmod{\Phi_n(q)^3}.$$

Noting that

$$\sum_{\substack{j=1 \\ 2|j}}^{n-1} \frac{j}{[j]_q} + \sum_{\substack{j=1 \\ 2\nmid j}}^{n-1} \frac{n-j}{[j]_q} = \sum_{\substack{j=1 \\ 2|j}}^{n-1} \frac{j}{[j]_q} + \sum_{\substack{j=1 \\ 2|j}}^{n-1} \frac{j}{[n-j]_q}$$

$$= \sum_{\substack{j=1 \\ 2|j}}^{n-1} \left(\frac{j}{[j]_q} + \frac{jq^j}{[n]_q - [j]_q}\right) \equiv \frac{n^2 - 1}{4}(1 - q) \pmod{\Phi_n(q)},$$

we have

$$\sum_{k=0}^{n-1}(-1)^k q^{2a\binom{k+1}{2}}\begin{bmatrix} n-1 \\ k \end{bmatrix}_q^{2a} - q^{a(n^2-1)/4}\sum_{k=0}^{n-1}(-1)^k q^{a((n-1)/2-k)^2}\begin{bmatrix} n-1 \\ k \end{bmatrix}_q^{2a}$$

$$\equiv \left(-\binom{a(n-1)/2}{2} - \frac{a(a+1)(n-1)}{4} + \frac{a^2(n^2-1)}{4}\right)\left(1 - q^n\right)^2$$

$$= \frac{a^2(n^2-1)}{8}\left(1 - q^n\right)^2 \pmod{\Phi_n(q)^3}.$$

In view of (1.7), this concludes the proof of (1.8). □

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Author details
[1]Department of Teaching Affairs, Nanjing Institute of Technology, Nanjing, 211167, People's Republic of China.
[2]Department of Mathematics, Nanjing University, Nanjing, 210093, People's Republic of China. [3]Department of
Mathematics and Physics, Nanjing Institute of Technology, Nanjing, 211167, People's Republic of China.

References

1. Calkin, NJ: Factors of sums of powers of binomial coefficients. Acta Arith. **86**, 17-26 (1998)
2. Cusick, TW: Recurrences for sums of powers of binomial coefficients. J. Comb. Theory, Ser. A **52**, 77-83 (1989)
3. McIntosh, RJ: Recurrences for alternating sums of powers of binomial coefficients. J. Comb. Theory, Ser. A **63**, 223-233 (1993)
4. Perlstadt, MA: Some recurrences for sums of powers of binomial coefficients. J. Number Theory **27**, 304-309 (1987)
5. Gould, HW: Combinatorial Identities: A Standardized Set of Tables Listing 500 Binomial Coefficient Summations. Henry W. Gould, Morgantown (1972)
6. de Bruijn, NG: Asymptotic Methods in Analysis. Dover, New York (1981)
7. Andrews, GE, Askey, R, Roy, R: Special Functions. Cambridge University Press, Cambridge (1999)
8. Morley, F: Note on the congruence $2^{4n} \equiv (-1)^n (2n)!/(n!)^2$, where $2n + 1$ is a prime. Ann. Math. **9**, 168-170 (1895)
9. Pan, H: A q-analogue of Lehmer's congruences. Acta Arith. **128**, 303-318 (2007)
10. Cai, T-X, Granville, A: On the residues of binomial coefficients and their products modulo prime powers. Acta Math. Sin. Engl. Ser. **18**, 277-288 (2002)